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Dual graphs and Schensted correspondences

Abstract

A graph is said to be graded if its vertices are divided into *levels* numbered by integers, so that the endpoints of any edge lie on consecutive levels.

The following three types of problems are considered:

(1) path counting in graded graphs, and related combinatorial identities;
(2) bijective proofs of these identities;
(3) design and analysis of algorithms establishing corresponding bijections.

The R.P.Stanley's [St88,St90] linear-algebraic approach to (1) is extended to cover a wide range of graded graphs. The main idea is to consider the *pairs* of graded graphs with the common set of vertices and common rank function. Such graphs are said to be *dual* if the associated linear operators satisfy a certain *commutation relation* (the "Heisenberg" one). The algebraic consequences of these relations are then interpreted as combinatorial identities. (This idea is also implicit in [St90].)

Applications include various examples of graded graphs, e.g., the Young, Fibonacci, Young-Fibonacci and Pascal lattices, the graph of shifted shapes, the r-nary trees, the subword order, the lattice of finite binary trees, etc. Many enumerative identities (both known and unknown) are obtained.

These identities can also be derived in a purely combinatorial way by generalizing the Robinson-Schensted correspondence to the class of graphs under consideration(cf. [Fo86]). The same tools can be applied to permutation enumeration, including Ferrers boards and involution counting. The bijective correspondences mentioned above are effected by the RSK-type algorithms a general approach to which is given. As particular cases of the construction we rederive the classical algorithm of Robinson, Schensted, and Knuth [Sc61,Kn70,Sc77], the Sagan-Worley [Sa87,Wo84] and Haiman's [Ha89] algorithms, the algorithm for the Young-Fibonacci graph [Fo86, Ro91]. Among new applications there are RSK-analogues for the infinite binary tree, the Pascal graphs, etc.

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An extended version of this contribution appeared in the Mittag-Leffler Institute preprint series [Fo1, Fo2]; the paper [Fo3] presents a related approach to Knuth-type correspondences and respective Schur functions.

1. Graded Graphs

A *graded graph* is a triple $G = (P, \rho, E)$ where

(1) P is a discrete set of *vertices*,

(2) $\rho : P \to \mathbb{Z}$ is a *rank function*,

(3) E is a multiset of *arcs/edges* (x, y) where $\rho(y) = \rho(x) + 1$.

The set $P_n = \{x : \rho(x) = n\}$ is called a *level* of G. We shall always assume that the levels are finite and G has a zero $\hat{0}$, i.e.,

$$P_0 = \{\hat{0}\}, \quad P_{-1} = P_{-2} = P_{-3} = \cdots = \phi.$$

Various examples of graded graphs can be found in [St86,Pr89,etc.].

G can be regarded either as oriented or as non-oriented graph. The accessibility relation in an oriented graph defines a *partial order* on P. If there are no multiple arcs, G turns out to be the Hasse diagram of this poset. Therefore non-oriented paths (i.e. paths in non-oriented graph) are often called *Hasse walks.*

Let $e(x \to y)$ denote the number of *oriented* paths between x and y; also write $e(y) = e(\hat{0} \to y)$. So $e(y)$ is the number of paths going from $\hat{0}$ to y ("a generalized binomial coefficient"). One can similarly define $e(x \to y \to z)$ et al.

The main results we are going to obtain are combinatorial identities involving $e(x)$ and similar enumerative characteristics. The typical example is the Young-Frobenius identity

$$\sum_{x \in P_n} e(x)^2 = n! \tag{1.1}$$

for the Young's lattice. This is not an isolated result. Surprisingly, the same formula proves to be valid for the so-called Fibonacci lattices [St75,Fo86,St88]. Similar identity (with additional coefficients) is known for the graph of shifted shapes (see (4.3)). Another examples are the lattice of binary trees and the binary subword order where

$$\sum_{x \in P_n} e(x) = n! \quad . \tag{1.2}$$

Each of these facts is known to have both "computational" and "bijective" proofs. However these proofs use *individual* properties of the graphs (except for the proof of (1.1) in [St88,Fo86]).

We develop combinatorial techniques providing general results of this type for a wide class of graded graphs. Unified proofs of (1.1), (1.2), and many other both known and unknown enumerative identities (cf., e.g., [St88] and [Sa90]) are given. Then we present a general approach to the Robinson-Schensted-type algorithms *for the same class of graphs.* It allows to provide bijective proofs to the enumerative results of this paper.

2. Linear-Algebraic Approach

A graded graph is completely determined by the *adjacency matrices* of the bipartite graphs formed by consecutive levels and the arcs joining them. Thus graded graphs can be studied as linear-algebraic objects.

Fix a field K of zero characteristic. The finitary K-valued functions on P (or, equivalently, the formal linear combinations of the vertices) form the vector space KP. The space KP can also be equipped with a Euclidean structure by declaring the vertices to form an orthonormal system. However we don't exploit this Euclidean structure; only linear techniques are used.

2.1 Definition. Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with the common set of vertices and common rank function. Define an *oriented graded graph* $G = (P, \rho, E_1, E_2)$ by directing the G_1-edges "upwards" and the G_2-edges "downwards" (i.e. according to the rank increase or decrease, respectively). Now it is natural to introduce the *up* and *down operators* $U, D \in End(KP)$ associated with the graph G (or with the pair (G_1, G_2)) by

$$Ux = \sum_y a_1(x,y)y \quad ,$$
$$Dy = \sum_x a_2(x,y)x$$

where $a_i(x, y)$ is the multiplicity (weight) of the edge (x, y) in the [non-oriented] graph G_i.

Many path counting characteristics can be easily expressed in terms of operators U and D. For example, let $x \in P_n$. Then

$$U^k x = \sum_{y \in P_{n+k}} e_1(x \to y)y \quad ,$$
$$D^k x = \sum_{y \in P_{n-k}} e_2(y \to x)y$$

where indices in e_1 and e_2 refer to path counting in G_1 and G_2, respectively.

Non-oriented paths (Hasse walks) having arbitrary (but fixed) structure can be dealt with in the same manner.

2.2 Definition. Let G be defined by Definition 2.1. Assume w is a word in the alphabet $\{U, D\}$ ($\{U, D\}$-word, for short). A path p in G is said to *have a structure* w (or to be a w-path) if its consecutive arcs are directed upwards/downwards in accordance with the word w. So U's correspond to up-directed arcs (remind they should be the arcs of G_1) and D's to the down-directed arcs (those of G_2). We emphasize that in this definition a word w is to be read *from the right to the left* since it will be later interpreted as an operator.

Using this terminology, we can say that the number of w-paths in G from x to y is a coefficient of y in the expansion of wx where w is interpreted as an operator in KP (a product of corresponding U's and D's). For example, the coefficient of $\hat{0}$ in $D^nU^n\hat{0}$ is

$$\sum_{x\in P_n} e_1(x)e_2(x) \ .$$

We study the situations when the maps U and D satisfy some algebraic conditions. Algebraic consequences of those conditions can be then restated as combinatorial identities.

2.3 Definition. Let $r \in \mathbb{P}$. Graded graphs G_1 and G_2 with common set of vertices and common rank function (see Definition 2.1) are said to be *r-dual* (or simply *dual* when $r = 1$) if

$$DU = UD + rI \ . \tag{2.1}$$

It means combinatorially that

(1) if x and y are *different* vertices of the same rank then the number of DU-paths from x to y equals the number of UD-paths joining x and y;
(2) for any vertex x the number of DU-paths (loops) from x to x equals the number of UD-loops plus r.

Note that (2.1) is symmetric (invariant) with respect to an interchange of the initial graphs G_1 and G_2. Indeed, a change of D and U into U^* and D^*, respectively, transforms (2.1) into *equivalent* relation

$$U^*D^* = D^*U^* + I$$

(the * stands for a conjugation with respect to a natural pairing in KP).

The case $U^* = D$ or, equivalently, $D^* = U$, $G_1 = G_2$, corresponds to self-dual graphs (differential posets; see Sec.3).

In what follows we assume that the graphs G_1, G_2, G, the operators U and D, etc., are as in Definition 2.1 and satisfy (2.1).

Let $r \in \mathbb{P}$. Define an associative graded algebra $\mathfrak{A}_r$ with identity I generated by elements U and D satisfying $DU = UD + rI$. Given a pair of r-dual graded graphs G_1 and G_2, a natural representation of $\mathfrak{A}_r$ arises. Thus any equality (identity) in $\mathfrak{A}_r$ can be reinterpreted in combinatorial terms. The following are typical examples of such identities.

2.4 Lemma. [St88] *Let k and l be nonnegative integers. Then*

$$D^lU^{k+l} = U^k(UD + (k+1)r)\dots(UD + (k+l)r) \ . \tag{2.2}$$

This lemma can be generalized in the following way.

2.5 Theorem. *For any $\{U, D\}$-word $w = w(U, D)$ with m entries of D and n entries of U*

$$w(U, D) = \sum_k r^k d_k(w) U^{n-k} D^{m-k}$$

where $d_k(w)$ is the k'th coefficient of the rook polynomial of the Ferrers board (cf. [St86, Sec.2.4]*) whose boundary is defined by w.*

Identities of this type can be converted into enumerative formulae concerning path counting (taking in mind the natural representation of the algebra $\mathfrak{A}_r$ in the space KP). Consider Lemma 2.4 as a typical example.

2.6 Corollary. *For any vertex $x \in P$ of rank k*

$$\sum_{y \in P_{k+l}} e_1(y) e_2(x \to y) = e_1(x) r^l (k+l)!/k! \ .$$

Proof. Apply both sides of (2.2) to $\hat{0}$ (note $D\hat{0} = 0$!) and take the coefficient of x.□
Now we state separately the special case $x = \hat{0}$ of Corollary 2.6.

2.7 Corollary.

$$\sum_{y \in P_l} e_1(y) e_2(y) = r^l \ l! \qquad (2.3)$$

We shall demonstrate that (2.3) generalizes (1.1), (1.2), and (4.3). Respective versions of Corollary 2.6 reduce to enumerative formulae involving "skew tableaux".

3. Self-Dual Graphs (Differential Posets)

Let Q be a countable poset. The distributive lattice $J(Q)$ of finite order ideals of Q (cf. [St86, Sec.3.1]) is a graded graph with zero; rank of an ideal is its cardinality.

3.1 Example. *Young graph* [St86, etc.] Let $\mathbb{P}^2 = \{(i,j) : i > 0, \ j > 0; \ i, j \in \mathbb{Z}\}$ be the two-dimensional integral quadrant with the usual (i.e. coordinatewise) partial order. The graph $\mathbb{Y} = J(\mathbb{P}^2)$ is called the *Young graph/lattice*; see Fig.1. The numbers $e(x)$ are the dimensions of irreducible representations of symmetric groups S_n (see, e.g., [St71, Sec.17]).

The Young graph is a *self-dual* graph = differential poset[St88]. Roughly, it is because for each shape there is one more box one can add than one can delete.

3.2 Theorem. [St88, Prop.5.5] *The Young graph is the only self-dual distributive lattice.*

3.3 Example. *Young-Fibonacci graph* [Fo86,St88]. Let $\{1,2\}^*$ denote the set of all the words in the alphabet $\{1,2\}$. Define the *Young-Fibonacci graph* $\mathbb{YF}$ (see Fig.2) as follows:

(1) $\{1,2\}^*$ is the set of its vertices (i.e., vertices are $\{1,2\}$-words);

(2) w' covers w iff either $w' = 1w$ (concatenation) or $w' = 2v$ where w covers v.

So the rank of a $\{1,2\}$-word is the sum of its "digits". We have mentioned yet that (1.1) is known to hold for the Young-Fibonacci graph. In [Fo86,Ro91] the complete analogue of Robinson-Schensted for $\mathbb{YF}$ was also constructed. The *self-duality* of the Young-Fibonacci graph follows from observing that

(1) $\mathbb{YF}$ is a modular lattice [Fo86,St88];

(2) every vertex of $\mathbb{YF}$ has one more successors than predecessors.

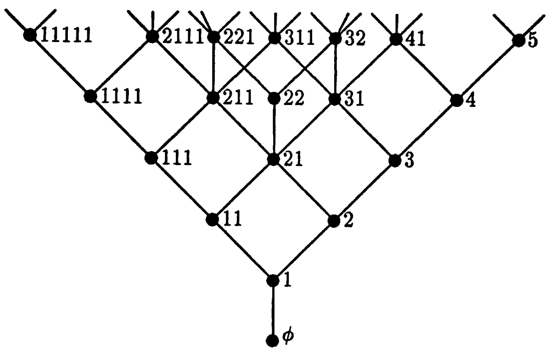

Fig.1 Young graph

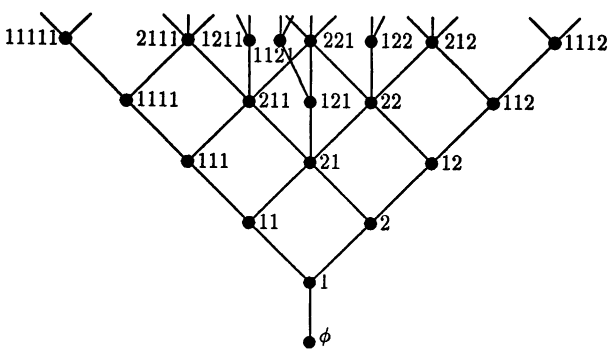

Fig.2 Young-Fibonacci graph

4. Weighted Distributive Lattices

Assume $G_1 = (P, \rho, E_1)$ is a distributive lattice: $P = J(Q)$. Let us try to construct the r-dual graph G_2 as follows. Let $w : Q \to K$ be a weight function (recall K is the basic field). For any edge $(x, y) \in E_1$ consider the only element $q \in y \setminus x$ and define $w(q)$ to be a multiplicity (weight) of an edge (x, y) in G_2. Thus G_2 is a *weighted distributive lattice.* (Generally, G_2 is not even a graph but a graded network.)

We want G_1 and G_2 to satisfy (2.1). Observe that whichever weight function w you choose the matrix $DU - UD$ is *diagonal.* Thus we only need to prove the equality of the diagonal elements of both sides of (2.1).

4.1 Lemma. *Assume $G_1 = J(Q)$. A weight function $w : Q \to K$ defines a weighted distributive lattice G_2 which is dual to G_1 if and only if the following condition holds for any finite order ideal x of Q:*

(4.1) *the total weight of the elements $q \in Q$ able to be added to x (i.e. such that $x \cup \{q\}$ is also an ideal) is r more than the total weight of the elements $q \in Q$ able to be deleted from x.*

The latter condition is very strong; only some special posets Q allow a weight function w to be defined satisfying (4.1).

In the remainder of this section we give several examples.

4.2 Example. *The chain* $\mathbb{N} = \{0, 1, 2, \dots\}$. This graph can be treated as a lattice of ideals of $\mathbb{P} = \{1, 2, 3, \dots\}$ with the usual ordering. The only weight function w on $\mathbb{P}$ satisfying (4.1) with $r = 1$ is $w(q) = q$. Thus the graph dual to $\mathbb{N}$ is the same chain having its qth edge multiplied by q.

4.3 Example. *Pascal graphs.* Let $Q = r\mathbb{P}$, i.e., Q is a union of r disjoint copies of $\mathbb{P}$. Then $\mathbb{N}^r = J(Q)$ is an r-dimensional *Pascal graph.* So $\mathbb{N}^r$ is the lattice of r-dimensional vectors with nonnegative integer coordinates.

To construct an r-dual graph for $\mathbb{N}^r$ (the coincidence of these two r's is *not* accidental) let us make the following general observation.

4.4 Lemma. *Assume the graphs G_1 and G_2 are r-dual, and the graphs H_1 and H_2 are s-dual. Then $G_1 \times H_1$ and $G_2 \times H_2$ are $(r + s)$-dual.*

Thus the r-dual graph for $\mathbb{N}^r$ can be obtained by cartesian multiplication of r copies of a graph dual to $\mathbb{N}$ (see Example 4.2).

By means of Lemma 4.4 one can easily construct new pairs of r-dual graphs from old ones; e.g., the graph $\mathbb{Y}^r$ (the rth cartesian power of the Young graph) is self-r-dual, as is $\mathbb{YF}^r$.

4.5 Example. *Diagrams with $\leq r$ rows.* Let Q be a direct product of an infinite chain $\mathbb{P}$ and a finite chain $[r] = \{1, \dots, r\}$. The distributive lattice $J(Q) = J(\mathbb{P} \times [r])$ is the lattice of Young diagrams containing r rows or less. It is of course a sublattice and an order ideal of the Young lattice. So the constants $e(x)$ are the same as in $\mathbb{Y}$.

4.6 Lemma. *A weight function* $w : \mathbb{P} \times [r] \to K$ *defined by*

$$w((q_1, q_2)) = r + q_1 - q_2 \tag{4.2}$$

satisfies (4.1).

Thus we have constructed a weighted lattice that is r-dual to $J(\mathbb{P} \times [r])$. Note that the r's involved in $\mathbb{P} \times [r]$, (4.2), and (2.1) do coincide.

4.7 Example. *Shifted shapes* [Sa87,Wo84,St90]. Let $Q = J(\mathbb{P} \times [2]) = \mathit{SemiPascal}$ (cf. Example 4.5). The graph $\mathbb{SY} = J(Q)$ is the *graph of shifted shapes* which are the Young diagrams with *strictly* decreasing row lengths; see Fig.3. Since $\mathbb{SY}$ is *not* an order ideal of the Young lattice, the values $e(x)$ in $\mathbb{SY}$ differ from those in $\mathbb{Y}$, x being a shifted shape. The main identity involving $e(x)$'s in $\mathbb{SY}$ is the following [Sc11,Sa79]:

$$\sum_{x \in P_n} e(x)^2 2^{n-h(x)} = n! \tag{4.3}$$

where $h(x)$ is the *height* (=number of rows) of the shape x.

The only weight function on *SemiPascal* satisfying (4.1) with $r = 1$ is

$$w(q) = \begin{cases} 1\ , & \text{if } q \text{ is a diagonal element} \\ 2\ , & \text{otherwise} \end{cases} \tag{4.4}$$

Thus the dual graph for $\mathbb{SY}$ is the same graph having doubled the edges which correspond to adding non-diagonal elements; see Fig.3.

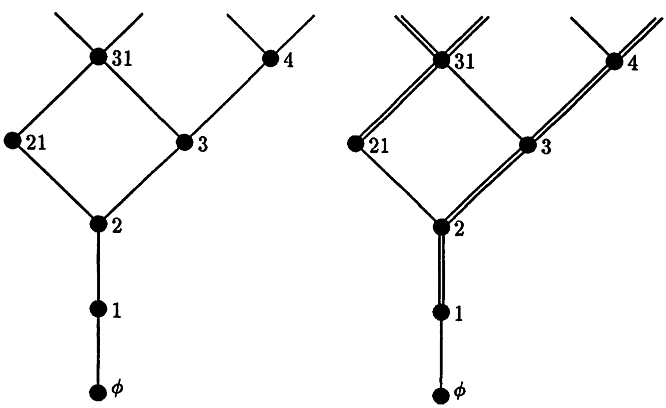

Fig.3 Graph of shifted shapes and its dual

5. Derivations in Graded Algebras

Let A be a graded associative K-algebra with identity; so A (as a vector space) is a direct sum of "homogeneous" subspaces A_n, $n \in \mathbb{N}$; and if $a \in A_n$ and $b \in A_m$ then $ab \in A_{n+m}$. Assume $D : A \to A$ is a derivation, i.e., a linear endomorphism satisfying

$$D(ab) = aD(b) + D(a)b \quad ;$$

assume, in addition, that $rank(D(a)) = rank(a) - 1$. Suppose $t \in A$ is such that $D(t) = r \cdot id$ where id stands for an identity element of A; of course $t \in A_1$. (Informally, D is a "derivation with respect to t/r".) Hence the operator $U \in End(A)$ defined by $U(a) = ta$ and the derivation operator D satisfy the condition (2.1) where $I \in End(A)$ is an identity transformation. (One can also take the *right* multiplication $U(a) = at$ instead.) In case the homogeneous subspaces A_n are finite-dimensional we can fix arbitrary bases in them to obtain a pair of r-dual networks/graphs.

Now we re-construct several examples of Sec.3-4 by means of this tool.

5.1 Example. *The chain* $\mathbb{N} = \{0, 1, 2, \dots\}$ (cf. Example 4.2). Let $A = K[t]$ be the algebra of polynomials in the variable t, and $rank(t^n) = n$. Then U is the operator of multiplication by t, and D is $\frac{d}{dt}$. Now take t^n as the basic vectors to obtain Example 4.2.

5.2 Example. *Young graph* (cf. Example 3.1). Let A be the algebra of symmetric functions, i.e., the symmetric formal power series of bounded degree in commuting variables $x_1, x_2, \dots$ This algebra can be alternatively described as the algebra $K[t_1, t_2, \dots]$ of polynomials in the variables t_n satisfying $rank(t_n) = n$. Now let D be the derivation with respect to t_1 and U the multiplication by t_1 (so $t = t_1$). Since $D(t_1) = 1$, we have a pair of dual graded graphs/networks; to make the construction explicit bases in A_n's should be chosen. In case $t_n = \sum x_i^n$ the basis of the *Schur functions* (see, e.g., [Ma79]) gives rise to the construction of Example 3.1 (the Young graph).

5.3 Example. *Pascal graphs* (cf. Example 4.3). Let A be the algebra $K[t_1, \dots, t_r]$ of polynomials in r commuting variables. Define

$$Uf(t_1, \dots, t_r) = (t_1 + \dots + t_r) f(t_1, \dots, t_r) \quad , \tag{5.1}$$

$$D = \frac{\partial}{\partial t_1} + \dots + \frac{\partial}{\partial t_r} \quad . \tag{5.2}$$

The condition (2.1) holds, and we get the pair of Example 4.3.

5.4 Example. *Diagrams with* $\leq r$ *rows* (cf. Example 4.5). Here A is the algebra of symmetric polynomials in r commuting variables $t_1, \dots, t_r$. Like in Example 5.3, define U and D by (5.1) and (5.2). The result coincides with that of Example 4.5.

6. Bracket Tree, Binary Trees, and Subword Order

This section is devoted to constructing dual graphs for two remarkable rooted trees.

6.1 Example. *Lifted binary tree.* This is the infinite binary tree $\mathbb{T}_2$ with an additional edge attached below the old zero. The vertices of this tree can be naturally labelled by the binary notations of the nonnegative integers: 0, 1, 10, 11, 100, 101, 110, ... so that 0 is the root; 1 is the only vertex of rank 1; 10 and 11 are the vertices of rank 2, etc. (see Fig.5). Generally, the rank is the length of the notation, except for the case $\rho(0) = 0$.

Now define the graph *BinWords* (a *lifted binary subword order*; cf. [Bj91]; see Fig.5) with the same set of vertices, the same rank function, and the following covering relation: x covers y iff x can be obtained by deleting a single symbol from y. In addition, 1 covers 0. (For example, 101001 covers 11001, 10001, 10101, and 10100.) We emphasize that *BinWords* is a graph *without multiple edges.*

6.2 Lemma. *BinWords and the lifted binary tree are dual.*

6.3 Example. *Bracket Tree.* This tree (see Fig.4) is defined as follows. The vertices of rank n are the "syntactically correct" formulae defining different versions of calculation of *non-associative* product $x \cdot x \cdot \ldots \cdot x$ containing $n+1$ entries of x. We call such sequences the *bracket schemes.* Two schemes are linked in the tree if one of them results from another by deleting the first entry of x and subsequent removing the pair of "unnecessary" brackets. Indeed, it is a tree.

6.4 Lemma. *The Bracket Tree is dual to the distributive lattice $J(\mathbb{T}_2)$ of the finite order ideals of the infinite binary tree.*

This statement needs explanation: we should demonstrate that, in a sense, the Bracket Tree and $J(\mathbb{T}_2)$ have the same set of vertices (see, e.g., [SW86,Sec.3.1] for another proof). To do that, associate any bracket scheme with an appropriate parsing tree. Remove the leaves of this tree (they correspond to the entries of x) as well as the edges incident to them. As a result we obtain a *marked* binary tree, i.e., an order ideal of the infinite binary tree $\mathbb{T}_2$. See [St75] for additional information concerning $J(\mathbb{T}_2)$.

Now we can apply the enumerative results of Sec.2 to each of the examples listed above (the same is true concerning the generalized Schensted construction of Sec.8).

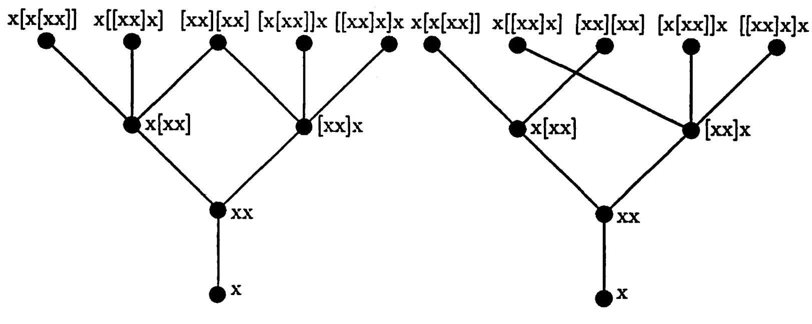

Fig.4 Lattice of binary trees and Bracket Tree

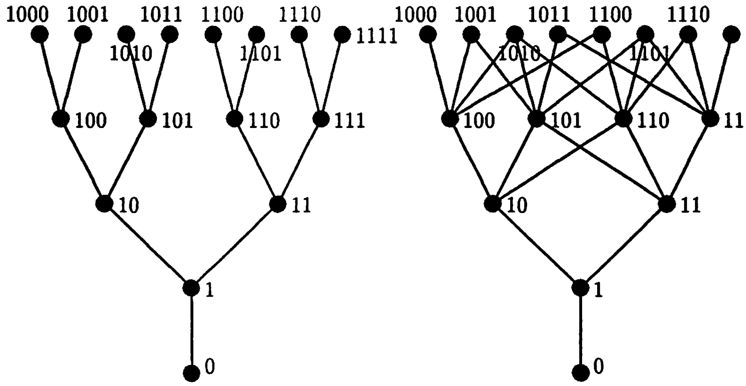

Fig.5 Lifted binary tree and BinWords

7. Some Applications

The identities of Sec.2 can be applied to each of the examples given above. The versions of Corollary 2.6 are stated below.

7.1 Corollary. *Assume G is a self-dual graph. Then*

$$\sum_{x \in P_n} e(x)^2 = n!$$

In particular, this is true for the Young graph $\mathbb{Y}$ *and the* Young-Fibonacci graph $\mathbb{YF}$.

7.2 Corollary. *Assume G_1 and G_2 are dual graphs, and G_2 is a tree. Then*

$$\sum_{x \in P_n} e_1(x) = n! \quad .$$

In particular, this is the case when G_1 is either BinWords *or* $J(\mathbb{T}_2)$.

7.3 Corollary. *Assume a weight function w defined on a poset Q satisfies the conditions of Lemma 4.1. Then in the distributive lattice $P = J(Q)$ one has*

$$\sum_{x \in P_n} e(x)^2 \prod_{q \in x} w(q) = r^n n! \ . \tag{7.1}$$

In particular, this identity holds in the following distributive lattices:

(1) The graph of shifted shapes $\mathbb{SY}$: *w is defined by* (4.4);
(2) The graph of the diagrams with $\leq r$ rows: *w is defined by* (4.2);
(3) Pascal graph $\mathbb{N}^r$:

$$w(q) = num(q)$$

where $q \in r\mathbb{P}$ is an integer $num(q)$ taken from a certain chain $\mathbb{P}$.

7.4 Comments.

1. In the case of the graph of shifted shapes $\mathbb{SY}$ the identity (7.1) reduces to (4.3).

2. In the case of the graph of r-row diagrams (7.1) turns into

$$\sum_{x=(x_1,\ldots,x_r)\in P_n} e(x)^2 \prod_{j=1}^{r} \frac{(r+x_j-j)!}{(r-j)!} = r^n n!$$

3. In the case of the Pascal graph $\mathbb{N}^r$ a direct simplification transforms (7.1) into the classical $\sum e(x) = r^n$, where $e(x)$ denotes the appropriate multinomail coefficient.

8. Generalized Schensted

All the enumerative identities that can be derived from

$$DU = UD + rI \tag{8.1}$$

can also be proved combinatorially, i.e., by establishing certain explicit bijections. To do this, one can start with fixing a bijection between the objects which are counted in the left-hand and the right-hand sides of (8.1). Then, to get a bijective proof of any identity involving U's and D's (say, (2.2)), one can write a sequence of elementary algebraic transformations which prove this identity (that is, at each step a single substitution $DU \leftarrow UD + rI$ is performed) and then proceed step by step along the lines of this algebraic proof by applying "bijective version" of $DU = UD + rI$ that we fixed in advance. Since some of the involved elementary transformations commute, the resulting algorithm is *parallel.*

This construction, when applied to the identity

$$D^n U^n = (UD + r)(UD + 2r)\ldots(UD + nr)$$

(cf. (2.2)), results in a generalized Schensted algorithm which establishes bijection between pairs of paths and permutations (in case $r > 1$ the latters are colored in r colors). In other words, we obtain bijective proofs of respective versions of Corollary 2.7.

We conclude by describing the basic construction in explicit algorithmic terms.

8.1 Definition. Assume, as before, that G_1 and G_2 are r-dual graded graphs. Let $\Phi = \{\Phi_{xy}\}_{x,y\in P}$ be a family of bijections between the objects counted by "the (x, y)'th matrix elements" of DU and $UD + rI$, respectively, i.e., bijections from

$$B_{xy} = \{(b_1, b_2) : b_1 \in E_1, b_2 \in E_2, end(b_1) = end(b_2), start(b_1) = y, start(b_2) = x\}$$

to

$$A_{xy} = \{(a_1, a_2) : a_1 \in E_1, a_2 \in E_2, start(a_1) = start(a_2), end(a_1) = x, end(a_2) = y\} \cup \{1, \ldots, r\} .$$

Such a family is called an *r-correspondence*. Since x and y are determined by either a_i or b_i, we may unambigously regard Φ as a single bijection

$$\Phi: \bigcup B_{xy} \longrightarrow \bigcup A_{xy} .$$

Once an r-correspondence Φ is fixed for a given pair of dual graded graphs, a bijective correspondence between pairs of paths and colored permutations arises.

8.2 Algorithm. *("Generalized Schensted: tableaux to permutations")*

Input:

(i) edges $t_1(1)$, $t_1(2)$, ... , $t_1(n)$ forming a path in G_1 starting at $\hat{0}$;

(ii) edges $t_2(1)$, $t_2(2)$, ... , $t_2(n)$ forming a path in G_2 starting at $\hat{0}$ and having common endpoint with the first path.

Output: colored permutation σ (matrix $n \times n$ with exactly one nonzero element in each row and column; this element should be one of 1, ... ,r).

```
var
  φ1 : array [1..n,0..n] of ( edge of G1  or nil) ;
  φ2 : array [0..n,1..n] of ( edge of G2  or nil) ;
  σ : array [1..n,1..n] of integer;
  a1, a2, b1, b2, k, l : integer;
begin
  for k := 1 to n do φ1(k,n) := t1(k);
  for l := 1 to n do φ2(n,l) := t2(l);
  for (k,l) := (n,n) downto (1,1) do
    begin
      b1 := φ1(k,l) ;   b2 := φ1(k,l);
      case
        b1 = nil or b2 = nil  ⟹  a1 := b1;  a2 := b2;  σ(k,l) := 0;
        Φ(b1,b2) ∈ E1 × E2  ⟹  (a1,a2) := Φ(b1,b2);  σ(k,l) := 0;
        Φ(b1,b2) = i ∈ {1,...,r}  ⟹  a1 := nil;  a2 := nil;  σ(k,l) := i
      endcase ;
      φ1(k,l−1) := a1 ;   φ2(k−1,l) := a2
    end
end
```

Comments: In the third **for**-cycle (n^2 repetitions) a pair (k_1, l_1) should be treated *after* (k_2, l_2) whenever $k_1 \leq k_2$ and $l_1 \leq l_2$. Calculations may be done in parallel regarding this condition.

One can also write, in the same manner, an algorithm realizing the reverse bijection.

8.3 Algorithm. *("Generalized Schensted: permutations to tableaux")*

Input = Output of Algorithm 8.2.

Output = Input of Algorithm 8.2.

```
var
  φ1, φ2 : ... ; {see Algorithm 8.2}
  a1, a2, b1, b2, k, l : integer;
begin
  for k := 1 to n do φ1(k,0) := (0̂,0̂);
  for l := 1 to n do φ2(0,l) := (0̂,0̂);
  for (k,l) := (1,1) to (n,n) do
    begin
      a1 := φ1(k, l − 1) ;   a2 := φ1(k − 1, l);
      case
        (a1 = nil or a2 = nil) and σ(k,l) = 0 ⟹ b1 := a1;  b2 := a2;
        (a1 = nil and a2 = nil) and σ(k,l) ≠ 0 ⟹ (b1, b2) := Φ⁻¹(σ);
        (a1 ≠ nil and a2 ≠ nil) ⟹ (b1, b2) := Φ⁻¹(a1, a2)
      endcase ;
      φ1(k,l) := b1 ;   φ2(k,l) := b2
    end;
  for k := 1 to n do t1(k) := φ1(k,n);
  for l := 1 to n do t2(l) := φ2(n,l)
end
```

Both algorithms are essentially parallel. To get a sequential version of, e.g., Algorithm 8.2, replace

for $(k, l) := (n, n)$ **downto** $(1, 1)$ **do**

by

for $k := n$ **downto** 1 **do**

  for $l := n$ **downto** 1 **do**

where the last two loops can be interchanged as well. The interior **for**-loop is an analogue of the Schensted insertion and reduces to the latter in the case of the Young lattice.

For the graph of shifted shapes the "row-wise" and "column-wise" sequential versions coincide with the algorithms of Sagan-Worley [Sa87,Wo84] and Haiman [Ha89], respectively (provided the natural r-correspondence is chosen).

The applications to rim hook tableaux are given in [FS92].

For other pairs of dual graphs (including those listed above) respective Schensted analogues can be constructed by specializing the general scheme; see [Fo2]. A unified approach to Knuth-type algorithms (cf. [Kn70]) is suggested in [Fo3]. Other identities in the algebra $\mathfrak{A}_r$ give rise to corresponding modifications of "generalized Schensted", like its "skew version" that extends the constructions of [SS90] (see [Fo2,Fo3] for details).

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