RECENT PROGRESS
on
THE MACDONALD q,t-KOSTKA CONJECTURE

A. M. Garsia (*)

ABSTRACT. In this lecture I shall present recent progress, in a joint effort with Mark Haiman, toward proving the Macdonald q,t-Kostka conjecture. The main thrust of our work has been towards the construction of a representation theoretical setting for the Macdonald basis $\{P_\mu(X,q,t)\}_\mu$. The original goal was to obtain new methods for attacking some of the problems and conjectures arising from Macdonald work [14]. This effort has been met with success beyond our best expectations. In particular, it has already brought to light some truly remarkable properties and facts concerning these polynomials. It has also opened up a new area of investigation with a wide variety of exciting algebraic and combinatorial problems and conjectures. The feeling prevails that this is only the the tip of a mathematical iceberg that could keep many investigators occupied for a few years to come. I can give here only a sample of the results and problems that stem from this development. We refer to [3], [4] and [5] for a more complete treatment.

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We recall that Macdonald in [14] shows the existence of a family of polynomials $\{P_\lambda(x;q,t)\}$ which are uniquely characterized by the following conditions

$$\begin{aligned} a)\quad & P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\mu\, \xi_{\mu\lambda}(q,t) \\ b)\quad & \langle P_\lambda\,,\, P_\mu\rangle_{q,t} \;=\; 0 \quad for \quad \lambda \neq \mu \end{aligned} \tag{1}$$

Where S_λ denotes the Schur function indexed by λ and $\langle\ ,\ \rangle_{q,t}$ denotes the scalar product of symmetric polynomials defined by setting for the power basis $\{p_\rho\}$

$$\langle p_{\rho_1}\,,\, p_{\rho_2}\rangle_{q,t} \;=\; \begin{cases} z_\rho\, p_\rho\left[\frac{1-q}{1-t}\right] & \text{if } \rho_1=\rho_2=\rho \text{ and} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Here we use λ-ring notation and z_ρ is the integer that makes $n!/z_\rho$ the number of permutations with cycle structure ρ. There are a number of outstanding conjectures concerning these polynomials (see [14]). Here we shall be dealing with those involving the so called ***integral forms*** $J_\mu(x;q,t)$ and their associated Macdonald-Kostka coefficients $K_{\lambda\mu}(q,t)$. We shall use the same notation as in [14]. In particular $\{Q_\lambda(x;q,t)\}$ denotes the basis dual to $\{P_\lambda(x;q,t)\}$ with respect to the scalar product $\langle\ ,\ \rangle_{q,t}$. Clearly, (1) b) gives

$$Q_\lambda(x;q,t) \;=\; d_\lambda(q,t)\, P_\lambda(x;q,t)\ , \tag{3}$$

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for a suitable rational function $d_\lambda(q,t)$. However in [14] it is shown that

$$d_\lambda(q,t) = \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}$$

with

$$h_\lambda(q,t) = \prod_{s\in\lambda}(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}) \quad , \quad h'_\lambda(q,t) = \prod_{s\in\lambda}(1-q^{a_\lambda(s)+1}t^{l_\lambda(s)})$$

where s denotes a generic lattice square and $a_\lambda(s)$, $l_\lambda(s)$ respectively denote the *arm* and the *leg* of s in the Ferrers' diagram of λ.

We recall from [14] that

$$J_\mu(x;q,t) = h_\mu(q,t)P_\mu(x;q,t) = h'_\mu(q,t)Q_\mu(x;q,t) \ , \tag{4}$$

and the coefficients $K_{\lambda\mu}(q,t)$ are defined through an expansion which in λ-ring notation may be written as

$$J_\mu(x;q,t) = \sum_\lambda S_\lambda[X(1-t)]\, K_{\lambda\mu}(q,t) \tag{5}$$

Macdonald conjectures that these coefficients are polynomials in q and t with non-negative integer coefficients. We shall refer to this here and after as the MPK conjecture. Macdonald derives a number of properties of the $K_{\lambda\mu}(q,t)$; in particular he shows that for any partition μ

$$K_{\lambda\mu}(1,1) = f_\lambda \tag{6}$$

where f_λ denotes the number of standard tableaux of shape λ. This given, the MPK conjecture is equivalent to the statement that for each μ there exists an S_n-module M_μ yielding a bigraded version of the left regular representation whose character has the expansion

$$char\ M_\mu = \sum_\lambda \chi^\lambda\, K_{\lambda\mu}(q,t) \ . \tag{7}$$

More precisely, if $\mathcal{H}_{h,k}(M_\mu)$ denotes the submodule of M_μ consisting of its bihomogeneous elements of bidegree (h,k) and we set

$$p^\mu(q,t) = \sum_{h,k\geq 0} q^h t^k\, char\ \mathcal{H}_{h,k}(M_\mu) \tag{8}$$

then (7) should hold true with $char\ M_\mu = p^\mu(q,t)$. In this vein, the symmetric polynomial

$$H_\mu(x;q,t) = \sum_\lambda S_\lambda\, K_{\lambda\mu}(q,t) = J_\mu[X/(1-t);q,t] \tag{9}$$

may be viewed as the Frobenius characteristic of M_μ, while the expression

$$F_\mu(q,t) = \sum_\lambda f_\lambda\, K_{\lambda\mu}(q,t) \tag{10}$$

should give its Hilbert series, that is the polynomial

$$F_\mu(q,t) \;=\; \sum_{h,k\geq 0} q^h t^k \, dim\, \mathcal{H}_{h,k}(M_\mu) \; . \tag{11}$$

For technical reasons it is preferable to work with the modified versions of $H_\mu(x;q,t)$ and $F_\mu(q,t)$ obtained by setting

$$\tilde{H}_\mu(x;q,t) \;=\; H_\mu(x;q,1/t)t^{n(\mu)} \quad , \quad \tilde{F}_\mu(q,t) \;=\; F_\mu(q,1/t)t^{n(\mu)} \; . \tag{12}$$

It will also be convenient to set

$$\tilde{K}_{\lambda\mu}(q,t) \;=\; K_{\lambda\mu}(q,1/t)t^{n(\mu)} \; ,$$

where, for $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$

$$n(\mu) \;=\; \sum_{i=1}^{k} (i-1)\mu_i \; . \tag{13}$$

In the fall of 1989 I set myself the task of finding a module M_μ whose bigraded character, as defined by (8), has the expansion

$$char\, M_\mu \;=\; \sum_\lambda \chi^\lambda \, \tilde{K}_{\lambda\mu}(q,t) \; . \tag{14}$$

Since setting $q = 0$ in $K_{\lambda\mu}(q,t)$ yields the Kostka-Foulkes coefficients $K_{\lambda\mu}(t)$ a good starting point appeared to be some early work of the algebraic geometers (see [6] and the references quoted there) which yielded the first proof of the analogous positivity result for $K_{\lambda\mu}(t)$. The basic ingredient that may be extracted from this literature is a certain graded S_n-module $\mathbf{R}_\mu$ in whose character the coefficients $K_{\lambda\mu}(t)$ appear as t-multiplicities of irreducibles. To be precise, let p^μ_m denote the character of the action of S_n on the m^{th} graded component of $\mathbf{R}_\mu$, and set

$$p^\mu(t) \;=\; \sum_{m\geq 0} t^m \, p^\mu_m \; . \tag{15}$$

Expanding in terms of the the irreducible characters χ^λ we may also write

$$p^\mu(t) \;=\; \sum_\lambda \chi^\lambda C_{\lambda\mu}(t) \; , \tag{16}$$

where $C_{\lambda\mu}(t)$ is the polynomial whose coefficient of t^m gives the multiplicity of χ^λ in p^μ_m. Now a sequence of deep developments (see [13] II §3 ex. 1 p. 92 and III §7 ex. 9 p.136) yields that

$$K_{\lambda\mu}(t) \;=\; C_{\lambda,\mu}(t^{-1})t^{n(\mu)} \; . \tag{17}$$

However, although $\mathbf{R}_\mu$ was later shown by Kraft [7] and DeConcini-Procesi [2] to have an elementary direct definition as a quotient of the polynomial ring $\mathbf{Q}[x_1,\ldots,x_n]$, the relation in (17) had only been established in a setting that not only required t to be the power of a prime, but also relied on some of the deepest results and tools of Algebraic Geometry. This given, I set myself the task of providing a purely representation theoretical setting for the study of $\mathbf{R}_\mu$ and its graded character $p^\mu(t)$. This program was successfully carried out in joint work with Procesi in [6]. In particular a new proof of (17) was obtained which only used elementary representation theoretical tools. In an earlier work [8]-[12] Lascoux and Schützenberger stated without proof that that a suitable submodule $\mathbf{R}_\mu[X]$ of the S_n-harmonic polynomials had also the same graded character. Subsequently, Bergeron and Garsia [1] were able to provide a proof of this fact that used only elementary tools of commutative algebra. This done, the next task was to try and see to what extent these methods could be applied to the q,t-case. This program is in the process of being carried out in joint work with Mark Haiman. The starting point is the brilliant idea of Haiman to try and construct the desired bigraded module by working in an analogous manner with polynomials in two sets of variables $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. This viewpoint led to the construction of an S_n-module $M_D[X, Y]$ which yields a bigraded version of the left regular representation of S_n for each lattice square diagram D. During more than a year we have been involved in an intensive study of the module M_D and the various problems that have arisen from our efforts to compute its character. During this period we have gathered overwhelming evidence that when D is the Ferrers' diagram of a partition μ the resulting module $M_\mu[X, Y]$ has a bigraded character given by (14). To describe this evidence we need some notation. For a given diagram D we let $p^D(q,t)$ denote the bigraded character of M_D, and let $G[D](x; q, t)$ be its Frobenius characteristic. We also set

$$p^D(q,t) \;=\; \sum_\lambda \chi^\lambda \, C_{\lambda D}(q,t) \; . \tag{18}$$

Clearly we must also have then

$$G[D](x;q,t) \;=\; \sum_\lambda S_\lambda(x)\, C_{\lambda D}(q,t) \; . \tag{19}$$

When D is the Ferrers' diagram of a partition μ, $p^D(q,t)$, $G[D](x;q,t)$ and $C_{\lambda D}(q,t)$ will be simply represented by $p^\mu(q,t)$, $G_\mu(x;q,t)$ and $C_{\lambda\mu}(q,t)$ respectively. Our efforts have been directed towards proving that

$$C_{\lambda\mu}(q,t) \;=\; \tilde{K}_{\lambda\mu}(q,t) \tag{20}$$

So far this has now been proved by Garsia and Haiman ([4],[5]) in the following cases

(1) For all μ when λ is a hook,
(2) For all λ when μ is a hook,
(3) For all λ when μ any two row or two column partition,
(4) For all λ when μ is a partition of $n \leq 6$.
(5) Verified by computer for all λ when μ a partition of $n \leq 7$

The completion of this work, and a proof of the conjecture in full generality, hinges on the establishment of a number of properties of the modules $\mathbf{R}_\mu[X, Y]$ which have emerged from theoretical considerations combined with computer data. I can give a brief view of some the work that has been done. In the one parameter case, the modules $\mathbf{R}_\mu$ and $\mathbf{R}_\mu[X]$ studied in [1] and [6] respectively have different definitions but they are shown in [1] to be equivalent as graded S_n-modules. The module $\mathbf{R}_\mu$ studied in [6] is a quotient of the ring of polynomials in $x_1, x_2, \ldots, x_n$, while

$\mathbf{R}_\mu[X]$ is defined in [1] as the linear span of the derivatives of the Garnir polynomials corresponding to standard tableaux of shape μ. The module $\mathbf{R}_\mu[X,Y]$ referred to above is a natural bigraded extension of $\mathbf{R}_\mu[X]$. It may be defined as the linear span of the derivatives of a single bihomogeneous polynomial $\Delta_\mu(x,y)$ in the variables X and Y. Extending the definition of the ring $\mathbf{R}_\mu$ given in [6] to the two variable case, leads to two separate constructions and two additional spaces ${}^{bg}\mathbf{R}_\mu$ and ${}^{sg}\mathbf{R}_\mu$. Both these spaces are obtained by working on polynomials in X and Y. While ${}^{bg}\mathbf{R}_\mu$ is bigraded, it is not defined as a quotient ring, in contrast ${}^{sg}\mathbf{R}_\mu$ is a quotient ring but only known to be singly graded. All three spaces $\mathbf{R}_\mu[X,Y]$, ${}^{bg}\mathbf{R}_\mu$ and ${}^{sg}\mathbf{R}_\mu$ are S_n modules under the diagonal action of S_n. This is the action defined by setting, for $\sigma = \sigma_1 \cdots \sigma_n \in S_n$:

$$\sigma P(x_1, x_2, .., x_n; y_1, y_2, .., y_n) \;=\; P(x_{\sigma_1}, x_{\sigma_2}, .., x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, .., y_{\sigma_n}) \;. \tag{21}$$

The first task is to prove (as in the one parameter case), that these three spaces are equivalent as graded S_n-modules. All evidence gathered so far supports this conclusion. It is interesting to have a look at some of the properties that have been established concerning these three spaces. We shall also agree to use the same notation for the bigraded characters of $\mathbf{R}_\mu[X,Y]$ and ${}^{bg}\mathbf{R}_\mu$. Let us keep in mind that (due to (6)) the validity of (20) implies that $\mathbf{R}_\mu[X,Y]$ and ${}^{bg}\mathbf{R}_\mu$ should be bi-graded versions of the left regular representation of S_n. In particular, all the three spaces should have dimension $n!$. Under this conjecture, the polynomial

$$F_\mu(q,t) = \sum_\lambda f_\lambda \, \tilde{K}_{\lambda\mu}(q,t) \tag{22}$$

should give a bigraded version of the Hilbert series of these two modules, while

$$F_\mu(q) \;=\; F_\mu(q,q) \;=\; \sum_\lambda f_\lambda \, \tilde{K}_{\lambda\mu}(q,q)$$

should give the Hilbert series of ${}^{sg}\mathbf{R}_\mu$ and the singly graded one for $\mathbf{R}_\mu[X,Y]$ and ${}^{bg}\mathbf{R}_\mu$.

Macdonald also shows a number of identities relating the coefficients $K_{\lambda\mu}(q,t)$ for various values of λ, μ, q, t. For instance, from the results in [14] it can be derived that

1) $F_\mu(q,t)$ is symmetric, that is $F_\mu(q,t) = F_\mu(q^{-1},t^{-1}) q^{n_\mu} t^{n_{\mu'}}$.
2) $K_{\lambda\mu}(0,t) \;=\; K_{\lambda\mu}(t)$,
3) $\tilde{K}_{\lambda\mu}(q,t) \;=\; \tilde{K}_{\lambda\mu'}(t,q)$,
4) $\tilde{K}_{\lambda\mu}(q,t) \;=\; q^{n_{\mu'}} t^{n_\mu} \, \tilde{K}_{\lambda'\mu}(q^{-1},t^{-1})$,

where, priming a partition here represents conjugation. This given, the following table summarizes some of the results presented in [4] and [5], which yield further evidence in support of (20).

PROPERTY	$\mathbf{R}_\mu[X,Y]$	${}^{bg}\mathbf{R}_\mu$	${}^{sg}\mathbf{R}_\mu$
Symmetric Hilbert series	yes	yes	?
Quotient ring	$\asymp$	?	yes
Regular representation	?	yes	yes

Dimension $n!$	?	yes	yes
Bigraded	yes	yes	?
$C_{\lambda\mu}(q,1) = \tilde{K}_{\lambda\mu}(q,1)$	?	yes	$\asymp$
$C_{\lambda\mu}(q,0) = \tilde{K}_{\lambda\mu}(q,0)$	yes	yes	$\asymp$
$C_{\lambda\mu}(q,t) = \tilde{C}_{\lambda\mu'}(t,q)$	yes	yes	$\asymp$
$C_{\lambda\mu}(q,t) = q^{n_{\mu'}} t^{n_\mu} C_{\lambda'\mu}(q^{-1},t^{-1})$	yes	yes	$\asymp$

The symbol $\asymp$ is to signify here that the property in question is not applicable to the given module. The question mark "?" represents the conjecture that we should have a *yes*. Note that each property holds true for at least one of the modules. Remarkably, it can be shown that the removal of a single question mark "?" (that is replacing it by a *yes*) in this table removes them all and forces all three constructions to yield the same bigraded S_n-module.

However the strongest evidence supporting the validity of (20), is that the modules $M_D[X,Y]$ have suggested us identities involving the polynomials $G[D](x;q,t)$ which we were in fact able to prove within the theory of Macdonald polynomials. We shall only give a brief view of this development and refer the reader to [3] for a more detailed presentation.

We shall say that two lattice square diagrams D_1 and D_2 are equivalent and write $D_1 \approx D_2$ if and only if D_2 can be obtained from D_1 by a sequence of row and column rearrangements. If D is a lattice square diagram, the diagram obtained by reflecting D across the diagonal line $x = y$ will be called the *conjugate* of D and denoted by D'. Similarly, the reflection of a lattice square s across $x = y$, will be denoted by s'. Finally, if D may be decomposed into the union of two diagrams D_1 and D_2 in such a manner that no square of D_2 is in the rook domain of a square of D_1, then we shall say that D is *decomposable* and we write $D = D_1 \times D_2$. This given, the construction of the module M_D suggests that the family of polynomials $\{G[D](x;q,t)\}_D$ has the following basic properties

$$\begin{cases} (1)\quad G[D_1](x;q,t) = G[D_2](x;q,t) & \text{if } D_1 \approx D_2 \\ (2)\quad G[D_1](x;q,t) = G[D_2](x;t,q) & \text{if } D_2 \approx D_1' \\ (3)\quad G[D](x;q,t) = G[D_1](x;q,q)G[D_2](x;q,t) & \text{if } D \approx D_1 \times D_2 \end{cases} \tag{23}$$

The validity of (20) yields the further relation

$$G[D](x;q,t) = \tilde{H}_\mu(x;q,t) \quad \textit{if } D \textit{ is the diagram of } \mu \tag{24}$$

which may be interpreted as an initial condition. Moreover, a study of the behavior of the S_n-module M_D under restriction to S_{n-1} suggests recursions for some of the polynomials $G[D](x;q,t)$ which, together with equation (24) above completely determine them as linear combinations of the polynomials $\tilde{H}_\mu(x;q,t)$. This permits the study of the the polynomials $G[D](x;q,t)$ independently of the validity of (20). In particular by recursing through this extended family, in [3], we were able to rederive some of the Stanley-Macdonald [15]-[14] Pieri rules in a manner which unravels their original intricacy as a combination of successive, simple elementary steps. The remarkable agreement of the resulting identities with those that can be derived from the theory of Macdonald polynomials, offers what is so far the best evidence of the validity of (20). In our lecture we shall present some of the latter developments in a *Viennotique* of lattice square diagrams manipulations.

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