The numbers game and Coxeter groups

KIMMO ERIKSSON
Department of Mathematics
Royal Institute of Technology
S-100 44 Stockholm, Sweden
kimmo@nada.kth.se

1 Introduction

In 1988, Professor Anders Björner in Stockholm gave to me, his brand new student, the following problem to work on: What can be said about the numbers game? Three years later new answers to the question are still popping up, as the subject tends to get both deeper and wider the more is extracted from it. I will here mention some basic properties of the game and some connections with Coxeter groups.

The numbers game is a one-player game played on a graph. It was defined by Shahar Mozes [7] as follows. Let G be a simple graph with N nodes. Place a real number on each node. A move now consists of first picking a node i with a negative number p_i, then adding the number p_i to the number at each neighbor j of i, and finally reversing the sign at node i. If no move is possible the game has terminated.

Mozes showed that the process is what I will call *strongly convergent*: Given a starting position either every play sequence diverges (*i.e.* can be continued forever), or every play sequence will converge to the same terminal position in the same number of moves. In other words, the length and result of the game is independent of what choices are made.

Possible questions to ask about the numbers game are for example:

- Are there any natural generalizations that preserve the strong convergence property?
- From which positions does the game converge, and in how many steps?
- Given two positions p and q, is it decidable whether q can be reached by playing from p?

These questions all have sort of a "global" aspect on the game; we need to know every possible way to play from p, and every position reached in this way. Therefore it is natural to study the *game graph* obtained by taking all positions reachable from p as vertices, and taking the moves as directed edges, each labeled with the node fired. The sequences of legal moves from p define a language $\mathcal{L}_p$ in the alphabet of nodes. But now some additional questions arise.

- Is the language $\mathcal{L}_p$ a greedoid? This was known for a related game.
- Is the game graph, viewed as the partially ordered set of positions, a lattice?

In order to get some algebraic tools for dealing with such questions, it is natural to represent a position on a graph with d nodes as a point $p = (p_1, p_2, \ldots, p_d)$ in real d-dimensional space, where the coordinates are given by the numbers on the nodes. A move is then seen to be equivalent to a reflection of the point in some hyperplane [4] [5] [7]. This is the background to why Coxeter groups, which have a standard representation as groups generated by reflections, are interesting in this context.

2 The polygon idea

Björner had observed the following relations in the numbers game. Suppose two nodes x and y are both playable. If x and y are not neighbors, then playing xy or yx leads to equal positions. If x and y are neighbors, then xyx and yxy are legal play sequences leading to equal positions. Thus we get polygons in the game graph.

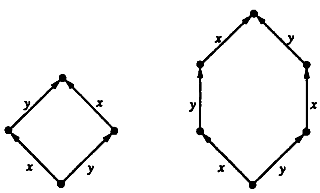

Figure 1: Polygons: $xy = yx$ (left) and $xyx = yxy$ (right)

These polygons imply strong convergence by the following theorem, Eriksson 1991 [4].

Theorem 2.1 *A game has the strong convergence property if and only if whenever two different moves are possible, they define a polygon, that is, two legal play sequences of equal length (possibly infinite) which end (if finite) in the same position.*

Let $(xyx...)_l$ denote the alternating play sequence of length l.

3 Generalizations by weighted graphs

The first generalization we shall make is the introduction of *weights* on the *edges* of the graph. Specifically, to each edge (i, j) we assign two strictly positive weights, k_{ij} and k_{ji}, such that if p_i denotes the number on node i, then the new position after firing node i is computed from the previous position by

$$\begin{cases} p_j := p_j - k_{ij}p_i & \text{for each vertex } j \neq i; \\ p_i := -p_i & \text{(as in the unweighted game).} \end{cases}$$

By geometric considerations one can show that if there are integers $m(i,j) > 2$ for every edge (i,j) such that the weight product $k_{ji}k_{ij} = 4\cos^2(\pi/m(i,j))$ then we get alternating polygons: $(xyx...)_{m(x,y)} = (yxy...)_{m(x,y)}$ which are legal if both x and y are playable. If $k_{ji}k_{ij} \geq 4$, then we get an infinite polygon, *i.e.* both sequences $(xyx...)$ and $(yxy...)$ are playable forever. Other weight products than these do not give polygons, hence it follows that the edge weighted numbers game is strongly convergent if and only if every weight product satisfies one of the conditions above.

Next, it is natural to explore what happens when we define a *node weight*, $w_i > 0$, for each node i, such that firing node i affects the number at the same node by $p_i := -w_i p_i$. In this case things get more complicated, but a thorough analysis along the same lines as in the edge weighted case gives:

Theorem 3.1 *A node weighted game is strongly convergent if and only if for each edge* (i,j) *the corresponding weight product satisfies either* $k_{ji}k_{ij} \geq 2\sqrt{w_i w_j} + w_i + w_j$, *or* $k_{ji}k_{ij} = 2\sqrt{w_i w_j}\cos(2\pi/n) + w_i + w_j$ *for some integer* $n \geq 3$, *and* $w_i = w_j$ *if* n *is odd.*

Note that if all node weights are equal to 1, then we get back to the conditions for the purely edge weighted case. From now on, by an *N-game* I will mean a strongly convergent, edge weighted numbers game.

4 Coxeter groups

Let V be a finite index set. A Coxeter group (W,S) is a group W with a distinguished set of involutory generators, $S = \{s_x : x \in V\}$, and relations $(s_x s_y)^{m(x,y)} = e$ (the identity of the group), with integers $m(x,y) \geq 2$, or $m(x,y) = \infty$, by which we mean that $s_x s_y$ has infinite order in W.

Now think about the moves of an N-game as linear transformations of the position vector. These transformations are clearly involutions, and by the polygon property $(xyx...)_{m(x,y)} = (yxy...)_{m(x,y)}$ we have the Coxeter type of relations. Indeed, the moves generate a group of linear transformations that is isomorphic to the corresponding Coxeter group defined by the $m(x,y)$, as is demonstrated by the following equivalence between the numbers game and the standard representation of a Coxeter group as acting on an $|S|$-dimensional space X with some basis $\{e_x : x \in V\}$ and geometry given by a bilinear form $B(e_x, e_y) = -\cos(\pi/m(x,y))$, and generators $S = \{\sigma_x : x \in V\}$ where every σ_x is a reflection in the hyperplane perpendicular to unit vector e_x, [5]. Hence, for $\lambda \in V$, $\sigma_x(\lambda) = \lambda - 2B(\lambda, e_x)e_x$.

Given a Coxeter group (W,S), construct a graph G with $|S|$ nodes on which to play the numbers game in the following way. For any pair x, y of nodes, put an undirected edge (x,y) in G if the relation exponent $m(x,y) > 2$. Let the edge weights be $k_{xy} = k_{yx} = 2\cos(\pi/m(x,y))$. Then the numbers game on G is an N-game, by Theorem 3.1.

To every vector $\lambda \in V$ we now associate the position in the game where the number written on a node x is $\lambda_x \overset{\text{def}}{=} B(\lambda, e_x)$. A reflection σ_x gives a new vector $\sigma_x(\lambda)$, in which the numbers of the associated game position are given by

$$[\sigma_x(\lambda)]_y = \begin{cases} -\lambda_x & \text{if } x = y \ ; \\ \lambda_y + 2\cos(\pi/m(x,y))\lambda_x & \text{if } x \neq y. \end{cases}$$

Hence a hyperplane reflection σ_x from the negative to the positive halfspace is equivalent to the firing of the corresponding node x in the numbers game.

Indeed, if Γ_p is the game graph from a position p where all numbers are negative, then Γ_p is isomorphic to the Cayley graph of the corresponding Coxeter group.

5 Length of game

Now let W be the Coxeter group of linear transformations generated by the set of moves, which we denote S, in the N-game played on $G = (V, E)$. Let $p = (p_1, p_2, ...)$ be a position in the game. Let α_i be the functional on positions that returns the number at node i, i.e. $\alpha_i(p) = p_i$. Thus the multiset $\{\alpha_i(w(p)) : w \in W, i \in V\}$ contains all numbers that can ever arise in any node when playing the game from p (backwards and forwards). This can be interpreted as the values obtained by applying the set of functionals $\Phi = \{\alpha_i(w(\cdot)) : w \in W, i \in V\}$ on the start position p. One can see that Φ is (isomorphic to) a *root system*, see [6], of the Coxeter group, where the α_i are the primitive roots of Φ.

It is known that the roots can be partitioned in $\Phi = \Phi^+ \cup \Phi^-$, where every $\phi \in \Phi^+$ is a nonnegative linear combination of the α_i, and every $\phi \in \Phi^-$ is a nonpositive linear combination of the α_i. Further, if $s_i \in S$ is the move of firing node i, then an important property of root systems is that $\Phi^+ \circ s_i = (\Phi^+ - \{\alpha_i\}) \cup \{-\alpha_i\}$. If i is playable, then $p_i = \alpha_i(p)$ is negative, while $-\alpha_i(p)$ is positive. Hence, $\Phi^+ \circ s_i(p)$ has exactly one negative value less than $\Phi^+(p)$. In a terminal position t, $\Phi^+(t)$ has only positive values. Consequently, the length of a game from p is equal to the number of negative values in $\Phi^+(p)$, if finite. If this number is infinite, then the game from p is divergent.

By the above, we can also state a comparison test.

Theorem 5.1 *If $\bar{p} \leq p$ (that is, $\bar{p}_i \leq p_i$ for every node i) then every play sequence that is legal from p is also legal from $\bar{p}$.*

This is immediately clear from the fact that the linear combinations in Φ^+ have nonnegative coefficients.

6 The language of play sequences is a greedoid

Definition. A language $\mathcal{L}$ is a greedoid if it is left-hereditary, which means that

$$\alpha\gamma \in \mathcal{L} \Rightarrow \alpha \in \mathcal{L} \qquad \text{(G1)}$$

and $\mathcal{L}$ satisfies the following exchange condition.

$$\alpha, \beta \in \mathcal{L}, |\beta| > |\alpha| \Rightarrow \exists x \in \beta : \alpha x \in \mathcal{L} \qquad \text{(G2)}$$

It is known that in another vertex-firing game, the chips game of Björner, Lovász and Shor [2], the language of legal play sequences is a greedoid. We now prove this for any N-game.

Lemma 6.1 *Let $G = (V, E)$ be an edge weighted graph that defines an N-game.*

(a) *Given a subset $V' \subseteq V$, let G' be the subgraph of G induced by the nodes in V'. Then The numbers game on G' is strongly convergent, that is, an N-game..*

(b) *If x is a legal move and α is a legal play sequence in a position p on G, such that $x \notin \alpha$, then $x\alpha$ is a legal play sequence from p.*

PROOF. (a) This is obvious from the characterization of N-games in Section 3, since the weight product of any edge in G' is the same as for the corresponding edge in G which defines an N-game.

(b) Let $V' = V - \{x\}$. Let $\bar{p}$ be the position after playing x. Then α is playable from $p|_{G'}$ (the restriction to G'), and $\bar{p}|_{G'} \leq p|_{G'}$, so by Theorem 5.1 α is playable from $\bar{p}|_{G'}$. Accordingly, $x\alpha$ is playable from p. □

In the following, fix $G = (V, E)$, and fix a subset $V' \subseteq V$ with induced subgraph G'. Let $d(\alpha)$ denote the number of moves in play sequence α which are firings of nodes that are not in V'.

Lemma 6.2 *Let p be a position on G where β and α are legal play sequences such that $d(\beta) = 0$ and $d(\alpha) \geq |\alpha| - |\beta|$. Then there exists a play sequence γ such that $d(\gamma) = 0$, $|\gamma| = |\beta| - |\alpha| + d(\alpha)$ and $\alpha\gamma$ is legal from p.*

PROOF. If $d(\alpha) = 0$ we can restrict the game to G' and strong convergence on G' implies the existence of the desired γ. Suppose the theorem has been proved when $d(\alpha) < n$, and suppose this $d(\alpha) = n$. Then α can be written $\alpha_1 y \alpha_2$ where $y \notin V'$, $d(\alpha_1) = n - 1$ and $d(\alpha_2) = 0$. By hypothesis there exists some γ_1 such that $d(\gamma_1) = 0$, $|\gamma_1| = |\beta| - |\alpha_1| + n - 1$ and $\alpha_1\gamma_1$ is legal from p. By Lemma 6.1(b), $\alpha_1 y \gamma_1$ is legal from p. Let q be the position after $\alpha_1 y$. In q we have γ_1 and α_2 legal, $d(\gamma_1) = 0$ and $d(\alpha_2) = 0 \geq |\alpha_2| - |\gamma_1|$ by the above. Thus there exists a γ with $d(\gamma) = 0$, such that $\alpha_1 y \alpha_2 \gamma = \alpha\gamma$ is legal from p, and $|\gamma| = |\gamma_1| - |\alpha_2| = |\beta| - |\alpha| + d(\alpha)$. □

Theorem 6.3 *$\mathcal{L}_p$, the language of legal play sequences from position p in an N-game, is a greedoid.*

PROOF. Obviously, $\mathcal{L}_p$ is left-hereditary, so we only have to verify (G2), the greedoid exchange property. Suppose α and β are legal play sequences with $|\beta| > |\alpha|$. Let V' be the set of vertices fired in β. Then $d(\beta) = 0$ and $d(\alpha) \geq |\alpha| - |\beta|$, so by Lemma 6.2 there is a play sequence γ of length at least 1 and firing only nodes in V', such that $\alpha\gamma$ is legal. Let x be the first move of γ. Then $x \in \beta$ and αx is legal. □

7 Final words

Much more can be said about the numbers game than was possible here. I refer the interested reader to my doctoral thesis, due in spring 1993.

References

[1] A. Björner, On a combinatorial game of S. Mozes, preprint, 1988.

[2] A. Björner, L. Lovasz and P. Shor, Chip-firing games on graphs, *Eur. J. Comb.* **12** (1991), 283–291.

[3] K. Eriksson, Convergence of Mozes's game of numbers, 1990, to appear in *Linear Algebra Appl.*

[4] K. Eriksson, Strong convergence and a game of numbers, 1991, preprint.

[5] K. Eriksson, Games and the weak order of Coxeter groups, 1992, in preparation.

[6] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[7] S. Mozes, Reflection processes on graphs and Weyl groups, *J. Combin. Theory*, Ser. A, **53** (1990), 128–142.