Some Combinatorial Aspects of Time-Stamp Systems

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Abstract

Many problems in distributed computing are solved using the paradigm of bounded time-stamps. The basic object of this powerful technique is a finite directed graph called a time-stamp system. The vertices of this graph are used as labels for processes in the system, and the arcs between the labels encode the order of creation of two processes. The requirements generally considered lead to a time-stamp system whose size is an exponential function of the maximal number k of living processes in the system. In the following, we restrict the requirements and we construct time-stamp systems of linear size in k for these restricted problems.

0 Introduction

Recently, many problems in the coordination of concurrent processes were shown to be solvable by using a bounded set of time-stamps [5, 8, 11]. Let us begin by describing the main features of these systems.

In a system, processes are created and die. When a process is created, a time-stamp is assigned to it; it indicates the "logical time" when this creation took place. The aim of this time-stamp is to allow the determination of the most recently created process between any pair of living processes. Generally [10], the set of natural numbers is used as the set of time-stamps. In [8] Israëli and Li proved that, when the number of living processes is assumed to be bounded by an integer k, a time-stamp system with a finite number of elements may be used. They observed the analogy with a combinatorial problem adressed by Erdös [6] and others [7, 12] in the sixties. Although proving the existence of such a system with $(2+\epsilon)^k$ elements, they constructed explicitly a system with 3^{k-1} elements. More recently, Zielonka [14] improved their solution by giving a time-stamp system with $k2^{k-1}$ elements; he also proved that this time-stamp system is optimal if it is required that any stamp also contains in a certain sense the name of the process. A generalization of time-stamp systems was considered in [4] as an application of the construction of a certain family of automata; these may be considered as distributed time-stamp systems.

In the following paper we consider two new problems consisting in building a restricted time-stamp system and we give solutions with a set whose size is a linear function of the maximal number k of living processes. The first restriction concerns the order in which processes die: we restrict the processes dying to the p older ones. We call these systems *p-restricted* time stamp systems. Using lexicographic product on graphs we obtain a p-restricted time-stamp system with $p2^{p-1}(2k-2p+1)$ elements.

The second restriction is obtained by weakening the information asked to the system. It is assumed that there are always exactly k living processes (immediately after the death of any process another one is created), and the determination of the last process created is required when the whole set of labels is given. We call these systems *weak* time-stamp systems. The determination of weak time-stamp systems was already considered [11] and a solution with k^2 time-stamps was given. We improve this result by giving a weak time-stamp system with $2k-1$ elements. Our construction uses a matching from the family of $(k-1)$-subsets of $\{1,\ldots,2k-1\}$ onto the family of its k-subsets. This matching was considered by many authors [1, 2, 9, 13]. Note that weak time-stamp systems may be used to solve the mail-box problem stated in [3].

Only sequential time-stamps are examined here, concurrent ones being left for further investigation.

1 Time-stamps

In this section, we give the definitions and some combinatorial results on time-stamp systems, most of them being due to Israëli & Li. Let us begin giving some notation.

A *directed graph* is defined as a finite set X of *vertices* together with a set of *arcs* which is a subset E of $X \times X$. If (x,y) is an arc, the vertex y is said to dominate x.

The set of all dominators of a vertex x is denoted by $\Gamma_G(x)$.

$$\Gamma_G(x) = \{y \mid (x,y) \in E\}$$

For a subset $Y \subset X$, $\Gamma_G(Y)$ denotes the set of vertices which are dominators of all the elements of Y.

$$\Gamma_G(Y) = \bigcap_{y \in Y} \Gamma_G(y).$$

In the whole paper we only consider loopless and antisymmetric graphs. They satisfy

$$\forall\, x,y \in X,\ (x,x) \notin E \quad \text{and} \quad (x,y) \in E \Rightarrow (y,x) \notin E$$

A sequence $(y_1, y_2, \ldots, y_p)$ of vertices is an *ordered sequence* if for any $1 \leq i < j \leq p$, y_j is a dominator of y_i.

Definition 1.1 *A time-stamp system of order k is a directed graph, in which any ordered sequence having less than k elements has a dominator.*

In such a graph, any vertex belongs to an ordered sequence of cardinality k. A related notion was considered by many authors after Erdös [6, 7, 12] namely that of a tournament (i.e. a directed graph in which for any pair of vertices $\{x,y\}$ one is the dominator of the other) satisfying the so-called property $S(p)$. For such a tournament, any subset of cardinality p has a dominator. Hence, any tournament with property $S(p)$ is a time-stamp system of order $p+1$ but the converse is not true. An example of a tournament which is a time-stamp system of order 4 and which does not satisfy $S(3)$ is given below. The lower bounds found for the number of vertices a tournament must have in order to satisfy $S(p)$, are hence not valid for time-stamps systems; however, similar constructions hold.

For any graph $G = (X, E)$ and any vertex x let us denote by G_x the graph whose vertex set is $\Gamma_G(x)$, and whose edge set is $E \cap (\Gamma_G(x) \times \Gamma_G(x))$. We get :

Proposition 1.2 *If G is a time-stamp system of order k, then for any x in X, G_x is a time-stamp system of order $k-1$.*

Proof. If $(y_1, y_2, \ldots, y_p)$ is an ordered sequence in G_x then $(x, y_1, y_2, \ldots, y_p)$ is an ordered sequence in G. If $p < k-1$, since G is a time-stamp system of order k, the sequence $(x, y_1, \ldots, y_p)$ has a dominator which is in $\Gamma_G(x)$. □

Corollary 1.3 *The number of vertices of a time-stamp system of order k is not less than $2^k - 1$.*

Proof. We use induction on k. For $k = 0, 1$ there is nothing to prove. The first non trivial case is $k = 2$ and the smallest time-stamp system of order 2 is the circuit C_3 with 3 vertices. Let G be a time-stamp system of order $k+1$ having n vertices. By the induction hypothesis and by proposition 1.2 each of the G_x's has not less than $2^k - 1$ vertices. Hence the number of arcs $|E|$ of G satisfies $|E| \geq n(2^k - 1)$. Since G is antisymmetric and loopless $|E| \leq \frac{n(n-1)}{2}$ and the result follows. □

Note that the converse of proposition 1.2 holds:

Proposition 1.4 *Let G be an antisymmetric graph such that for any vertex x, G_x is a time-stamp system of order $k-1$. Then G is a time-stamp system of order k.*

Proof. Let $(x_1, x_2, \ldots, x_l)$, $l < k$ be an ordered sequence in G. Then $(x_2, \ldots, x_l)$ is an ordered sequence in G_{x_1}. By the hypothesis it has a dominator x in G_{x_1}, and x is a dominator of $(x_1, x_2, \ldots, x_l)$. □

The following classical notion in graph theory is useful in order to build time-stamp systems.

Definition 1.5 *Let $G = (X, E)$ and $H = (Y, F)$ be two directed graphs. The lexicographic product $G \otimes H$ has vertex set $X \times Y$ and its set of arcs is given by*

$$(x', y') \in \Gamma_{G\otimes H}(x, y) \text{ iff } (x, x') \in E \quad \text{or} \quad (x = x' \text{ and } (y, y') \in F)$$

Proposition 1.6 *If G and H are time-stamp systems of respective order k and l, then $G \otimes H$ is a time-stamp system of order $k + l - 1$.*

Proof. Let $(u_1, u_2, \ldots, u_m)$ be an ordered sequence in $G \otimes H$, such that $m < k + l - 1$. Let $u_i = (x_i, y_i)$, then the sequence of x_i's is an ordered sequence in G. Note that the x_i's are not necessarily distinct. If the number of distinct x_i's is less than k, they have a dominator x in X and for any $y \in Y$, (x, y) is a dominator of $(u_1, u_2, \ldots, u_m)$. If the number of distinct x_i's is not less than k, then the number of those equal to x_m is less than l. Let $(y_j, \ldots, y_m)$ be such that $x_j = x_m$ and $x_{j-1} \neq x_m$. This sequence is an ordered sequence in H with less than l elements, thus it has a dominator y and (x_m, y) is a dominator of $(u_1, u_2, \ldots, u_m)$. □

From this proposition follows a method for the construction of time-stamp systems of arbitrary order. Using the graph C_3, it is possible to get a time-stamp system of order k with 3^{k-1} vertices [8]. Other time-stamp systems are known; for little values of k the smallest are given by the tournaments satisfying $S(p)$ and for greater values

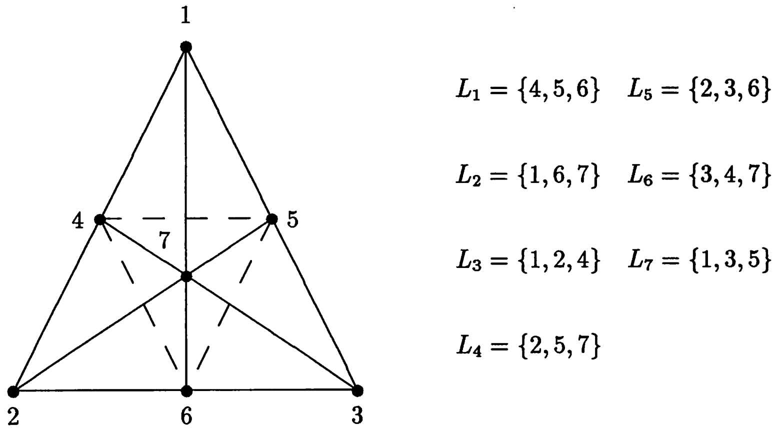

Figure 1: The Fano plane.

by a construction due to Zielonka [14]. We recall here these results, one of them being that the Fano plane of order 7 gives a time-stamp system of order 3. Consider the dominators of vertex i as a line L_i of this plane. Since a time-stamp system is loopless and antisymmetric, the lines have to be numbered such that

$$i \notin L_i \text{ and } j \in L_i \Rightarrow i \notin L_j$$

This is possible for the Fano plane and this numbering is given in Figure 1.

The corresponding graph F_7 is the smallest time-stamp system of order 3, it has 7 vertices and is given by $\Gamma_{F_7}(i) = L_i$. E. & G. Szekeres [12] gave a tournament satisfying property $S(3)$ with 19 vertices, it is the smallest time-stamp system of order 4, already known. Note that $C_3 \otimes C_3 \otimes C_3$ is a time-stamp system of order 4 which is a tournament but which does not satisfy $S(3)$. The following construction, due to Zielonka [14], gives a time-stamp system of order k with $k2^{k-1}$ vertices; for $k \geq 9$ no time-stamp system with a smaller number of vertices is known.

Consider the subset X_k of $\{1 \dots k\} \times \{0,1\}^k$ consisting of elements $(\alpha, x_1, \dots, x_k)$ such that $x_\alpha = 0$, as a set of vertices of a graph $G = (X_k, E_k)$ and let E_k be such that

$$(\beta, y_1, \dots, y_k) \in \Gamma_G(\alpha, x_1, \dots, x_k) \text{ if } (\alpha > \beta \text{ and } x_\beta \neq y_\alpha) \text{ or } (\alpha < \beta \text{ and } x_\beta = y_\alpha)$$

Proposition 1.7 *G is a time-stamp system of order k having $k2^{k-1}$ vertices.*

Proof. Clearly the number of vertices of G is $k2^{k-1}$. Since there are no arcs between two vertices with the same first component, any ordered sequence U of G must have vertices in which all the first components are distinct. Now, if U has less than k elements then at least one $\alpha \in \{1, 2, \ldots, k\}$ is available for the first component of a dominator x of U. To end the proof, it is necessary to define the other components $x_1, \ldots, x_k$ of x. Of course $x_\alpha = 0$; to obtain x_β, if there exists an element $y \in U$ with β as first component take

$$x_\beta = y_\alpha \text{ if } \beta < \alpha \quad \text{and} \quad x_\beta = 1 - y_\alpha \text{ if } \alpha < \beta$$

If no such element exists take $x_\beta = 0$. □

2 Restricted Time-Stamp Systems

Consider the family $\Im$ of ordered sequences having k elements in a time-stamp system of order k; then, for any U and any $u_i \in U$, there exists $v \in X$ such that $U \backslash \{u_i\} \cup \{v\} \in \Im$.

Returning to the labeling of processes, this means that when k processes are living and one dies, then a time-stamp can be given to a new process. Let us restrict the set of processes which may die to the elder ones, introducing the following notion.

Definition 2.1 *A p-restricted time-stamp system of order k is a loopless antisymmetric graph such that there exists a family $\Im$ of ordered sequences with k elements satisfying*

(2.1) *For any vertex* $x \in G, \;\; \exists\, U \in \Im$ *such that* $x \in U$.

(2.2) *If* $U = (u_1, \ldots, u_k)$ *and if* $i \leq p, \; \exists v$ *such that* $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, v) \in \Im$.

We will first build a 1-restricted time-stamp system of order k, then we will show that for a fixed p, there exists a p-restricted time-stamp system of order k with a number of vertices which is a linear function of k.

Definition 2.2 *Let G_k be the graph with vertex set $\{1, \ldots, 2k-1\}$ and for each vertex i, let $\Gamma_{G_k}(i) = \{i+1, i+2, \ldots, i+k-1\}$ where the sums are taken mod$(2k-1)$.*

This graph is a tournament, and moreover each vertex is the dominator of $k-1$ vertices and has $k-1$ dominators. It is not so difficult to verify that :

Proposition 2.3 *G_k is a 1-restricted time-stamp system of order k.*

Proof. Consider the family $\Im$ of all ordered sequences having k elements. Any $U \in \Im$ has the form $(i, i+1, \ldots, i+k-1)$, where the sums are taken $mod(2k-1)$. Since we are only checking the 1-restricted property, it is sufficient to find a dominator for $(i+1, \ldots, i+k-1)$, which is $i+k$. □

The graph G_k allows us to build p-restricted time-stamp systems for any arbitrary integer p since we have :

Proposition 2.4 *Let $H = (X, E)$ be a time-stamp system of order p. Then $H \otimes G_k$ is a p-restricted time-stamp system of order $k+p-1$.*

Proof. Let us first give some notation. Let Y_k denote the set of vertices of G_k and for any ordered sequence $V = (v_1, \ldots, v_m)$ in $H \otimes G_k$, where $v_i = (x_i, y_i)$, let $\alpha(V)$ be the subset of X consisting of the first components of the v_i's, and let $\beta(V)$ be the subset of Y_k consisting of the second components of the elements whose first one is equal to x_m :

$$\begin{aligned} \alpha(V) &= \{x_i | u_i = (x_i, y_i)\}, \\ \beta(V) &= \{y_i | x_i = x_m\}. \end{aligned}$$

Let $\Im$ be the family of ordered sequences V having $k+p-1$ elements and such that

$$\begin{array}{ll} (i) & card(\alpha(V)) \leq p, \\ (ii) & \beta(V) = y, y+1, \ldots, y+i \;\; mod(2k-1), \; i < k. \end{array}$$

Thus, $\beta(V)$ consists of consecutive elements in Y_k.

We first prove that a vertex (x, y) of $H \otimes G_k$ belongs to at least one element of $\Im$. Consider an ordered sequence U in H of order p and containing x as its last element,

$$U = (x_1, x_2, \ldots, x_{p-1}, x_p = x).$$

Then, the following sequence v is an element of $\Im$:

$$(v_1 = (x_1, y_1), v_2 = (x_2, y_2), \ldots, v_p = (x, y_p), v_{p+1} = (x, y_p{+}1), \ldots, v_{p+k-1} = (x, y_p{+}k{-}1))$$

where the y_i's, $(i = 1, \ldots, p)$ are abitrarily taken in Y_k.

Let now $V = (v_1, v_2, \ldots, v_m)$ be an element of $\Im$ where $m = p+k-1$, and consider $v_i \in V, i \leq p$. Since $\beta(V)$ is an ordered sequence in G_k we have $card(\beta(V)) \leq k$, hence either $v_i = (x_i, y_i)$ is such that $x_i \neq x_m$ or $i = p$ and $(x_j, y_j) = (x_m, y_i + j - i)$ for $j = i+1, \ldots, m$.

If $x_i = x_m$ or if $\beta(V)$ has less than k elements, let $v = (x_m, y_m)$. Then

$$(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m, v)$$

is an element of $\Im$.

If $x_i \neq x_m$ and $\beta(V)$ has k elements, then $\alpha(V)$ has no more than $m - k + 1 = p$ elements and $\alpha(V) \setminus \{v_i\}$ has less than p elements. Since H is a time-stamp system of order p there exists $x \in X$ such that $\alpha(V) \setminus \{x_i\} \cup \{x\}$ is an ordered sequence in H with x as last element. Let $v = (x, y)$ where y is any element of Y_k. Then $(V = v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m, v)$ is an element of $\Im$. □

Corollary 2.5 *There exists a p-restricted time-stamp system of order k with* $p2^{p-1}(2k - 2p + 1)$ *vertices.*

3 Weak time-stamp systems

A time-stamp system allows us to compare any pair of stamps. In many applications this strong request may be weakened to the determination of the last created process when the whole set of the k living processes (namely their time-stamps) is known. This informal requirement may be made precise by the following definition :

Let X be a finite set and let $\Im$ be a family of k-subsets of X. Then $\Im'$ denotes the family of $(k-1)$-subsets Y' of X such that $\exists Y \in \Im,\ \ Y' \subset Y$.

Definition 3.1 *A weak time-stamp system of order k on the family $\Im$ is given by two mappings α and β:*

$$\begin{aligned} \alpha &: \Im \to X \\ \beta &: \Im' \to X \end{aligned}$$

satisfying:

(1) $\forall x \in X,\ \exists Y \in \Im,\ \ x \in Y$

(2) $\alpha(Y) \in Y \quad \text{and} \quad \beta(Y') \notin Y'$

(3) $\alpha(Y' \cup \beta(Y')) = \beta(Y') \quad \text{and} \quad \beta(Y \setminus \alpha(Y)) = \alpha(Y).$

Note that the two parts of (3) are equivalent provided that $\forall\ Y \in \Im,\ \ \exists\ Y' \in \Im'$ such that $Y = Y' \cup \beta(Y')$ and $\forall\ Y' \in \Im',\ \exists\ Y \in \Im$ such that $Y' = Y \setminus \alpha(Y)$.

In the context of processes, $\alpha(Y)$ is the last created time-stamp where Y is a set of k living processes, and $\beta(Y')$ is the time-stamp which has to be assigned to a new process when the set of living processes is Y'.

With this definition, it is assumed that there are always k or $k-1$ living processes. If the determination of the last created process is required for any set of less than k processes, then we are lead to a situation more or less similar to that of ordinary time-stamp systems. To verify this fact it suffices to consider the algorithm allowing us to compare any pair of time-stamps contained in a same element Y of $\Im$ by deleting iteratively the last element of Y until one of the two time-stamps to compare is found.

The following proposition allows us to build weak time-stamp systems :

Proposition 3.2 *There exists a weak time-stamp system on* $< X, \Im >$ *if and only if (1) is satisfied and there exists a bijection* λ *of* $\Im$ *onto* $\Im'$ *such that*

$$\forall\, Y \in \Im, \quad \lambda(Y) \subset Y. \tag{4}$$

Proof. Let $\Im$ be a family of k-subsets of X satisfying (1), and let λ be a bijection of $\Im$ onto $\Im'$ satisfying (4). Define α and β by:

$$\begin{aligned} \alpha(Y) &= Y \setminus \lambda(Y), \\ \beta(Y') &= \lambda^{-1}(Y') \setminus Y'. \end{aligned}$$

Clearly, the definitions of α and β imply (2). The verification of (3) is straightforward:

$$\alpha(Y' \cup \beta(Y')) = \alpha(\lambda^{-1}(Y')) = \lambda^{-1}(Y') \setminus \lambda(\lambda^{-1}(Y')) = \beta(Y').$$

Conversely, let $(X, \Im, \alpha, \beta)$ be a weak time-stamp system and consider λ defined by $\lambda(Y) = Y \setminus \alpha(Y))$, it is easy to verify that λ' defined by $\lambda'(Y') = Y' \cup \beta(Y')$ is the inverse of λ. □

Corollary 3.3 *For any weak time-stamp system* $(X, \Im, \alpha, \beta)$, $|X| \geq 2k-1$.

Proof. Consider the bipartite graph whose vertex set is $\Im \cup \Im'$, and whose edge set is given by the pairs $\{Y, Y'\}$ satisfying $Y' \subset Y$. In this graph, every element $Y \in \Im$ has valency k and any element $Y' \in \Im'$ has valency at most $|X| - k + 1$. Thus, if m denotes the cardinality of $\Im$ we get

$$km \leq m(|X| - k + 1)$$

and the result follows. □

Proposition 3.4 *For any* k*, there exists a weak time-stamp system of order* k *with* $2k-1$ *elements. Moreover, the computation of the mappings* α *and* β *can be done with a number of operations which is a linear function of* k*.*

Proof. Let $X = \{1, \dots, 2k-1\}$ and let $\Im$ be the family of all k-subsets of X. Then $\Im'$ is the family of $(k-1)$-subsets of X. The existence of a matching from $\Im'$ into $\Im$ is a classical result of combinatorial theory. It may be obtained as a consequence of Hall's theorem, also known as the "marriage theorem". The following algorithms allow the computation of $\alpha(Y)$ and $\beta(Y')$; they use a last-in/first-out stack S.

Algorithm 1 : Determination of $\alpha(Y)$

```
. for i := 1 step 1 until 2k − 1 do
.   begin
.     if i ∉ Y then push(S, i)
.     else if notempty(S)
.       then pop(S) else x := i
.   end;
.   α(Y) := x
```

Algorithm 2 : Determination of $\beta(Y')$

```
. for i := 1 step 1 until 2k − 1 do
.   begin
.     if i ∉ Y' then push(S, i)
.     else if notempty(S) then pop(S)
.   end;
.   while notempty(S)
.     do begin x := top(S); pop(S) end;
.   β(Y') := x
```

□

These algorithms can be found in [9] (exercise 1 p 567); they are there attributed to Debruijn et al. [2]. Aigner [1] proposed another algorithm using lexicographic order on the k-subsets of $\{1, 2, \dots, 2k-1\}$ and Trehel [13] proved that these two algorithms give the same matching.

References

[1] M. Aigner, *Lexicographic matchings in Boolean algebras,* J. Comb. Theory, B, **14** (1973), 187 – 194.

[2] N.G. de Bruijn, van Ebbenhorst Tengbergen, Kuyswijk, *On the set of divisors of a number*, Nieuw Archief voor Wiskunde 2, **23** (1951), 191 – 193.

[3] R.Cori, M. Latteux, M. Roos, E. Sopena, *2-Asynchronous automata,* Theoretical Comp. Sci. **61** (1988), 93 – 102.

[4] R. Cori, Y. Metivier, W. Zielonka, *Asynchronous mappings and asynchronous cellular automata,* (1990), to appear in Information and Computation.

[5] D. Dolev, N. Shavit, *Bounded concurrent time-stamp systems are constructible,* ACM Symposium on Theory of Computing (1989), 454 – 466.

[6] P. Erdős, *On a problem in graph theory,* Math. Gaz. **47** (1963), 220 – 223.

[7] R.L. Graham, J.H. Spencer, *A constructive solution to a tournament problem,* Canad. Math. Bull **14** (1971), 45 – 48.

[8] A. Israëli, M. Li, *Bounded time-stamps,* Proc 28th IEE Symposium on Foundations of Computer Science (1987), 371 – 382.

[9] D.E. Knuth, *The art of computer programming,* Vol 3, *Sorting and searching,* Addison Wesley, Reading (1973).

[10] L. Lamport, *Time, clocks and the ordering of events in a distributed system,* Comm. ACM **21** (1978), 558 – 564.

[11] M. Li, P. Vitanyi, *How to share concurrent asynchronous wait free variables,* Proceedings ICALP 1989 , Lecture Notes in Computer Science **372** (1989), 488 – 507.

[12] E.& G. Szekeres, *On a problem of Schütte and Erdős,* Math. Gaz. **49** (1965), 290 – 293.

[13] M.Trehel, *Deux constructions équivalentes d'un jeu d'arrangements pour les fichiers inverses multi-indices,* R.A.I.R.O Informatique Théorique, **12** (1978), 3 – 14.

[14] W. Zielonka, *Time-stamp systems for a fixed set of agents,* (1990), submitted.