Invisible Permutations and Rook Placements on a Ferrers Board

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1 Introduction

Let λ be a partition of some integer. Write $\lambda = (\lambda_1, \dots, \lambda_m)$, $\lambda_1 \geq \cdots \geq \lambda_m \geq 1$. Sometimes, it is convenient to write $\lambda = (1^{\nu_1} 2^{\nu_2} \cdots n^{\nu_n})$ where ν_i is the number of λ_j's which are equal to i. In this paper, we view a Ferrers board F_λ of shape λ as a two dimensional subarray of an m by n matrix, where $n = \lambda_1$ and the k-th row has length λ_k, $1 \leq k \leq m$. For example,if $\lambda = (3,1)$, then

$$F_\lambda = \begin{pmatrix} * & * & * \\ & & * \end{pmatrix}.$$

We assume, contrary to the usual convention, that F_λ is right justified; the reason for this is explained below. Let $M(F_\lambda)$ be the set of all m by n matrices (a_{ij}) with a_{ij} in some field K such that $a_{ij} = 0$ for $(i,j) \notin F_\lambda$. Thus, for $\lambda = (3,1)$,

$$\mathbf{M}(F_\lambda) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{23} \end{pmatrix} \right\}.$$

Say that λ is *parabolic* of type $(\mu_1, \mu_2, \dots, \mu_k)$ if $m = n$ and there exist positive integers $\mu_1, \mu_2, \dots, \mu_k$ such that

$$\mathbf{M}(F_\lambda) \quad = \quad \left\{ \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{kk} \end{pmatrix} \right\}. \qquad (1)$$

Thus, $\lambda = (4,3,3,1)$ is parabolic with $(\mu_1, \mu_2, \mu_3) = (1,2,1)$. This explains our use of the word parabolic: if λ is parabolic then the invertible elements in $\mathbf{M}(F_\lambda)$ are a parabolic subgroup of $GL_n(K)$. It also explains why our Ferrers boards are right justified. An element of $\mathrm{M}(F_\lambda)$ is a rook placement of shape λ if it is a (0, 1) matrix with at most one 1 in each row and column. The 1's correspond to non-attacking rooks on the board F_λ. If there are r rooks, the matrix has rank r and we say that the rook placement has rank r. If there exists some rook placement σ of rank r on F_λ, then $\lambda_k \geq r-(k-1)$, $1 \leq k \leq r$. Let σ be a permutation of $[n]=\{1,2,\cdots,n\}$, $M(\sigma)$ be the corresponding n by n rook matrix with one 1 in the i-th row and the $\sigma(i)$-th column, $1 \leq i \leq n$. Thus, we have $M(\sigma\tau) = M(\tau)M(\sigma)$. An inversion of σ is a pair $((\sigma(i),\sigma(j))$ where $i<j$ and $\sigma(i)>\sigma(j)$.

Definition 1.1 *Let r,m,n be nonnegative integers such that $r \leq min\{m,n\}$. Let $\Omega^r_{m,n}$ be the set of all integer sequences*

$$(\mathbf{u},\mathbf{v},\mathbf{w}) = (u_1,u_2,\cdots,u_{n-r},v_1,v_2,\cdots,v_r,w_1,w_2,\cdots,w_{m-r})$$

satisfying

$$\begin{cases} 0 \leq u_1 \leq u_2 \leq \cdots \leq u_{n-r} \leq r \\ 0 \leq w_1 \leq w_2 \leq \cdots \leq w_{m-r} \leq r \\ 0 \leq v_i \leq i-1, \quad 1 \leq i \leq r \end{cases} \tag{2}$$

We call the sequences satisfying the inequalities above the inversion number sequences. When $m=n=r$, $\mathbf{u}$ and $\mathbf{w}$ do not exist and the sequence $\mathbf{v}$ defined above is an inversion sequence of an ordinary permutation; see [4]. Sometimes, we write $(\mathbf{u},\mathbf{v},\mathbf{w})$ as $(\mathbf{u}|\mathbf{v}|\mathbf{w})$ to mark the position between two consecutive components, clearly.

Let R^r_λ be the set of all the rook placements of rank r on a Ferrers board of shape λ. Let W_k be the symmetric group on k letters. Let $S(k)=\{(12),(23),\cdots,(k,k-1)\}$ be its distinguished generators, which we sometimes view as k by k matrices. Let $E_{ij} \in \mathrm{M}(F_\lambda)$ be the matrix whith 1 at (i,j) position and 0's elsewhere. Now, we introduce a length function l on R^r_λ.

Definition 1.2 *Let $\nu_r = \sum_{i=1}^r E_{i,n-r+i}$. For $\sigma \in R^r_\lambda$, the length function $l(\sigma)$ is defined by*

$$l(\sigma) = min\{k+h \mid \sigma = s_k\cdots s_1\nu_r s'_1\cdots s'_h\}.$$

where $s_i \in S(m)$ and $s'_j \in S(n)$ and

$$s_p\cdots s_1\,\nu_r\,s'_1\cdots s'_q \in R^r_\lambda$$

for each $p \in [k]$ and $q \in [h]$.

If $\lambda = (n^m)$ and $r=n$, then $R^r_\lambda = W_n$. This function l agrees with the usual length function on W_n in terms of the generators $S(n)$ which counts the number of inversions in a permutation. If $\sigma \in R^r_\lambda$ and $\sigma \in R^r_{\lambda'}$ for two different partitions λ and λ', it is not clear, without proof, that $l(\sigma)$ computed in R^r_λ is the same as $l(\sigma)$ computed in $R^r_{\lambda'}$. We will prove this in Lemma 5.1. Until then we will assume that $l(\sigma)$ is defined with respect to a rectangular board $\lambda=(n^m)$.

In [2], Garsia and Remmel considered another numerical function on R^r_λ.

Definition 1.3 *For each $\sigma \in R^r_\lambda$, place a dot in every cell that is above a rook or to the right of a rook and a circle in each of the remaining cells of F_λ. Let $GR(\sigma)$ denote the number of circles.*

Example 1.4 *Suppose $\lambda = (4,3,2)$ and $\sigma = E_{12} + E_{33}$. Then we get the configuration*

$$\begin{array}{cccc} \circ & 1 & \bullet & \bullet \\ & \circ & \bullet & \circ \\ & & 1 & \bullet \end{array}$$

Thus, $GR(\sigma) = 3$.

Definition 1.5 *Let*

$$R_r(\lambda, q) = \sum_{\sigma\ \in R^r_\lambda} q^{GR(\sigma)},$$

and

$$RL_r(\lambda, q) = \sum_{\sigma\ \in R^r_\lambda} q^{l(\sigma)}.$$

Call them rook polynomial and rook length polynomial, respectively.

The main results of this paper are the following three theorems.

Theorem 1.6 *There is a bijection $\Phi : R^r_{m,n} \longrightarrow \Omega^r_{m,n}$ such that if $\Phi(\sigma) = (\mathrm{u}|\mathrm{v}|\mathrm{w})$, then*

$$l(\sigma) = \sum_{i=1}^{n-r} u_i + \sum_{j=1}^{r} v_j + \sum_{k=1}^{m-r} w_k. \tag{3}$$

The triple $(\mathrm{u}, \mathrm{v}, \mathrm{w})$ may be described as follows: Suppose $\sigma = E_{x_1,y_1} + \cdots + E_{x_r,y_r}$ where $x_1 < x_2 < \cdots < x_r$. Call the rook at (x_i, y_i) the i-th rook or the i-th 1 of σ. Then,

- u_i is the number of rooks to the left of the i-th zero column in σ
- v_i is the number of rooks above and to the right (to the "northeast") of the i-th rook in σ
- w_i is the number of rooks below the i-th zero row in σ

In this paper, we introduce certain permutations, which we call *invisible permutations.* These permutations serve as a bridge between $R^r_{(n^m)}$ and $\Omega^r_{m,n}$. We will postpone the formal description of the invisible permutations to section 2.

Theorem 1.7 *If $\sigma \in R^r_\lambda$, then*

$$GR(\sigma) + l(\sigma) = \sum_{i=1}^{m} \lambda_i - r(r+1).$$

Theorem 1.8 *If λ is a partition, then*

$$RL_r(\lambda, q) = \sum_{1 \le i_1 < \cdots < i_r \le m} q^{\sum_{j=1}^{r}(i_j - j)} \prod_{j=1}^{r} (\lambda_{i_j} - r + j)_q \tag{4}$$

where $(k)_q = 1 + q + q^2 + \cdots + q^{k-1}$.

These two theorems give an explicit formula for Garsia-Remmel rook polynomial

$$R_r(\lambda, q) = q^C \sum_{1 \le i_1 < \cdots < i_r \le m} q^{-\sum_{j=1}^{r}(i_j - j)} \prod_{j=1}^{r} (\lambda_{i_j} - r + j)_{q^{-1}}. \tag{5}$$

where

$$C = \sum_{i=1}^{m} \lambda_i - r(r+1)$$

is the constant in Theorem 1.7.

Corollary 1.9 *For an m by n rectangular board, we have*

$$RL_r(n^m, q) = \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} [r]! \tag{6}$$

where $\left[{k \atop r}\right]$ is the Gaussian binomial coefficient and $[r]! = (1)_q (2)_q \cdots (r)_q$.

If $m = n$, this is proved in [8] by means of the root system of type A_{n-1}.

Corollary 1.10 *If $m = r$, then*

$$RL_r(\lambda, q) = (\lambda_1 - r + 1)_q (\lambda_2 - r + 2)_q \cdots (\lambda_r)_q \tag{7}$$

Corollary 1.11 *Suppose $m = r$ and λ is parabolic of type $(\mu_1, \cdots, \mu_k)$. Then,*

$$RL_r(\lambda, q) = \prod_{i=1}^{k} [\mu_i]!. \tag{8}$$

In particular, if λ and λ' are parabolic of types $(\mu_1, \cdots, \mu_k)$ and $(\mu'_1, \cdots, \mu'_k)$, respectively, where $(\mu'_1, \cdots, \mu'_k)$ is a permutation of $(\mu_1, \cdots, \mu_k)$, then $RL_r(\lambda, q) = RL_r(\lambda', q)$. Thus, the rook length polynomial is invariant under permutations of the diagonal blocks.

2 A Bijection: Maximum Rank Case

The main idea of this and the next section is to extend a rook placement σ on a m by n rectangular board to a permutation matrix $M(P(\sigma))$ of size $(m+n-r)$ which corresponds to a permutation $P(\sigma)$. We call the rows and the columns in $M(P(\sigma))$ which are not in σ the imaginary part of $M(P(\sigma))$, and call the submatrix of $M(P(\sigma))$ which is identical to σ the real part of $M(P(\sigma))$. Note that there are a lot of different ways to extend a rook placement to a permutation. We choose the permutation so that the column indices of the rooks in the imginary part of $M(P(\sigma))$ increase as the row indices increase. We call the permutation $P(\sigma)$ obtained in this way an *invisible permutation* of σ as this permutation is "behind the curtain". This explains the title of this paper. In the following, we give the definition of invisible permutations and then use an example to illustrate our idea.

Definition 2.1 *Let $\sigma \in R^r_{m,n}$. Let $b_1, b_2, \ldots, b_r$ and $c_1, c_2, \ldots, c_r$ be the column indices and the row indices of σ, respectively, such that the entries of σ at (c_i, b_i) are 1's, $i \in [r]$ and $c_1 < c_2 < \cdots < c_r$. Then, $P(\sigma)$ is a permutation in W_{m+n-r} defined by*

$$P(\sigma) \;=\; \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & n-r & n-r+c_1 & \cdots & n-r+c_r & d_1 & \cdots & d_{m-r} \\ a_1 & \cdots & a_{n-r} & b_1 & \cdots & b_r & n+1 & \cdots & n+m-r \end{array}\right)$$

where $\{a_1, a_2, \ldots, a_{n-r}\}$ is the complement of $\{b_1, b_2, \ldots, b_r\}$ in $[n]$ with $a_1 < a_2 < \cdots < a_{n-r}$, and $\{d_1, d_2, \ldots, d_{m-r}\}$ is the complement of $\{n-r+c_1.\cdots, n-r+c_r\}$ in $[n+m-r]-[n-r]$ with $d_1 < d_2 < \cdots < d_{m-r}$.

Example. Let $r=2$, $m=n=4$.

$$\sigma = E_{14} + E_{32} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

According to Definition 2.1, $c_1 = 1$, $c_2 = 3$, $b_1 = 4$ and $b_2 = 2$. Then, $\{a_1, a_2\} = [n] - \{b_1, b_2\} = \{1,3\}$. In particular, $a_1 = 1$ and $a_2 = 3$. Similarly, $\{d_1, d_2\} = [n+m-r] - [n-r] - \{n-r+c_1, n-r+c_2\} = \{4,6\}$. Thus, $d_1 = 4$ and $d_2 = 6$. So we have the following invisible permutation.

$$P(\sigma) = \left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 5 & 4 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{array}\right).$$

By the correspondence M, as mentioned in the introduction, the rook matrix corresponding to $P(\sigma)$ is

$$M(P(\sigma)) \;=\; (E_{11} + E_{23}) + (E_{34} + E_{52}) + (E_{45} + E_{66})$$

$$= \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \cline{1-4} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

In $P(\sigma)$, the columns to the left of the first vertical line are called the column imaginary part of $P(\sigma)$. The columns to the right of the second vertical line are called the row imaginary part of $P(\sigma)$. The columns between the two vertical lines are called the real part of $P(\sigma)$. In the example above, the real part corresponds to the 4 by 4 submatrix of $M(P(\sigma))$ at the southwest corner which is exactly the rook matrix σ itself.
Remark. It might seem "natural" to put the "real" submatrix σ in the northwest corner of $M(P(\sigma))$. But, we find that is not convenient.

In order to make this paper more accessible, we consider first the special case $r = m \le n$. The general bijection will be given in the next section. For a rectangular board, sometimes we write R^r_{nm} as $R^r_{m,n}$ to match the notation $\Omega^r_{m,n}$. We call the elements of $R^r_{r,n}$ maximum rank rook matrices. Recall the definition of invisible permutation. If $m = r$, then $c_i = i$ for $i = 1, \cdots, r$. So, Definiton 2.1 becomes

$$P(\sigma) = \left(\begin{array}{ccc|ccc} 1 & \cdots & n-r & n-r+1 & \cdots & n \\ a_1 & \cdots & a_{n-r} & b_1 & \cdots & b_r \end{array}\right)$$

Now, we give the main result of this section. If $r = n$, this is Hall's classical theorem and the map in our theorem is Hall's bijection [4]. It is clear that the following theorem is the special case $m = r$ of Theorem 1.6.

Theorem 2.2 *There exists a bijection $\Phi : R^r_{r,n} \to \Omega^r_{r,n}$ such that $\forall \sigma \in R^r_{r,n}$, if $(\mathrm{u}|\mathrm{v}) = \Phi(\sigma)$, then*

$$l(\sigma) = \sum_{i=1}^{n-r} u_i + \sum_{j=1}^{r} v_j.$$

Proof. For each $\sigma = \sum_{i=1}^{r} E_{i,b_i} \in R^r_{r,n}$, consider the permutation $P(\sigma) = (\mathrm{a}|\mathrm{b})$. We count the number of inversions in $P(\sigma)$ in two steps. First, note that the imaginary part a is increasing. Any inversion of $P(\sigma)$ with some a_i involved must be of the form (a_i, b_j). Let u_i be the number of inversions of the form (a_i, b_j), $1 \le i \le n-r$, $1 \le j \le r$. Next, let v_i be the number of inversions of the form (b_j, b_i), $1 \le i \le r$, $1 \le j \le r$. Thus, we get an integer sequence

$$(\mathrm{u}|\mathrm{v}) = (u_1, u_2, \ldots, u_{n-r}|v_1, v_2, \ldots, v_r).$$

Write $\Phi(\sigma) = (\mathrm{u}|\mathrm{v})$.

In the rest of this section, we will refer to the inversions of the form (a_i, b_j) and (b_j, b_i) as *column inversions* and *essential inversions*, respectively. Since the imaginary part of $P(\sigma)$ is increasing,

$$0 \le u_1 \le u_2 \le \dots \le u_{n-r} \le r. \tag{9}$$

Notice that for each i, the number of inversions of the form (b_j, b_i) is solely determined by the sign of each difference $(b_j - b_i)$ rather than the actual values of b_j and b_i. The v_i's satisfy the following inequalities, which are exactly the restrictions on the inversion sequences of elements of W_r (see [4], or page 90 in[1]).

$$0 \le v_k \le k-1, \quad 1 \le k \le r. \tag{10}$$

Hence, the sequence $\Phi(\sigma) = (\mathbf{u}|\mathbf{v})$ satisfies the inequalities in Definition 1.1. Thus, we have $\Phi(\sigma) \in \Omega^r_{r,n}$.

Next, we need to verify that $\Phi(\sigma)$ satisfies the equality in Theorem 2.2. By the definition of the length function l, we can first move all the zero columns of σ to the left of all the nonzero columns, keeping the relative positions of the nonzero columns unchanged. This takes exactly u_{n-r+i} adjacent column swaps for the $(i+1)$-th zero column, $i = 0, 1, \cdots, n-r-1$. Then, the last r columns form a r by r permutation matrix, which has $\sum_{i=1}^r v_i$ inversions. Hence the equality in Theorem 2.2 is true. So far, we have proved that Φ is a mapping from $R^r_{r,n}$ to $\Omega^r_{r,n}$ satisfying the equality in our theorem.

From the definitions of $R^r_{r,n}$ and $\Omega^r_{r,n}$, it is easy to see that they are of the same cardinality $\binom{n}{r} r!$. Therefore, in order to show that Φ is a bijection, we need only show that Φ is surjective. We will do this in two steps.

First, consider an arbitrary sequence $(\mathbf{u}|\mathbf{v}) \in \Omega^r_{r,n}$. We are going to construct an integer sequence $(\mathbf{a}|\mathbf{b})$ such that

1. $(\mathbf{a}|\mathbf{b}) \in W_n$,
2. $\Phi((\mathbf{a}|\mathbf{b})) = (\mathbf{u}|\mathbf{v})$, and
3. $\mathbf{a}$ is a strictly increasing sequence,

The construction can be done by the following algorithm.

Algorithm 2.3

1) *For $i = 1$ to $n-r$, let $a_i = u_i + i$, (Thus, by Definition 1.1, $\{a_1, \cdots, a_{n-r}\}$ is stricly increasing.)*

2) *Define a sequence of integers $b_r, b_{r-1}, \cdots, b_1$ and a sequence of subsets $B_r, B_{r-1}, \cdots, B_1$ of $[n]$ recursively as follows: Let $B_r = [n] - \{a_1, \cdots, a_{n-r}\}$. Let b_r be the (v_r+1)-th largest element in B_r. For $i \ge 1$, let $B_{r-i} = B_{r-i+1} - \{b_{r-i+1}\}$, let b_{r-i} be the $(v_{r-i}+1)$-th largest element in B_{r-i}. (This is possible since $v_i \le i-1$, by Definition 1.1.)*

Output $\mathbf{a} = (a_1, a_2, ..., a_{n-r})$ *and* $\mathbf{b} = (b_1, b_2, ..., b_r)$.

Clearly, $(\mathbf{a}|\mathbf{b})$ satisfies the conditions 1, 2 and 3, prior to the algorithm above.

The second step is to find an element σ of $R^r_{r,n}$ such that $P(\sigma) = (\mathbf{a}|\mathbf{b})$. To do this, we can simply throw away the imaginary part $\mathbf{a}$, and use the correspondence M, as mentioned in the introduction, on the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

Then, $P(\sigma) = (\mathbf{a}|\mathbf{b})$, by definition of $P(\sigma)$. Q.E.D.

In particular, the construction of Algorithm 2.3 gives an inverse of the mapping Φ. In order to illustrate the bijection given above, we consider the following example.

Example 2.4 *Let*

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then,

$$P(\sigma) = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{array}\right).$$

Thus, $(\mathbf{a}|\mathbf{b}) = (a_1, a_2|b_1, b_2) = (1, 3|4, 2)$.

First, consider the inversions of the form (a_i, b_j). Since there is no b_i which is less than $a_1 = 1$, we have $u_1 = 0$. Again, since there is only one term $b_2 = 2$ in the real part of $P(\sigma)$ which is less than $a_2 = 3$, we have $u_2 = 1$. Next, look at the real part (b_1, b_2). As there is no term in the real part of $P(\sigma)$ which is to the right of $b_1 = 4$ and bigger than 4, $v_1 = 0$. On the other hand, there is exactly one term $b_1 = 4$ in the real part of $P(\sigma)$, which is to the right of $b_2 = 2$ and bigger than 2. So, $v_2 = 1$. Hence, we have $\Phi(\sigma) = (u_1, u_2|v_1, v_2) = (0, 1|0, 1)$.

To recover $P(\sigma)$ from $\Phi(\sigma)$, we need only to add i to u_i for $i = 1, 2$, and then arrange the elements of $\{1, 2, 3, 4\} - \{u_1, u_2\}$ according to the inversion numbers $v_1 = 0$ and $v_2 = 1$. Hence, $u_1 = 0 + 1 = 1$ and $u_2 = 1 + 2 = 3$. Also,

$$B_2 = \{1, 2, 3, 4\} - \{u_1, u_2\} = \{2, 4\}.$$

As $v_2 = 1$, b_2 is the $(1 + 1)$st largest element in B_2, so $b_2 = 2$. Thus, $B_1 = B_2 - \{2\} = \{4\}$. Similarly, $v_1 = 0$ implies that b_1 is the $(0 + 1)$st largest element in B_1 so $b_1 = 4$. To recover σ from $P(\sigma)$, we can simply write $P(\sigma)$ as the permutation matrix $M(P(\sigma))$ and chop off the first two rows. Then, we are done.

3 The Bijection in General Form

In this section, we remove the restriction $r = m$ and prove Theorem 1.6 in general. The idea, here, is similar to that used in the previous section though the techniques are a little more complicated.

Remark. When $m = r$, the mapping Φ is exactly the mapping introduced in the previous section. Because of this, we use the same symbol Φ in this section.

Proof of Theorem 1.6. Consider the sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ and $\{d_i\}$ as defined in Definition 2.1. Now, define

$$\Phi(\sigma) = (\mathrm{u}|\mathrm{v}|\mathrm{w}) = (u_1, u_2, \cdots, u_{n-r}|v_1, v_2, \cdots, v_r|w_1, w_2, \cdots, w_{m-r}) \tag{11}$$

where

- u_i = the number of b_j's smaller than a_i,
- v_i = the number of b_j's larger than b_i with $j < i$,
- w_i = the number of $(n - r + c_j)$'s larger than d_i.

Mimic the terminology used in section 2. We call $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ the *column inversion number sequence*, the *essential inversion number sequence* and the *row imaginary inversion number sequence*, respectively. And, we call $CIN(\sigma) = \Sigma\, u_i$, $EIN(\sigma) = \Sigma\, v_i$ and $RIN(\sigma) = \Sigma\, w_i$ the *column inversion number* of σ , the *essential inversion number* of σ and the *row inversion number* of σ , respectively. By Definition 2.1, it is easy to check that $\Phi(\sigma)$ satisfies the inequalities in Definition 1.1. Further, since $\mathbf{a}$ and $\mathbf{d}$ are both increasing, all the inversions of $P(\sigma)$ are counted in the right hand side of the formula (3) exactly once. Thus,

$$l(P(\sigma)) = \sum_{i=1}^{n-r} u_i + \sum_{j=1}^{r} v_j + \sum_{k=1}^{m-r} w_k$$

where l is the usual length function on W_{m+n-r}. Since $\mathbf{a}$ and $\mathbf{d}$ are both increasing, we have $l(\sigma) = l(P(\sigma))$. Hence, formula (3) is true.

Thus, Φ is a mapping from $R^r_{m,n}$ to $\Omega^r_{m,n}$. Now, we need to show that Φ is a bijection. Clearly, the sets $R^r_{m,n}$ and $\Omega^r_{m,n}$ are of the same cardinality $\binom{m}{r}\binom{n}{r}r!$. Thus, we need only to show that Φ is surjective.

Here, we use the algorithm in section 2. Suppose we have a sequence $(\mathrm{u}|\mathrm{v}|\mathrm{w})$ from $\Omega^r_{m,n}$. First, use 1) of Algorithm 2.3 on $\mathbf{u}$. We get $\mathbf{a} = (a_1, a_2, \cdots, a_{n-r})$. Then, use 2) of the algorithm on $\mathbf{v}$ and $B_r = [n] - \{a_1, a_2, \cdots, a_{n-r}\}$. Then, we get the sequence $\mathbf{b} = (b_1, b_2, \cdots, b_r)$. Next, in a similar way, use 1) of the algorithm on $w_1 + n - r, w_2 + n - r, \cdots, w_{m-r} + n - r$. We get a strictly increasing sequence $\mathbf{d} = (d_1, d_2, \cdots, d_{m-r})$. Let

$$C = [n + m - r] - [n - r] - \{d_1, d_2, \cdots, d_{m-r}\}.$$

Arrange the elements of C, to form a strictly increasing sequence $\mathbf{c} = (n-r+c_1, n-r+c_2, \cdots, n-r+c_r)$. Then,

$$P(\sigma) = \left(\begin{array}{c|c|c} 1\,2\,3 \cdots\; n-r & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & n+1\;\, n+2\; \cdots\; n+m-r \end{array} \right).$$

If we delete the first $(n-r)$ rows and the last $(m-r)$ columns from $M(P(\sigma))$, we get σ. Thus, Φ is surjective and hence is bijective. Q.E.D.

4 The Poset $(R^r_{nm}, \leq)$

In this section, we define a partial order " $\leq$ " on R^r_{nm}.

Definition 4.1 *Define a graph with R^r_{nm} as its vertex set. If $\sigma, \tau \in R^r_{nm}$ and $\sigma \neq \tau$, we say that σ and τ are adjacent if there exists some $s \in S(m)$ such that $s\sigma = \tau$ or there exists some $s' \in S(n)$ such that $\sigma s' = \tau$ where $S(k) = \{(12), (23), \ldots, (k-1,\ k)\}$.*

Note that $S(k)$ generates W_k, and $W_m \times W_n$ acts on R^r_{nm}, transitively, by means of left and right multiplications. The graph defined as above is connected.

Definition 4.2 *Let l be defined as in Definition 1.2. For $\sigma, \tau \in R^r_{nm}$, define $\sigma \leq \tau$ if $\sigma = \tau$ or there exists a sequence of elements $\sigma_1, \sigma_2, \ldots, \sigma_k \in R^r_{nm}$ such that $\sigma = \sigma_1, \tau = \sigma_k$, each σ_i is adjacent to σ_{i+1} in the graph and $l(\sigma_i) < l(\sigma_{i+1})$, for $i \in [k-1]$. If, in particular, $\sigma = \nu_r$, then we call the sequence above a reduced sequence of τ.*

Lemma 4.3 *$(R^r_{nm},\ \leq)$ is a partially ordered set.*

Definition 4.4 *A level set of rank k in $(R^r_{nm}, \leq)$ is defined by*

$$L_k(R^r_{nm}) = \{\sigma \in R^r_{nm} \mid l(\sigma) = k\}.$$

Corollary 4.5 *The sequence $\{|L_k(R^r_{nm})|\}_{k\geq 0}$ of cardinalities of level sets is symmetric and unimodal.*

Proof. The best property of the bijection defined in formula (11) is that it maps a rook matrix to a triple of integer sequences. Each component of the triple satisfies a set of inequalities, independently. Thus, the enumeration of R^r_{nm} is decomposed into the enumeration of the three integer sequences.

By Definition 1.1, a sequence $\mathbf{u}$ satisfying $0 \leq u_1 \leq \cdots \leq u_{n-r} \leq r$ can be identified as a partition with its Ferrers board contained in a $n-r$ by r board, such that the i-th column of the rectangular board contains exactly u_i cells of the Ferrers board, $1 \leq i \leq n-r$. For example, if $n = 7$ and $r = 3$, then the sequence

$$(u_1, u_2, u_3, u_4) = (0, 1, 1, 3)$$

can be identified as a partition with the Ferrers board

$$\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Let $Y_{n,r}$ be the set of all partitions with their Ferrers boards contained in a $n-r$ by r board. Clearly, the inclusion relation of Ferrers boards is a partial order on the set $Y_{n,r}$. This poset has the following generating function function

$$\sum_{\mathbf{u}\in Y_{n,r}} q^{|\mathbf{u}|} = \begin{bmatrix} n \\ r \end{bmatrix}$$

where $|\mathbf{u}| = \sum_{i=1}^{n-r} u_i$. (This formula has many proofs. See for example [6, 3]). Similar is the case for the set $\{\mathbf{w}\}$. By Hall's argument, we know that the set $\{\mathbf{v}\}$ satisfying $0 \le v_i \le i-1$, $1 \le i \le r$ is in one to one correspondence with W_r and $\sum v_i$ counts the inversions in the permutation corresponding to $\{\mathbf{v}\}$. Thus, by Rodrigues' formula [7],

$$\sum_{\omega\in W_r} q^{l(\omega)} = \sum_{0\le v_i\le i-1} q^{|\mathbf{v}|} = [r]!$$

where $|\mathbf{v}| = \sum_{i=1}^{r} v_i$. Thus, by Theorem 1.6, we know that

$$\begin{aligned} \sum_{\sigma\in R^r_{nm}} q^{l(\sigma)} &= \sum_{\sigma\in R^r_{nm}} q^{CIN(\sigma)+EIN(\sigma)+RIN(\sigma)} \\ &= \Big(\sum_{\mathbf{u}\in Y_{n,r}} q^{|\mathbf{u}|}\Big)\Big(\sum_{0\le v_i\le i-1} q^{|\mathbf{v}|}\Big)\Big(\sum_{\mathbf{w}\in Y_{m,r}} q^{|\mathbf{w}|}\Big) \\ &= \Big(\sum_{\mathbf{u}\in Y_{n,r}} q^{|\mathbf{u}|}\Big)\Big(\sum_{\omega\in W_r} q^{l(\omega)}\Big)\Big(\sum_{\mathbf{w}\in Y_{m,r}} q^{|\mathbf{w}|}\Big) \\ &= \begin{bmatrix} n \\ r \end{bmatrix} [r]! \begin{bmatrix} m \\ r \end{bmatrix} \end{aligned}$$

This is the formula in Corollary 1.9. Since the coefficients of $\left[{k \atop r}\right]$ and $[r]!$ are symmetric and unimodal, the coefficients of $\sum_{\sigma\in R^r_{nm}} q^{l(\sigma)}$ is symmetric and unimodal.Q.E.D.

Lemma 4.6 *Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a partition with $\lambda_1 = n$. Then, R^r_λ is an order ideal of R^r_{nm}.*

Proof. Suppose for a fixed $\sigma \in R^r_\lambda$, there exists some $\tau \le \sigma$ such that $\tau \notin R^r_\lambda$. Choose such τ so that $l(\tau)$ is maximum. Then, by the definition of our poset $(R^r_{nm}, \le)$, there exists a sequence of elements $\tau_0, \tau_1, \ldots, \tau_t \in R^r_{nm}$ such that $\tau = \tau_0, \sigma = \tau_t$, each τ_i is adjacent to τ_{i+1}, and $l(\tau_i) < l(\tau_{i+1})$, $i \in [t-1]$. Since $l(\tau)$ is maximum, $\tau_i \in R^r_\lambda$, for all $i > 0$. And, there exists some $s \in S(m)$ such that $s\tau = \tau_1$ or some $s' \in S(n)$ such that $\tau s' = \tau_1$. Without loss of generality, we consider the first case, only. Then, there exists some k such that $s = (k, k+1)$. The fact that $\tau \notin R^r_\lambda$ and $\tau_1 \in R^r_\lambda$ implies the following two facts:

- there is a rook in the $(k+1, n-h)$ position of τ such that $n-h > \lambda_{k+1}$, i.e., in τ, this rook is outside the board F_λ.
- if the k-th row of τ is nonzero, the rook in the k-th row is to the right of the $(n-h)$-th column since otherwise, $\tau_1 \notin R^r_\lambda$.

Therefore, $l(\tau) > l(\tau_1)$ by the equality in Theorem 1.6 whether the k-th row is zero or not. Thus, we get a contradiction. Q.E.D.

5 Evaluation of Length Function l

Lemma 5.1 *Suppose λ and μ are partitions and $\sigma \in R^r_\lambda \bigcap R^r_\mu$. Then the length of σ viewed in R^r_λ is the same as the length of σ viewed in R^r_μ.*

Combine the lemma with Theorem 1.6, we have

Corollary 5.2 *If $\sigma \in R^r_\lambda$, then*

$$l(\sigma) = \sum_{i=1}^{n-r} u_i + \sum_{j=1}^{r} v_j + \sum_{k=1}^{m-r} w_k.$$

where $(\mathrm{u}, \mathrm{v}, \mathrm{w})$ *is obtained by the method described in the section before, viewing σ as a m by n rook matrix.*

Proof of Lemma 5.1. Write $l_\lambda(\sigma)$ and $l_\mu(\sigma)$ for the two lengths which we want to prove equal. We may assume that $\mu = (n^m)$ and write $l_\mu(\sigma)$ as $l(\sigma)$ as in Section 2, 3 and 4. For any given $\sigma \in R^r_\lambda$, if

$$\sigma = s_k s_{k-1} \cdots s_2 s_1 \, \nu_r \, s'_1 s'_2 \cdots s'_{h-1} s'_h$$

with $(k+h)$ being minimum, then Lemma 4.6 tells us that each

$$s_{k'} s_{k'-1} \cdots s_2 s_1 \, \nu_r \, s'_1 s'_2 \cdots s'_{h'-1} s'_{h'} \in R^r_\lambda,$$

for $1 \leq k' \leq k, 1 \leq h' \leq h$. Thus, each reduced sequence of σ in R^r_λ is also a reduced sequence of σ in $R^r_{n^m}$. Hence, our equality $l_\lambda(\sigma) = l(\sigma)$ is true. Q.E.D.

In fact, the length function l can be evaluated "locally", in the sense of the following corollary, according to the explanation of u_i, v_j, w_k as described in the introduction.

Corollary 5.3 *Let $\sigma \in R^r_\lambda$ such that $\sigma = \sum_{i=1}^r E_{x_i, y_i}$ and $x_1 < x_2 < \cdots < x_r$. Let α_i be the number of zero rows above the x_i-th row in σ, γ_i the number of zero columns to the right of the y_i-th column in σ,and β_i the number of 1's to the "northeast" of the i-th 1. Then,*

$$l(\sigma) = \sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i). \tag{12}$$

Proof. Clearly, $v_i = \beta_i$, for each i. Note that

$$\begin{aligned}\sum_{i=1}^{n-1} u_i &= \sum_{i=1}^{n-r} \text{number of rooks to the left of the } i\text{-th zero column in } \sigma \\ &= \sum_{i=1}^{r} \text{number of zero columns to the right of the } i\text{-th rook} \\ &= \sum_{i=1}^{r} \alpha_i.\end{aligned}$$

Similarly, we have $\sum_{i=1}^{m-r} w_i = \sum_{i=1}^{r} \gamma_i$. Q.E.D.

In [8], it was proved that on a square board, let $\{(x_i, y_i)\}_{1\leq r}$ be as above, then

$$l(\sigma) = \sum_{i=1}^{r} (x_i + n - y_i) + \text{Inv}(y_1, \cdots, y_r) - r(r-1). \tag{13}$$

Here,

$$\text{Inv}(y_1, \cdots, y_r) = \sum_{i=1}^{r} \mu_i$$

where μ_i is the number of y_j such that $y_j > y_i$, and $j < i$, $1 \leq i \leq r$.

Remark 5.4 *The sum in formula (13) can be realized by the following procedure. First, move the i-th 1 of σ to the north until it reaches the top row. Then, move the 1 to the right until it reaches the northeast corner of the Ferrers board. Count the total number of the steps taken by the 1. Now, sum over all i, $1 \leq i \leq r$. Next, delete all the zero rows and all the zero columns from σ. The array obtained in this way is a r by r permutation matrix with inversion number equal to $Inv(y_1, \cdots, y_r)$.*

In the following, we will prove that the formula (13) is true on arbitrary Ferrers boards. Although we could prove this by repeating the argument above, we will show directly that the right hand side of formula (13) equals the right hand side of formula (12).

Lemma 5.5 *Let $\sigma \in R^r_\lambda$. Then,*

$$\sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i) = \sum_{i=1}^{r} (x_i + n - y_i) + \text{Inv}(y_1, \cdots, y_r) - r(r-1).$$

Proof. Note that

$$\text{Inv}(b_1, \cdots, b_r) = \sum_{i=1}^{r} \gamma_i.$$

We need only show that

$$r(r-1) + \sum_{i=1}^{r} (\alpha_i + \beta_i) = \sum_{i=1}^{r} (x_i + n - y_i).$$

Note that we can obtain the right hand side as in the first step of the procedure in Remark 5.4. If we look at the procedure more carefully., we see that as the i-th 1 in σ, moves to the top row, it passes every non-zero row above the x_i-th row (say, there are k_i of them), and every non-zero column to the right of the y_i-th column (say, there are h_i of them). Meanwhile, the i-th 1 in σ passes through all the zero rows above the x_i-th row and all the zero columns to the right of the y_i-th column. Hence,

$$\sum_{i=1}^{r}(x_i + n - y_i) = \sum_{i=1}^{r}(\alpha_i + \beta_i + k_i + h_i).$$

Note that $(k_1, \cdots, k_r)$ and $(h_1, \cdots, h_r)$ are two permutations of the set $[r]$, so,

$$\sum_{i=1}^{r} h_i = \sum_{j=1}^{r} k_j = \binom{r}{2}.$$

Hence, our equality is proved, for all $\sigma \in R_\lambda^r$. Q.E.D. Remark. So far, we have four different ways to evaluate our length function l. They are given by Definition 1.2, Theorem 1.6, formula (13) and formula (12). The fourth is the only one which calculates the length function, "locally", i.e., it counts the contribution of each rook in σ, individually. This local property is very useful in the proof of Theorem 1.7 and Theorem 1.8. In fact, Theorem 1.7 gives another way to evaluate our length function.

Proof of Theorem 1.7. By Lemma 4.6, all rook placements of shape λ with r rooks can be obtained by adjacent row interchanges and adjacent column interchanges from the "canonical" rook placement ν_r so that all the intermediate rook placements stay inside F_λ. We need only show that $GR(\sigma) + l(\sigma)$ is an invariant under the action of the transposition $s_i = (i, i+1)$, on the rows or on the columns.

Without loss of generality, we prove that this is true for the action of s_i on the rows, only. The situation concerning the columns is exactly the same. We separate four cases.

a) the i-th and the $(i+1)$-th row are both non-zero,

b) the i-th row is non-zero and the $(i+1)$-th row is zero,

c) the i-th row is zero and the $(i+1)$-th row is non-zero,

d) the i-th and the $(i+1)$-th row are both zero.

In case a), there are two subcases.

a1) The i-th rook is to the right of the $(i+1)$-th rook.

a2) The i-th rook is to the left of the $(i+1)$-th rook.

In case a1), let k and j be the column indices of the i-th and the $(i+1)$-th rook, respectively. Then, $j < k$. Consider the two row array formed by the i-th row and the $(i+1)$-th row.

$$\begin{array}{cccccc} & (j) & & (k) & \\ \cdots & \bullet & \cdots & 1 & \cdots \\ \cdots & 1 & \cdots & \bullet & \cdots \end{array}$$

By Definition 1.3, if there is a column of the form $\substack{\bullet\\ \bullet}$ in this array, either all columns of the form $\substack{\bullet\\ \bullet}$ are to the right of the k-th column, i.e., they are determined by the i-th rook and the $(i+1)$-th rook or they are both determined by a rook in some row below the $(i+1)$-th row. In both of the cases, the action of s_i does not change this column $\substack{\bullet\\ \bullet}$. If there is a column of the form $\substack{\circ\\ \circ}$ in the array, the column must be to the left of the j-th column. Again, this column will not be affected by the action of s_i. If there is a column of the form $\substack{\bullet\\ \circ}$ then the dot on the top is determined by the rook of the i-th row, which implies that the column is to the right of the k-th column. But, this means the bottom cannot be a "o", by Definition 1.3. So, such a column does not exist. If there is a column of the form $\substack{\circ\\ \bullet}$ the column must be between the j-th and the k-th columns. Thus, the bottom dot is determined by the $(i+1)$-th rook, only. Hence, the action of s_i changes this column into the form $\substack{\bullet\\ \circ}$. At last, the j-th column $\substack{\bullet\\ 1}$ is changed into the form $\substack{1\\ \circ}$ and the k-th column $\substack{1\\ \bullet}$ is changed into the form $\substack{\bullet\\ 1}$. Therefore, the local property of l tells us that the action of s_i increases GR by 1 and decreases l by 1, so, their sum is not affected by the action of s_i.

By a similar argument, we can prove that the sum of l and GR is not affected by the action of s_i in the cases a2), b) and c). Clearly, in case d), the action of s_i does not change $GR(\sigma)$ or $l(\sigma)$. So, the sum $GR(\sigma) + l(\sigma)$ is an invariant under the action of the symmetric group of the rows. For the columns, the situation is exactly the same. Thus, $GR(\sigma) + l(\sigma)$ is a constant C. The constant C can be figured out from the canonical form ν_r as given in Definition 1.2, in which $l(\nu_r) = 0$ and hence

$$C = GR(\nu_r) = \sum_{i=1}^{l} \lambda_i - r(r+1).$$

Therefore, Theorem 1.7 is proved. Q.E.D.

6 Proof of Theorem 1.8

To illustrate the idea of our argument, we start with the case that $r = m = 2$. Consider a Ferrers board, in which the first row has λ_1 cells and the second row has λ_2 cells. Suppose the rook in the second row is in the k-th cell from the right. Now, consider the position of the first rook. there are two cases: Either the first rook is to the left of the second rook or the first rook is to the right of the second rook.

Case 1. The first rook is to the left of the second rook. In this case, by formula (12), the second rook contributes q^{k-1} to the generating function because it has $(k-1)$ zero columns to the right and there is no rook to the northeast. At the same time, the first rook has a

contribution q^i, $k-1 \leq i \leq \lambda_1 - 2$.
Case 2. The first rook is to the right of the second rook. In this case, there are $(k-2)$ zero columns to the right of the second rook and there is only one rook to the northeast. Again, by formula (12), the second rook contributes q^{k-1} to the generating function. Note that the first rook contributes q^j, $1 \leq j \leq k-2$.

Hence, combining the two cases, the generating function of the rook placements of this shape is given by the following expression.

$$\sum_{k=1}^{\lambda_2} q^{k-1}(1+q+\cdots+q^{\lambda_1-2}) = (\lambda_2)_q(\lambda_1-1)_q.$$

From the argument above, we can see that the local property of l makes it possible to look at the contribution of each rook to the generating function, individually. With this in our mind, we can go ahead to look at the general case. From the formula obtained above, we conjecture that for $r = m$, we have

$$\begin{aligned} RL_m(\lambda, q) &= (\lambda_1 - m + 1)_q(\lambda_2 - m + 2)_q \cdots (\lambda_{m-1} - 1)_q(\lambda_m)_q \\ &= \prod_{j=1}^{m} (\lambda_j - m + j)_q. \end{aligned} \tag{14}$$

This is the formula in Corollary 1.10. If $m = 1$, this is obvious. When m equals 2, the formula is proved above. Now, we use induction on m. Suppose that the rook in the last row (i.e., the m-th row) is in the k-th cell from the right. Since $m = r$, this rook is the m-th rook. Then, by formula (12), the m-th rook has $k-1-t$ zero columns to the right if there are t rooks living to the right of the m-th rook. Hence, the m-th rook contributes q^{k-1} to the generating function. This is independent of the configuration formed by the first $m-1$ rooks. Now, delete the k-th column (from the right side) and the last row from the Ferrers board and use the induction hypothesis on that board, we get the formula 14.

$$\begin{aligned} RL_m(\lambda, q) &= \textstyle\prod_{j=1}^{m-1}((\lambda_j - 1) - (m-1) + j)_q \sum_{h=1}^{\lambda_m} q^{h-1} \\ &= \textstyle\prod_{j=1}^{m}(\lambda_j - m + j)_q \end{aligned}$$

Finally, we consider the general case, i.e., $r \leq m$. Suppose that the r rooks live in the rows with indices $i_1, i_2, \cdots, i_r$. Thus, by formula (12), the zero rows give an extra factor $q^{\sum_{u=1}^{j}(i_u - i_{u-1} - 1)}$ to the contribution of the rook in the i_j-th row, which does not appear in the case $r = m$. Therefore, using formula (14),

$$RL_r(\lambda, q) = \sum_{1 \leq i_1 < \cdots < i_r \leq m} q^{\sum_{j=1}^{r}(r-j+1)(i_j - i_{j-1} - 1)} \prod_{j=1}^{r}(\lambda_{i_j} - r + j)_q$$

Here, we used the convention that $i_0 = 0$. Since $\sum_{j=1}^{r}(i_j - j) = \sum_{j=1}^{r}(r-j+1)(i_j - i_{j-1} - 1)$, the theorem is proved. Q.E.D.

Proof of Corollary 1.11 Let λ be a parabolic board of type $(\mu_1, \cdots, \mu_k)$. Then, $\lambda_s - (m - i) = j$, if $s = \sum_{i=1}^{t} \mu_i + j$ and $1 \le j \le \mu_{t+1}$ for some integer t, $1 \le t \le k$. Therefore, by formula (14), we have

$$RL_m(\lambda, q) = (\lambda_1 - m + 1)_q(\lambda_2 - m + 2)_q \cdots (\lambda_{m-1} - 1)_q(\lambda_m)_q \;=\; \prod_{i=1}^{k} [\mu_i]!.$$

Since the formula above does not change if we permute the μ_i's, the rook length polynomial is invariant under the permutations of diagonal blocks. Q.E.D.

Comments. The results obtained here have other consequences. For example, Corollary 1.9 and formula (14) tell us that the coefficients of rook (length) polynomials are a unimodal sequence for arbitrary rook placements on a rectangular board and those maximum ranked rook placements on any Ferrers board. These strongly support the general conjecture made by Garsia and Remmel that the sequence of coefficients of $R_r(\lambda, q)$ is unimodal.

The corollary of 1.10 tells us that if we swap the second and the third diagonal blocks in a parabolic Ferrers board of shape $\lambda = (6, 5, 5, 3, 3, 3)$, then the rook length polynomials for the boards with $\lambda = (6, 5, 5, 3, 3, 3)$ and $\lambda' = (6, 5, 5, 5, 2, 2)$ are exactly the same though the two boards are quite different!

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* * * * * *      * * * * * *
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  * * * * *        * * * * *
      * * *        * * * * *
      * * *              * *
      * * *              * *
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Masao Ishikawa pointed out that the bijection Φ in section 3 does not preserve the partial order and in general $(R^r_{m,n}, \le)$ is not isomorphic to the direct product $Y_{n,r} \times W_r \times Y_{m,r}$. It is my pleasure to thank him for this and many other suggestions.

References

[1] R. A. Brualdi, Introductory Combinatorics, 2nd edition, North-Holland, 1992, p. 90.

[2] A. M. Garsia and J. B. Remmel, q-counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41(1986) 246-275.

[3] I. M. Gessel, A q-analog of the exponential formula, Discrete Mathematics 40(1982) 69-80.

[4] M. Hall Jr, Proceedings Symposium in Pure Mathematics, AMS, Providence, 6(1963), 203.

[5] M. Ishikawa, a private letter, December 28, 1991.

[6] A. Nijenhuis, A. E. Solow and H. S. Wilf, Bijective methods in the theory of finite vector spaces, J. of Combin. Theory Ser. A, 37(1984) 80-84.

[7] O. Rodrigues, Note sur les inversions, ou Dérangements produits dans les permutations, J. de Math. 1 Série 4(1839) 236-239.

[8] L. Solomon, The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field, Geometriae Dedicata 36(1990) 15-49.

[9] M. Wachs and D. White, p,q-Stirling numbers and set partition statistics, J. Combin. Theory Ser. A 56:1(1991) 27-46.