Free Hyperplane Arrangements Interpolating Between Root System Arrangements

by
Tadeusz Józefiak
and
Bruce Sagan[1]

Abstract

Let R and S be root systems with $R \subset S$. By adding the roots of $S \setminus R$ to R one at a time, one obtains a sequence of subsets each of which determines a hyperplane arrangement. It turns out that these arrangements are often free and so the associated characteristic polynomials have non-negative integer roots. Zaslavsky [Zas 81] was the first to consider the family of hyperplane arrangements interpolating between D_n and B_n. These investigations were continued by Cartier [Car 82], Orlik and Solomon [O-S 83], Orlik-Solomon-Terao [J-T 80, Example 2.6], Ziegler [Zie 90], and Hanlon [Han pr]. Surprisingly, other interpolating families seem not to have been studied previously. In the present work we will show that some of these families are free by explicitly calculating bases for the corresponding modules of derivations. As immediate corollaries, we can read off the roots of their characteristic polynomials.

Let

$$\mathcal{A} = \{H_1, \cdots, H_k\} \tag{1}$$

be an arrangement (set) of hyperplane subspaces in the Euclidean space $\mathbf{R}^n$. Let $L = L(\mathcal{A})$ be the poset of intersections of these hyperplanes ordered by reverse inclusion. Thus L has a unique minimal element $\hat{0}$ corresponding to $\mathbf{R}^n$, an atom corresponding to each H_i, and a unique maximal element $\hat{1}$ corresponding to $\bigcap_{1\leq i \leq k} H_i$. It is well-known that L is a geometric lattice with rank function

$$\operatorname{rk} X = n - \dim X$$

for any $X \in L$. Let $\mu(X) = \mu(\hat{0}, X)$ denote the Möbius function of the lattice. Then the *characteristic polynomial of* L is

$$\chi(L,t) = \sum_{X \in L} \mu(X) t^{\dim X}.$$

Now consider the polynomial algebra $A = \mathbf{R}[x_1, \ldots, x_n] = \mathbf{R}[x]$ with the usual grading by total degree $A = \oplus_{i\geq 0} A_i$. A *derivation* is an $\mathbf{R}$-linear map

$$\theta : A \to A$$

[1]Both authors are currently at: Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027

satisfying

$$\theta(fg) = f\theta(g) + g\theta(f)$$

for any $f, g \in A$. The set of all derivations is a A-module. It is graded by saying that θ has degree d if $\theta(A_i) \subseteq A_{i+d}$. This module is also free with a basis given by the operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$. It will often be convenient to display a derivation as a column vector whose entries are its components with respect to this basis. Thus if

$$\theta = p_1(x)\partial/\partial x_1 + \cdots + p_n(x)\partial/\partial x_n$$

where $p_i(x) \in \mathbf{R}[x]$ for all i, then we write

$$\theta = \begin{bmatrix} p_1(x) \\ \vdots \\ p_n(x) \end{bmatrix} = \begin{bmatrix} \theta(x_1) \\ \vdots \\ \theta(x_n) \end{bmatrix}.$$

Let $e_1, \ldots, e_n$ denote the coordinate vectors in $\mathbf{R}^n$ with the variables $x_1, \ldots, x_n$ being considered as elements of the corresponding dual basis. So any hyperplane $H \subseteq \mathbf{R}^n$ is defined by an equation

$$p_H(x_1, \ldots, x_n) = 0$$

where p_H is a linear polynomial. Thus the arrangement $\mathcal{A}$ in (1) is defined by the form

$$Q = Q(\mathcal{A}) = \prod_i p_{H_i}.$$

Consider the associated *module of $\mathcal{A}$-derivations* defined by

$$D(\mathcal{A}) = \{\theta \mid \theta \text{ a derivation and } \theta(Q) \in Q \cdot \mathbf{R}[x]\}.$$

We say that $\mathcal{A}$ is a *free arrangement* if $D(\mathcal{A})$ is a free module. Terao first introduced free arrangements and proved the following fundamental theorem [Ter 81, Ter 83]. A simpler proof was obtained with Solomon [S-T 87].

Theorem 1 *If $\mathcal{A}$ is free then*

1. *$D(\mathcal{A})$ has a homogeneous basis $\theta_1, \ldots, \theta_n$,*

2. *the set*

$$\{d_1, \ldots, d_n\} = \{\deg \theta_1, \ldots, \deg \theta_n\}$$

   depends only on $\mathcal{A}$,

3. *the characteristic polynomial of $\mathcal{A}$ factors as*

$$\chi(L(\mathcal{A}), t) = \prod_i (t - d_i - 1). \blacksquare$$

In order to find such homogeneous bases, we use a result whose holomorphic version is due to Saito [Sai 80], and whose algebraic analogue comes from Terao [Ter 83] and Solomon-Terao [S-T 87]. Given any set of derivations $\theta_1, \ldots, \theta_n$, consider the rectangular matrix whose columns are the corresponding column vectors

$$\Theta = [\theta_1, \ldots, \theta_n] = [\theta_j(x_i)].$$

Theorem 2 *Suppose $\theta_1, \ldots, \theta_n \in D(\mathcal{A})$ where $\mathcal{A}$ has defining form Q. Then the following conditions are equivalent:*

1. $\det \Theta = cQ$ *where $c \in \mathbf{R}$ is non-zero,*
2. *$\mathcal{A}$ is free with basis $\theta_1, \ldots, \theta_n$.* ■

Thus we can prove that an arrangement $\mathcal{A}$ is free by constructing homogeneous derivations that

1. are in the submodule of $\mathcal{A}$-derivations and
2. have the proper determinant.

Often, the hardest part of the proof is showing that the scalar c in part 1 of Theorem 2 is non-zero. In some cases this step involves interesting new determinants related to those of Jacobi-Trudi [Sag ta].

To state our results, we will need a bit more notation. Any finite set $P \subseteq \mathbf{R}^n$ of vectors gives rise to the arrangement whose hyperplane subspaces are $H = p^\perp$ for $p \in P$. Let $\chi(P, t)$ and $\Theta(P)$ stand, respectively, for the corresponding characteristic polynomial and matrix for a basis of derivations. Also define column vectors

$$x^k = \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix} \qquad \text{and} \qquad \hat{x} = \begin{bmatrix} \hat{x}_1 x_2 \cdots x_n \\ x_1 \hat{x}_2 \cdots x_n \\ \vdots \\ x_1 x_2 \cdots \hat{x}_n \end{bmatrix}$$

where $\hat{x}_i$ means that x_i is omitted.

We first interpolate between the root systems D_n and B_n.

Theorem 3 *Let*

$$DB_{n,k} = D_n \cup \{e_1, \ldots, e_k\}.$$

Then $DB_{n,k}$ is free with basis matrix

$$\Theta(DB_{n,k}) = [x^1, x^3, \ldots, x^{2n-3}, \theta_n]$$

where

$$\theta_n = x_1 x_2 \cdots x_k \hat{x}.$$

Thus

$$\chi(DB_{n,k}, t) \text{ has roots } 1, 3, \ldots, 2n-3, n+k-1. \; ■$$

By symmetry, it is clear that adding the e_i in any order would produce a free arrangement with the same characteristic polynomial.

When interpolating between A_{n-1} and B_n, the order in which the roots are added matters. First we add $e_1, \ldots, e_n$. The remaining roots can be listed in a triangular array

$$\begin{array}{cccc} e_1 + e_2 & e_1 + e_3 & \cdots & e_1 + e_n \\ & e_2 + e_3 & \cdots & e_2 + e_n \\ & & & \vdots \\ & & & e_{n-1} + e_n \end{array}$$

We can add the $e_i + e_j$ *by columns* where we read each column from top to bottom, or *by rows* where we read each row from left to right. To describe the basis matrices, let

$$E_k(t) = t(t - x_1)(t - x_2)\cdots(t - x_k)$$

so that the coefficients of powers of t in $E_k(t)$ are elementary symmetric functions in the first k variables. The corresponding column vectors are

$$E_k = \begin{bmatrix} E_k(x_1) \\ \vdots \\ E_k(x_n) \end{bmatrix}.$$

Note that the first k entries of E_k are zero.

Theorem 4 *Interpolate from A_{n-1} to B_n by columns by letting*

$$AB^c_{n,k,l} = A_{n-1} \cup \{e_1, \ldots, e_n\} \cup \{e_1 + e_2, e_1 + e_3, e_2 + e_3, \ldots, e_k + e_l\}.$$

Then $AB^c_{n,k,l}$ is free with basis matrix

$$\Theta(AB^c_{n,k,l}) = [x^1, x^3, \ldots, x^{2l-3}, \theta_l, E_l, E_{l+1}, \ldots, E_{n-1})$$

where

$$\theta_l = (x_1 + x_l)(x_2 + x_l)\cdots(x_k + x_l)E_{l-1}$$

Thus

$$\chi(AB^c_{n,k,l}, t) \text{ has roots } 1, 3, \ldots, 2l - 3, k + l, l + 1, l + 2, \ldots, n. \blacksquare$$

To interpolate by rows, define

$$E_{k,l}(t) = t(t + x_1)(t + x_2)\cdots(t + x_k)(t - x_1)(t - x_2)\cdots(t - x_l).$$

with associated column vector

$$E_{k,l} = \begin{bmatrix} E_{k,l}(x_1) \\ \vdots \\ E_{k,l}(x_n) \end{bmatrix}.$$

Theorem 5 *Interpolate from A_{n-1} to B_n by rows by letting*

$$AB^r_{n,k,l} = A_{n-1} \cup \{e_1, \ldots, e_n\} \cup \{e_1 + e_2, \ldots, e_1 + e_n, e_2 + e_3, \ldots, e_2 + e_n, \ldots, e_k + e_l\}.$$

Then $AB^r_{n,k,l}$ is free with basis matrix

$$\Theta(AB^r_{n,k,l}) = [x^1, x^3, \ldots, x^{2k-1}, E_{k,k}, E_{k,k+1}, \ldots, E_{k,l-1}, E_{k-1,l}, E_{k-1,l+1}, \ldots, E_{k-1,n-1}].$$

Thus

$\chi(AB^r_{n,k,l}, t)$ *has roots* $1, 3, \ldots, 2k-1, 2k+1, 2k+2, \ldots, k+l, k+l, k+l+1, \ldots, n$. ∎

Similar theorems hold for interpolation between A_{n-1} and D_n. One can also get results for arrangement interpolating between a root system and itself, e.g., from D_n to D_{n+1}. It is interesting to note that many, although not all, of the results we have obtained can be generalized to the Dowling lattices (using hyperplanes of the form $x_i + \zeta x_j$ as ζ runs through all rth roots of unity).

Other methods for proving these results are also being investigated. One can compute individual Möbius values in the various families introduced above and prove that their characteristic polynomials factor directly as Hanlon did for $DB_{n,k}$. Finally, Curtis Bennett and Sagan have developed a generalization of the notion of supersolvability which can be used to combinatorially prove factorization of $\chi(DB_{n,k}, t)$ though the lattices are not supersolvable for $k < n$. This method should extend to the other cases under consideration as well.

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References

[Car 82] P. Cartier, Les arrangements d'hyperplans: un chapitre de géométrie combinatoire, in "Séminaire Bourbaki, Volume 1980/81, Exposés 561–578," Lecture Notes in Math., Vol. 901, Springer-Verlag, New York, NY, 1982, 1–22.

[Han pr] P. Hanlon, A combinatorial construction of posets that intertwine the independence matroids of B_n and D_n, preprint.

[J-T 80] M. Jambu and H. Terao, Free arrangements of hyperplanes and supersolvable lattices, *Adv. in Math.* **52** (1984), 248–258.

[Orl 89] P. Orlik, "Introduction to Arrangements," Regional Conference Series in Math., No. 72, American. Math. Soc., Providence, RI.

[O-S 83] P. Orlik and L. Solomon, Coxeter arrangements, in Proc. Symp. Pure Math., Vol. 40, part 2, Amer. Math. Soc., Providence, RI, 1983, 269–291.

[Sag ta] B. E. Sagan, Log concave sequences of symmetric functions and analogs of the Jacobi-Trudi determinants, *Trans. Amer. Math. Soc.*, to appear.

[Sai 80] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sec. 1A Math.* **27** (1980), 265–291.

[S-T 87] L. Solomon and H. Terao, A formula for the characteristic polynomial of an arrangement, *Adv. in Math.* **64** (1987), 305–325.

[Ter 80] H. Terao, Arrangements of hyperplanes and their freeness, *J. Fac. Sci. Univ. Tokyo Sec. 1A Math.* **27** (1980), 293–312.

[Ter 81] H. Terao, Generalized exponents of a free arrangement of hyperplanes and the Shepherd-Todd-Brieskorn formula, *Invent. Math.* **63** (1981), 159–179.

[Ter 83] H. Terao, Free arrangements of hyperplanes over an arbitrary field, *Proc. Japan Acad. Ser. A Math* **59** (1983), 301–303.

[Zas 81] T. Zaslavsky, The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88** (1981), 88–105.

[Zie 90] G. Ziegler, Matroid representations and free arrangements, *Trans. Amer. Math. Soc.* **320** (1990), 525–541.