The asymptotic behaviour of coefficients of large powers of functions.

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Abstract

We review existing results on the asymptotic approximation of the coefficient of order n of a function $f(z)^d$, when n and d grow large while staying roughly proportional. Then we present extensions of these results to allow more general relationships between n and d and to take into account a multiplicative factor $\psi(z)$.

1 Introduction

Generating functions of the type $\phi(z) = f(z)^d$, where f is a given function with positive coefficients and d is a parameter which tends to infinity, appear in several problems of discrete probability theory, combinatorial enumeration, etc. These problems often require an estimate of the n^{th} coefficient of f^d, which we denote by $[z^n]\{f(z)^d\}$, for large n and d.

For example, let $X_1, \cdots, X_d$ be d random variables, independent and with the same probability distribution defined by the generating function $f(z)$. Their sum $S_d = \sum_{i=1}^{d} X_i$ has for generating function $f^d(z)$, whose coefficient of order n is the probability $\Pr(S_d = n)$. The average value of S_d is $d\, f'(1)$, and its variance is also of order d. The situations where $n = d\, f'(1) + o(\sqrt{d})$, $n = o(d)$ or $d = o(n)$ describe the behaviour of the sum respectively close to the mean (in a range where the central limit theorem applies), before or beyond the mean (in an area of large deviations).

Coefficients of the type $[z^n]\{f^d(z)\}$ appear for example in asymptotic coding theory [10], in the evaluation of some parameters on forests of trees [17, 21], in the evaluation of diagonal coefficients of some bivariate functions $F(z, u)$, and in a class of asymptotic

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distributions related to urn models, which require computing the coefficient $[y^n]\{f(x,y)^d\}$ of a bivariate function: See [14, 15] for a survey of results on urn models and [7, 9] for some applications to relational database theory. Related problems also appear in the evaluation of trie parameters [5] or of the number of lattice points in a ball [16, 20], and in the analysis of a random walk on an hypercube [4].

The present paper is intended as a survey of results on the asymptotic estimation of coefficients of the type $[z^n]\{f^d(z)\}$; it also presents some as yet unpublished results. Its plan is as follows: In order to unify the presentation, we introduce some notations, then recall the basis of the main technic (a saddle-point approximation) in Section 2. Section 3 presents results pertaining to the asymptotic approximation of $[z^n]\{f(z)^d\}$, for large n and d growing at a similar rate. We then extend these results to allow different growth rates for n and d. In Section 4, we allow a multiplicative factor $\psi(z)$ and study the coefficient $[z^n]\{f(z)^d\ \psi(z)\}$. Finally we indicate some applications, mostly related to urn models and Stirling numbers, in Section 5.

2 Notations and methods

2.1 Notations

We consider in this paper functions of one variable which have a power series expansion $f(z) = \sum_{k\geq 0} f_k z^k$. We assume in the sequel that the function f satisfies the following property:

> **Assumption $\mathcal{A}_1$:**
> *The function f has real positive coefficients with $f_0 \neq 0$ and $f_1 \neq 0$, and a strictly positive, possibly infinite, radius of convergence R. Its coefficients are such that $GCD\{k : f_k \neq 0\} = 1$.*

The condition on the GCD can be stated in an equivalent form: There exists no entire function g and no integer $m \geq 2$ such that $f(z) = g(z^m)$. The condition on f_0 simply means that when $f(z)$ has valuation p, we can factor out z^p: If $f(z) = z^p(f_0 + f_1 z + \cdots)$, then $f^d(z) = z^{dp}(f_0 + f_1 z + \cdots)^p$. The restriction on f_1 is a technical one, which might be removed, but this extension implies more restrictive conditions on the relative growths of n and d than those given in some theorems of this paper.

To simplify the notations in the sequel, we define two operators on a function f:

$$\Delta f(z) = z\frac{f'}{f}(z); \qquad \delta f(z) = \frac{f''}{f}(z) - \frac{f'^2}{f^2}(z) + \frac{f'(z)}{zf(z)}.$$

These operators are related by: $z\delta f(z) = (\Delta f)'(z)$. When the function f has real positive coefficients, it is not difficult to show that, for all real positive z smaller than R, the radius of convergence of f, *the value of $\delta f(z)$ is strictly positive and the function Δf is increasing.*

2.2 The saddle-point approximation

Before studying a function $f^d(z)$, we first recall results valid for any analytic function ϕ. Its coefficient of order n is given by Cauchy's formula, where the integration contour is a closed curve around the origin of the complex plane which stays inside the convergence domain:

$$[z^n]\phi(z) = \frac{1}{2i\pi}\oint \phi(z)\frac{dz}{z^{n+1}}.$$

We immediately deduce from it an upper bound $|[z^n]\phi(z)| \leq (1/2\pi)\oint|\phi(z)z^{-n-1}|dz$. Integrating on a circle of radius ρ smaller than the radius of convergence of ϕ gives $|[z^n]\phi(z)| \leq \phi(\rho)\rho^{-n}$, and the best (smallest) upper bound is obtained, when possible, for ρ such that $\rho\phi'(\rho)/\phi(\rho) = n$.

For example, let us assume that ϕ is the generating function of a random variable X, of mean μ, and let $n = (1+\delta)\mu$. Then $\Pr(X = n) = [z^n]\phi(z)$ is bounded from above by $\phi(\rho)\rho^{-n}$. Setting $\rho = e^t$ and using the fact that $\phi(e^t) = E(e^{tX})$, we get:

$$\Pr(X = (1+\delta)\mu) \leq \frac{E(e^{tX})}{e^{t(1+\delta)\mu}}.$$

This is Chernoff's bound, which often gives useful information on the probability that a random variable is at distance at least $1+\delta$ of its mean.

Now assume that X is itself obtained by summing d independent random variables with a common distribution: $\phi(z) = f^d(z)$. Then we have that $\Pr(X = n) \leq f^d(\rho)\rho^{-n}$, and this bound is tightest for ρ such that $\rho f'(\rho)/f(\rho) = n/d$.

The upper bound can be refined to give an approximation of $[z^n]\phi(z)$: Instead of bounding ϕ on the integration circle, we look closely at the points which give the main contribution to the integral. This is the basis of the saddle point method (see for example [3] for a general presentation and Hayman [13][22, Ch.5] for applications to the approximation of generating function coefficients). It turns out that, if we can choose for radius of the integration circle the point ρ defined by the equation $\rho\phi'(\rho)/\phi(\rho) = n$, the main part of the integral often comes from the vicinity of ρ, which is a *saddle point*. Defining $h(z) = \log\phi(z) - (n+1)\log z$, we get:

$$[z^n]\phi(z) = \frac{1}{2i\pi}\oint e^{h(z)}dz \approx \frac{e^{h(\rho)}}{\sqrt{2\pi h''(\rho)}} = \frac{\phi(\rho)}{\rho^{n+1}\sqrt{2\pi h''(\rho)}}.$$

This approximation holds for a large class of functions $\phi(z) = f^d(z)$, when n/d belongs to an interval $[a,b]$ $(0 < a < b)$ and $n, d \to +\infty$ [2, 11, 12].

The application of the saddle-point method to the asymptotic evaluation of coefficients of a function is closely related to Laplace's method for approximating an integral. Although this method is usually applied to integrals depending on one parameter, Fulks [6] and Pederson [19] have studied integrals depending on two parameters, which are in the same vein as the problem of evaluating $[z^n]\{f^d(z)\}$, where we have two parameters n and d.

3 Asymptotic approximations of coefficients

3.1 The case n constant

We include this case for the sake of completeness, although it presents no real difficulty. If n is constant, the saddle-point method does not work; however a direct analysis can give some information. For example, the following result simply means that the first coefficients of $f(z)^d$ behave as those of $(f_0 + f_1 z)^d$.

Theorem 1:
If the function f has real positive coefficients, such that $f_0 \neq 0$ and $f_1 \neq 0$, then, for $d \to +\infty$ and for any fixed n:

$$[z^n]\{f^d(z)\} = \binom{d}{n} f_0^{d-n} f_1^n (1 + O(1/d)).$$

This is proved by expanding the coefficient into a sum of (a fixed number of) multinomial coefficients, which are themselves easily approximated. If we allow n to grow, both the number of terms in the sum and the terms themselves are unbounded, and this proof no longer holds.

3.2 A general formula when $n = \Theta(d)$

The problem of finding the asymptotic value of $[z^n]\{f(z)^d\}$, when $n, d \to +\infty$ and n and d are roughly proportional, was studied for example by Daniels [2] and Greene and Knuth [12], mostly for probability generating functions. As noted by Good [11], this result is actually valid for a larger class of functions, such as entire functions or functions defined on an open disk; moreover it can be improved to give further terms of an asymptotic development. We give below the main result [2][11, p.868].

Theorem 2:
Let f be a function satisfying the assumption $\mathcal{A}_1$ of Section 2.1, and let R be its radius of convergence. Assume that n/d belongs to an interval $[a, b]$, $0 < a < b$, and that $n, d \to +\infty$. Define ρ and σ^2 by $\Delta f(\rho) = n/d$ and $\sigma^2 = \rho^2 \delta f(\rho)$. If $\rho < R$, then:

$$[z^n]\{f(z)^d\} = \frac{f(\rho)^d}{\sigma \rho^n \sqrt{2\pi d}}(1 + o(1)).$$

We can simplify Theorem 2 further if n/d has a finite, non null, limit:

Corollary 1:
Under the assumptions of Theorem 1, if there exist two real strictly positive constants k and m such that $n = kd + m$, then:

$$[z^n]\{f(z)^d\} = \frac{A^d}{B \rho_0^m \sqrt{d}}(1 + o(1)),$$

for suitable constants $A = f(\rho_0)\rho_0^{-k}$ *and* $B = \sigma\sqrt{2\pi}$*, and with* ρ_0 *the solution (independent of* n *and* d*) of* $\Delta f(z) = k$*. Note that* σ *too is a constant:* $\sigma^2 = \rho_0^2\,\delta f(\rho_0)$*. If* $n = kd$*, i.e.* $m = 0$*, then we have the simpler formula:*

$$[z^n]\{f(z)^d\} = \frac{A^d}{B\sqrt{d}}(1 + o(1)).$$

It is possible to get some information on the variation of the term A when the quotient n/d is finite and bounded away from 0. Let $A = A(k)$ with $k = n/d$. *On a closed interval of* $]0, +\infty[$ *including* $\Delta f(1)$*,* $A(k)$ *is a unimodal function of* k*, first increasing then decreasing , with a maximum* $A(\Delta f(1)) = f(1)$*.*

3.3 Function defined by an implicit equation

A recent paper by Meir and Moon [17] deals with the approximation of the coefficient of z^n in $f(z)^d$, when $d, n \to +\infty$ and $d = O(n)$, and with f defined by an implicit equation: $f(z) = z\phi(f(z))$ and $f(0) = 0$. This improves on a former result by Flajolet and Steyaert [21], which was proved for $d = o(n)$, more precisely for $d \leq \sqrt{n}/\log^3(n)$. Meir and Moon give the following result:

Theorem 3:
Let ϕ *be a function satisfying the assumption* $\mathcal{A}_1$ *of Section 2.1 and define a function* f *by* $f(z) = z\phi(f(z))$ *and* $f(0) = 0$*. Let* $d = \alpha n + \lambda\sqrt{n} + o(\sqrt{n})$*, with* α *a constant such that* $0 \leq \alpha < 1$ *and that* $(\Delta\phi)^{-1}(1-\alpha)$ *exists, and with* λ *a finite (positive or negative, possibly null) constant. Then, for* $n, d \to +\infty$

$$[z^n]\{f(z)^d\} = \frac{d}{n\sigma\sqrt{2\pi n}} e^{-\lambda^2/2\sigma^2} \rho^{d-n} \phi(\rho)^n (1 + o(1)),$$

where ρ *is defined by* $\Delta\phi(\rho) = 1 - \alpha$ *and* σ^2 *by*

$$\sigma^2 = \rho^2 \frac{\phi''(\rho)}{\phi(\rho)} + \alpha(1-\alpha) = \rho^2 \delta\phi(\rho).$$

Meir and Moon actually prove in passing the following result:

$$[t^n]\{\phi(t)^d\} = \frac{e^{-\lambda^2/2\sigma^2}}{\sigma\sqrt{2\pi d}} \frac{\phi(\rho)^d}{\rho^n}(1 + o(1)),$$

and their range of validity is for $n = (1-\alpha)d + \lambda\sqrt{d} + o(\sqrt{d})$. For $\alpha > 0$, this is basically an extension of Theorem 2 (to allow $\lambda \neq 0$) applied to $n = kd + O(\sqrt{d})$ with a constant $k = 1 - \alpha$ in $]0, 1]$. Theorem 3 is then obtained by an application of the Lagrange inversion formula:

$$[z^n]\{f(z)^d\} = \frac{d}{n}[t^{n-d}]\{\phi(t)^n\}.$$

When $\alpha = 0$ but $\lambda > 0$, Theorem 3 gives an approximation valid for $d = \lambda\sqrt{n}(1 + o(1))$, i.e. $d^2 = \lambda^2 n(1 + o(1))$. If $\alpha = \lambda = 0$, the result holds for $d = o(\sqrt{n})$ i.e. $d^2 = o(n)$. This means that Meir and Moon have results for $d \approx \alpha n$ (when $\alpha \neq 0$) or for $d^2 = O(n)$ (when $\alpha = 0$). If $\lambda = 0$ and $\alpha \neq 0$, and if we have $f(z) = z^q g(z)$ with $g(0) \neq 0$, then either one of Theorem 2 or Theorem 3 can be applied indifferently to evaluate the coefficient $[z^n]\{f^d(z)\} = [z^{n-qd}]\{g^d(z)\}$, for $d = \alpha n + o(\sqrt{n})$.

3.4 The case $n = o(d)$

We study now the case where d and n both grow large, but n stays much smaller than d. Theorem 4 is an extension of Theorem 2 to the case $n = o(d)$.

Theorem 4:
Let f satisfy the assumption $\mathcal{A}_1$ of Section 2.1 and let $n = o(d)$, with $n, d \to +\infty$. Define ρ as the unique real positive solution of $\Delta f(z) = n/d$. Then:

$$[z^n]\{f(z)^d\} = \frac{f(\rho)^d}{\rho^n\sqrt{2\pi n}}(1 + o(1)).$$

Theorem 4 is proved by integrating on a circle going through the saddle point ρ, which becomes $o(1)$ for $n = o(d)$. The singularities are beyond the integration contour as soon as n and d are large enough. The detailed proof can be found in [8].

Theorem 4 is closely related to a result of Odlyzko and Richmond [18], which holds for a class of polynomials f, and for n and d such that, with q denoting the degree of f, $qd - n \to +\infty$. Their result covers the case $n = o(d)$, when f is the generating function of a probability distribution with finite support.

If we have more information on the respective orders of growth of n and d, we can obtain a useful approximation of the saddle point ρ and give a more precise form of Theorem 4 . The following corollary, for example, deals with the cases when $n = o(\sqrt{d})$ or $n = o(d^{2/3})$.

Corollary 2:
If f satisfies the assumption $\mathcal{A}_1$ of Section 3.1 and if $n = o(\sqrt{d})$, with $n, d \to +\infty$, then:

$$[z^n]\{f^d(z)\} = \frac{f_0^d}{\sqrt{2\pi n}}\left(\frac{ef_1 d}{f_0 n}\right)^n (1 + o(1) = \frac{f_0^d}{n!}\left(\frac{f_1 d}{f_0}\right)^n (1 + o(1)).$$

If we only have the weaker condition $n = o(d^{2/3})$, then:

$$[z^n]\{f^d(z)\} = \frac{f_0^d}{\sqrt{2\pi n}}\left(\frac{ef_1 d}{f_0 n}\right)^n \exp\left(\frac{n^2}{d}(\frac{f_2}{f_1} - \frac{1}{2})\right)(1 + o(1)).$$

Intuitively, Corollary 2 means that, when $n = o(\sqrt{d})$, the first two coefficients of f determine the main term in the asymptotic expression of the coefficients of f^d. This result can be compared to the relevant one for an affine function (although an affine function does not satisfy assumption $\mathcal{A}_1$): $[z^n]\{(f_0 + f_1 z)^d\} = \binom{d}{n} f_0^{d-n} f_1^n$; Stirling's formula for the factorials gives an approximation equivalent to the first one of Corollary 2. When n increases with respect to d, the other coefficients are progressively introduced. As long as some relationship $n^l = o(d^q)$ holds, it is possible to get a result similar to Corollary 2. This requires a good approximation of the saddle point ρ, and might become quite involved according to which coefficients of f are null, but it would be possible to work it out for a given function f. However, if for example $n = d/\log d$, we cannot find a relationship $n^l = o(d^q)$ and we have to take all the coefficients of f into account.

3.5 The case $d = o(n)$

When the function f satisfies some functional equation, the result of Meir and Moon presented in Section 3.3 can sometimes be applied. More generally, we can prove analogs of Theorems 2 and 4 for some classes of functions, using similar technics.

Theorem 5:
Let $f(z) = e^{P(z)}$, where $P(z) = \sum_{0 \le i \le q} P_i z^i$ is a polynomial of degree q with positive coefficients. Assume that the coefficients P_0 and P_1 are nonnull. If $n, d \to +\infty$ in such a way that $d = o(n)$, define ρ as the unique real positive solution of $zP'(z) = n/d$. Then:

$$[z^n]\{e^{dP(z)}\} = \frac{e^{dP(\rho)}}{\rho^n \sqrt{2\pi n}}(1 + o(1)).$$

If the function f is not entire, its singularities become important. For example, we can prove the following result for a meromorphic function with one pole on its circle of convergence:

Theorem 6:
Let f be a meromorphic function with positive coefficients, whose singularity of smallest modulus is a pole in 1: $f(z) = g(z)/(1 - z)$, where g is a function analytic for $|z| \le 1$. Assume that $f_1 \ne 0$, and define ρ by $\Delta f(\rho) = n/d$. Then, if $d = o(n)$ and $n = o(d^{10/9})$, we have:

$$[z^n]\{f^d(z)\} = \sqrt{\frac{d}{2\pi}} \cdot \frac{f^d(\rho)}{n\rho^n}(1 + o(1)).$$

4 Introducing a factor $\psi(z)$

We now allow a multiplicative factor $\psi(z)$ and study $[z^n]\{f^d(z)\psi(z)\}$. The function ψ may itself depend on d or on other parameters, as long as the following property is satisfied:

Assumption $\mathcal{A}_2$:
The function ψ has positive coefficients, such that $\psi(0) \neq 0$, has a strictly positive radius of convergence, and either is fixed, or is a product of "large" powers of functions. In this case, it has the following form, where p is any fixed integer and the $d_i \to +\infty$:

$$\psi(z) = \prod_{i=1}^{p} g_i(z)^{d_i} \qquad \text{with} \qquad d_i = o(\frac{d}{\sqrt{n}}), 1 \leq i \leq p. \tag{1}$$

We can justify the condition on ψ as follows: An extra factor $\psi(z)$ moves the saddle-point away from the value ρ_0 obtained for f^d; this does not matter as long as the new saddle-point ρ stays close enough, within $o(1/\sqrt{n})$ of ρ_0. The difference $\rho - \rho_0$ is $\Theta(\rho_0(\sum_i d_i/d))$, hence the condition (1).

We now present some theorems which extend the former ones to allow an extra factor ψ. Theorem 7 is an obvious extension of Theorem 1:

Theorem 7:
If f is a function with positive coefficients such that $f_0 \neq 0$ and $f_1 \neq 0$, and if the function ψ satisfies the assumption $\mathcal{A}_2$, then for n constant and $d \to +\infty$:

$$[z^n]\{f^d(z)\psi(z)\} = \binom{d}{n} f_0^{d-n} f_1^n \psi(0)(1 + O(1/\sqrt{d})).$$

When n and d have the same growth rate, we can prove the following result, which is roughly Theorem 2 of [10] (the condition on ψ below is stronger than the assumption $\mathcal{A}_2$):

Theorem 8:
Let f satisfy the assumption $\mathcal{A}_1$ of Section 2.1, and let ψ be a function with positive coefficients and a strictly positive radius of convergence. Assume that the equation $\Delta f(z) = n/d$ has a real positive solution ρ smaller than the radius of convergence of f. Then, for $n, d \to +\infty$ and $n = \Theta(d)$:

$$[z^n]\{f^d(z)\psi(z)\} = \frac{f(\rho)^d \psi(\rho)}{\rho^{n+1}\sqrt{2\pi d\, \delta f(\rho)}}(1 + o(1)).$$

The case $n = o(d)$ is settled by the following theorem, whose proof can be found in [8]:

Theorem 9:
Let f and ψ satisfy repectively the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ and let $n = o(d)$, with $n, d \to +\infty$. Define ρ as the unique real positive solution of $\Delta f(\rho) = n/d$. Then:

$$[z^n]\{f(z)^d \psi(z)\} = \frac{f(\rho)^d \cdot \psi(\rho)}{\rho^n \sqrt{2\pi n}}(1 + o(1)).$$

If some relationship $n^l = o(d^q)$ holds, then we have the analog of Corollary 2:

Corollary 3:
If f satisfies the assumptions $\mathcal{A}_1$ of Section 2.1, if ψ satisfies $\mathcal{A}_2$ but with the stronger conditions $d_i = o(d/n)$, and if $n = o(\sqrt{d})$:

$$[z^n]\{f^d(z)\psi(z)\} = \psi(0)\frac{f_0^d}{\sqrt{2\pi n}} \cdot \left(\frac{ef_1d}{f_0n}\right)^n (1 + o(1)).$$

If we only have $n = o(d^{2/3})$, then:

$$[z^n]\{f^d(z)\psi(z)\} = \psi(0)\frac{f_0^d}{\sqrt{2\pi n}} \left(\frac{ef_1d}{f_0n}\right)^n \exp\left(\frac{n^2}{d}(\frac{f_2}{f_1} - \frac{1}{2})\right) (1 + o(1)).$$

5 Some applications

An easy check of our formulæ is provided by the function $f(z) = e^z$. The saddle point is $\rho = n/d$ and we get, for $d \to +\infty$ and for n either fixed or going to infinity

$$[z^n]\{e^{dz}\} = d^n/n! = \frac{e^n d^n}{n^n\sqrt{2\pi n}}(1 + o(1)),$$

which is simply Stirling's formula for $n!$.

One of the basic constructions for obtaining combinatorial structures is to take a sequence of simpler objects. Let $f(z)$ be the generating function enumerating these objects according to their size; the generating function enumerating the sequences of d basic objects, according to their global size, is $f(z)^d$, and the coefficient $[z^n]\{f(z)^d\}$ enumerates the number of sequences of d basic objects of size n. The same approach can also be used to analyze the abelian partitional complex, whose bivariate generating function has the form $\exp(xf(y))$.

However, we do not count the structures of size 0 and we have $f(0) = 0$ and $n \geq d$. Let us define $f(y) = yg(y)$ with $g(0) \neq 0$; we have that $[z^n]\{f^d(z)\} = [z^{n-d}]\{g^d(z)\}$. The results presented above can now be applied to evaluate the number of composed objects of size $n \geq d$ which are a sequence of d simpler objects.

Classical examples are the Stirling numbers of the first and the second type. Stirling numbers of the first type enumerate, among other things, the number of permutations of n objects with k cycles; their exponential generating function is $\sum_{n,k} s_{n,k}x^k y^n/n! = \exp(x\log(1/1-y))$; hence

$$s_{n,k} = \frac{n!}{k!}[y^{n-k}]\{f(y)^k\} \qquad \text{with} \qquad f(y) = \frac{1}{y}\log\frac{1}{1-y} = \sum_{n\geq 0}\frac{y^n}{n+1}.$$

For example, we can get an asymptotic equivalent for $n = k + o(k)$, or equivalently $k = n - o(n)$, but still $n - k \to +\infty$. The saddle point ρ is approximately $2(n-k)/k$ and

Corollary 2 gives, for $n = k + o(\sqrt{k})$:

$$s_{n,k} = \frac{n!}{k!\sqrt{2\pi(n-k)}} \left(\frac{ek}{2(n-k)}\right)^{n-k} (1 + o(1)),$$

i.e., using Stirling's approximation for $(n-k)!$ backwards:

$$s_{n,k} = \binom{n}{k} (k/2)^{n-k} (1 + o(1)).$$

Let $S_{n,k}$ be a Stirling number of second type, enumerating for example the number of partitions of n objects into k blocks. These numbers have for exponential generating function $\sum_{n,k} S_{n,k} x^k y^n / n! = \exp(x(e^y - 1))$, hence

$$S_{n,k} = \frac{n!}{k!}[y^{n-k}]\{f(y)^k\} \qquad \text{with} \qquad f(y) = \frac{e^y - 1}{y} = \sum_{n \geq 0} \frac{y^n}{(n+1)!}.$$

For $n = k + o(\sqrt{k})$ and $n - k \to +\infty$, Corollary 2 applied to $f(y)$ gives

$$S_{n,k} = \frac{n!}{k!\sqrt{2\pi(n-k)}} \left(\frac{ek}{2(n-k)}\right)^{n-k} (1 + o(1)) = \binom{n}{k} (k/2)^{n-k} (1 + o(1)).$$

This asymptotic expression, which is also given for example in [1, p. 825], is the same as the one for the Stirling numbers of the first type: From Corollary 2, only f_0 and f_1 are important if $n - k = o(\sqrt{k})$. However, if the difference $n - k$ is of order at least $\sqrt{k}$, the next coefficients become important. For example, if $n - k \to +\infty$ with only $n - k = o(k^{2/3})$, then the second part of Corollary 2 shows that the Stirling numbers of the first and second type have a different behaviour:

$$\begin{aligned} s_{n,k} &= \binom{n}{k} (k/2)^{n-k} e^{(n-k)^2/6k} (1 + o(1)); \\ S_{n,k} &= \binom{n}{k} (k/2)^{n-k} e^{-(n-k)^2/6k} (1 + o(1)). \end{aligned}$$

Stirling numbers of the second type also appear in a classical occupancy problem of discrete probability theory: *We throw n balls into k urns randomly and independently; what is the number of urns with at least one ball?* Let $N_{n,d}$ be the number of ways of assigning the n balls to exactly d urns. If the balls are undistinguishable and if the urns have unbounded capacity, the associated generating function is [14]:

$$\Phi(x, y) = \sum_{n,d} N_{n,d}\, x^d \,\frac{y^n}{n!} = (1 + x(e^y - 1))^k.$$

Let $f(y) = (e^y - 1)/y$; we have $N_{n,d} = n! \binom{k}{d} [y^{n-d}]\{f(y)^d\}$, which can be expressed using Stirling numbers of the second type: $N_{n,d} = d! \binom{k}{d} S_{n,d}$.

6 Conclusion

We have presented results on the asymptotic approximation of coefficients of the type $[z^n]\{f^d(z)\}$, with applications, and on the asymptotic approximation of the coefficient $[z^n]\{f^d(z)\psi(z)\}$; examples of applications using such coefficients can be found in [10]. Possible extensions include:

- Allowing the second coefficient of f to be null: $f_1 = 0$. This corresponds to a function $f(z) = 1 + f_2 z^2 + \cdots$. Preliminary studies indicate that such an extension considerably restricts the respective ranges of n and d.

- Allowing $d = o(n)$ for more general functions than those considered in Section 3.5. Here again we may have to introduce further growth restrictions on n and d, depending on the singularities of the funtion $f(z)$; we may also have to use a technic more adapted to the nature of the singularities than the saddle point method.

- Removing the restriction that ψ has positive coefficients. This does not seem to pose any real difficulty, as opposed to the fact that the similar condition on f is essential.

- Obtaining further terms of an asymptotic expansion. This is similar to the extension of the results of Daniels [2] by Good [11], and should not introduce major difficulties.

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