CONSEQUENCES OF THE A_ℓ AND C_ℓ BAILEY TRANSFORM AND BAILEY LEMMA

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1. INTRODUCTION

The purpose of this talk is to discuss some applications of the higher-dimensional generalization of the Bailey Transform and Bailey Lemma in the setting of basic hypergeometric series very well-poised on unitary A_ℓ or symplectic C_ℓ groups in [Lil91, LM91, Mil91a-f, ML91]. The derivation of the C_ℓ case in [LM91, ML91] is closely related to the previous analysis of the unitary A_ℓ, or equivalently $U(\ell+1)$ case from [Mil91a,c,d]. This program is based upon the A_ℓ and C_ℓ terminating very well-poised ${}_6\phi_5$ summation theorems which are extracted from [Mil85, Mil87, Mil91a] and [Gus89], respectively. Both types of very well-poised series are directly related [Gus89, Mil85] to the corresponding Macdonald identities. The classical case of all this work, corresponding to A_1 or equivalently $U(2)$, contains an immense amount of the theory and application of one-variable basic hypergeometric series [And76, And86a, Bai35, GR90, Sla66], including elegant proofs of the Rogers-Ramanujan-Schur identities. The ordinary ($q = 1$) case of some of the multiple series in [Mil87] first appeared in certain applications of mathematical physics and the unitary groups $U(n+1)$, or equivalently A_n. This earlier work on the theory of Wigner coefficients for $SU(n)$ was due to Biedenharn, Holman, and Louck [BL68–BL81b, Hol80, HBL76]. They showed in [Hol80, HBL76] how the classical work on ordinary hypergeometric series is intimately related to the irreducible representations of the compact group $SU(2)$. Their work was done in the context of the quantum theory of angular momentum [BL81a-b] and the special unitary groups $SU(n)$.

The classical A_1 Bailey Transform [And86a] and Bailey Lemma [And86a] were ultimately inspired by Rogers' [Rog17] second proof of the Rogers-Ramanujan-Schur identities [And76, And86a, GR90, Rog94, Rog95]. The Bailey Transform was first formulated by Bailey [Bai47, Bai49], utilized by Dyson in [Dys43], applied by Slater in [Sla51–Sla66], and then recast by Andrews [And79] as a fundamental matrix inversion result. This last version of the Bailey Transform has immediate applications to connection coefficient theory and "dual" pairs of identities [And79, And84, And86a, GS83, GS86], and q-Lagrange inversion and quadratic transformations [GS83, GS86]. The most important application of the Bailey Transform is the Bailey Lemma. This result was mentioned by Bailey [Bai49;

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§4], and he described how the proof would work. However, he never wrote the result down explicitly and thus missed the full power of *iterating* it. Andrews first established the Bailey Lemma explicitly in [And84] and realized its numerous possible applications in terms of the iterative "Bailey chain" concept. This iteration mechanism enabled him to derive many q-series identities by "reducing" them to more elementary ones. For example, two iterations of the Bailey Lemma reduce the Rogers-Ramanujan-Schur identities to the q-binomial theorem [And84, And86a]. The process of iterating Bailey's Lemma has led to a wide range of applications in additive number theory, combinatorics, special functions, and mathematical physics. For example, see [And84–And86b, ABF84, ADH88, Bax82, Pau82, Pau85, Sla51–Sla66]. The Bailey Transform is a consequence of the terminating very well-poised ${}_4\phi_3$ summation theorem. The Bailey Lemma is derived in [AAB87] directly from Rogers' [Rog95] terminating very well-poised ${}_6\phi_5$ summation theorem and the matrix inversion formulation [And79, GS83, GS86] of the Bailey Transform. The terminating very well-poised ${}_6\phi_5$ summation theorem is crucial to this entire program.

At this point, it is useful to survey the classical Bailey Transform and Bailey Lemma.

Let q be a complex number such that $|q| < 1$. Define

$$(\alpha)_\infty \equiv (\alpha;\, q)_\infty := \prod_{k\geq 0}(1-\alpha q^k) \tag{1.1a}$$

and, thus,

$$(\alpha)_n \equiv (\alpha;\, q)_n := (\alpha)_\infty \,/\, (\alpha q^n)_\infty \,. \tag{1.1b}$$

We then have Andrews' [And79] matrix inversion in

Theorem 1.2 (Classical Bailey Transform for A_1). *Let a be indeterminate and $i, j \geq 0$ be integers. Let the matrices M and M^* be defined as in*

$$M(i;\, j;\, A_1) := (q)^{-1}_{i-j}\, (aq)^{-1}_{i+j}\,; \tag{1.3a}$$

and

$$M^*(i;\, j;\, A_1) := (1-aq^{2i})\, (aq)_{i+j-1}\, (q)^{-1}_{i-j}\, (-1)^{i-j} q^{\binom{i-j}{2}}. \tag{1.3b}$$

Then M and M^* are inverse, infinite, lower-triangular matrices. That is,

$$\delta(i,\, j) = \sum_{j\leq y\leq i} M(i;\, y;\, A_1)\, M^*(y;\, j;\, A_1), \tag{1.4}$$

where $\delta(r,\, s) = 1$ if $r = s$, and 0 otherwise.

Theorem 1.2 follows from the terminating very well-poised ${}_4\phi_3$ summation theorem and a termwise rewriting of the (i, j) entry in the matrix product MM^*. Earlier, Carlitz [Car73; Theorem 5], and then later Al-Salam and Verma [AV84] had obtained bibasic matrix inversion results whose $p = q$ case is equivalent to Theorem 1.2. More recently, Gessel and Stanton [GS83; Theorem 1.2] proved several q-series identities using Theorem 1.2. Gasper

[Gas89] recently derived bibasic extensions and analogs of Theorem 1.2, and the earlier work of Carlitz, Al-Salam, and Verma. Bressoud [Bre83] has deduced an elegant extension of Theorem 1.2 for matrices $M_{a,b}$, with two free parameters, from the terminating very well-poised ${}_6\phi_5$ summation theorem. He proved that $M_{a,b}$ and $M_{b,a}$ are inverse, infinite, lower-triangular matrices. All of this work, as well as [AAB87, And79], provides a natural setting for Theorem 1.2.

Equation (1.3) motivates the definition of the A_1 Bailey Pair.

Definition 1.5 (A_1 Bailey Pair). Let $n \geq 0$ and $y \geq 0$ be integers and $\alpha = \{\alpha_y\}$ and $\beta = \{\beta_y\}$ be sequences. Let M and M^* be as in (1.3). Then we say that α and β form an A_1 Bailey Pair if

$$\beta_n = \sum_{0 \leq y \leq n} M(n;\, y;\, A_1)\, \alpha_y, \tag{1.6}$$

for all $n \geq 0$.

The study of A_1 Bailey Pairs $\{\alpha_n, \beta_n\}$ satisfying (1.6) goes back to L. J. Rogers' [Rog94, Rog17] proofs of the Rogers-Ramanujan-Schur identities, and more recently to L. J. Slater [Sla51–Sla66], D. M. Bressoud [Bre81], and G. E. Andrews [And84].

Equation (1.4) and Definition 1.5 immediately give

Corollary 1.7 (A_1 Bailey Pair Inversion). *α and β satisfy equation* (1.6) *if and only if*

$$\alpha_n = \sum_{0 \leq y \leq n} M^*(n;\, y;\, A_1)\, \beta_y. \tag{1.8}$$

Corollary 1.7 is responsible for the dual pairs of identities in [And79, And86a, GS86]. For example, with suitable α_n and β_n, it follows that (1.6) and (1.8) correspond to Rogers' [Rog95] terminating very well-poised ${}_6\phi_5$ summation [Bai35, GR90], and Jackson's [Jack10] terminating balanced ${}_3\phi_2$ summation [Bai35, GR90], respectively.

Andrews' explicit formulation of the Bailey Lemma is provided by

Theorem 1.9 (Classical Bailey Lemma for A_1). *Let the sequences $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ form an A_1 Bailey Pair. If $\alpha' = \{\alpha'_n\}$ and $\beta' = \{\beta'_n\}$ are defined by,*

$$\alpha'_n := \frac{(\rho)_n\,(\sigma)_n}{(aq/\rho)_n\,(aq/\sigma)_n}\,(aq/\rho\sigma)^n\,\alpha_n \tag{1.10a}$$

and

$$\beta'_n := \sum_{0 \leq y \leq n} \frac{(\rho)_y\,(\sigma)_y\,(aq/\rho\sigma)_{n-y}}{(q)_{n-y}\,(aq/\rho)_n\,(aq/\sigma)_n}\,(aq/\rho\sigma)^y\,\beta_y, \tag{1.10b}$$

then α' and β' also form an A_1 Bailey Pair.

Andrews notes in [And84] that Watson's [Wat29] q-analog of Whipple's transformation is an immediate consequence of the second iteration of Theorem 1.9, starting from one of the simplest A_1 Bailey Pairs. In fact, Andrews' infinite family of extensions of Watson's q-Whipple's transformation in [And75] is just a consequence of continued iteration of this same case of Theorem 1.9. Even Whipple's original work in [Whi24, Whi26] fits into the $q = 1$ case of this analysis. Paule [Pau82, Pau85] independently discovered important special cases of Theorem 1.9 and observed how these results could be iterated. Essentially all the depth of the classical Rogers-Ramanujan-Schur identitites and their iterations is embedded in the A_1 Bailey Lemma.

We organize the rest of this talk as follows. Let G denote A_ℓ or C_ℓ. In §2 we state the G terminating very well-poised ${}_6\phi_5$ summations from [LM91, Mil87, Mil91a] which we need in our subsequent work. We indicate in §3 how the G Bailey Transform of [LM91, Mil91a] is obtained from a suitably modified G terminating very well-poised ${}_4\phi_3$ summation theorem and termwise transformations. It is then interpreted as a matrix inversion result for two infinite, lower-triangular matrices. This provides a higher-dimensional generalization of Theorem 1.2. As in Definition 1.5 and Corollary 1.7, the concept of a G Bailey Pair is introduced, and then inverted. This G inversion applied to the G terminating very well-poised ${}_6\phi_5$ summations from §2 yields the G terminating balanced ${}_3\phi_2$ summations in §4. This is just a sample of the new A_ℓ terminating balanced ${}_3\phi_2$ summations from [Mil91a]. We describe in §5 how the G Bailey Lemma from [LM91, Mil91c] is obtained directly from a G terminating very well-poised ${}_6\phi_5$ summation theorem and the matrix inversion formulation of the G Bailey Transform. It shows how to construct another G Bailey Pair from an arbitrary G Bailey Pair, and thus extends Theorem 1.9. The concepts of an ordinary G Bailey Chain and a bilateral G Bailey Chain are introduced. Finally, appealing to the second iterate of the G Bailey Lemma, if time permitts, we will state , as an example, one A_ℓ and one C_ℓ q-Whipple transformation. These examples will appear in §6 of our longer paper based on this talk. Several A_ℓ q-Whipple transformations, including this one, are derived in [Mil91b-c]. Many other consequences of the G Bailey Transform and Lemma appear in [Lil91, LM91, Mil91a-f, ML91].

2. Background Information

The main results in this talk depend upon an A_ℓ and a C_ℓ terminating very well-poised ${}_6\phi_5$ summation theorem from [Mil85, Mil87, Mil91a] and [Gus89, LM91], respectively. Here, we state these two ${}_6\phi_5$ summations in a form convenient for our applications. The $\ell = 1$ case of each is the classical terminating ${}_6\phi_5$ summation in equation (II.21) of [GR90; pp. 238].

We start with

Theorem 2.1 (An A_ℓ terminating ${}_6\phi_5$ summation theorem). *Let a, b, c and $x_1, \dots, x_\ell$ be indeterminate, let N_i be non-negative integers for $i = 1, 2, \dots, \ell$ with $\ell \geq 1$, and suppose that none of the denominators in* (2.2) *vanishes. Then*

$$\left\{ \frac{(aq/bc)_{N_1+\cdots+N_\ell}}{(aq/b)_{N_1+\cdots+N_\ell}} \prod_{k=1}^{\ell} \frac{\left(\frac{x_k}{x_\ell} aq\right)_{N_k}}{\left(\frac{x_k}{x_\ell} aq/c\right)_{N_k}} \right\} \tag{2.2a}$$

$$= \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\dots,\ell}} \Bigg\{ \prod_{1\le r<s\le \ell}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{k=1}^{\ell}\left[\frac{1-\frac{x_k}{x_\ell}aq^{y_k+(y_1+\cdots+y_\ell)}}{1-\frac{x_k}{x_\ell}a}\right]$$

$$\times \prod_{r,s=1}^{\ell}\left[\frac{\left(\frac{x_r}{x_s}q^{-N_s}\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right] \prod_{k=1}^{\ell}\left[\frac{\left(\frac{x_k}{x_\ell}a\right)_{y_1+\cdots+y_\ell}}{\left(\frac{x_k}{x_\ell}aq^{1+N_k}\right)_{y_1+\cdots+y_\ell}}\right]$$

$$\times \frac{(c)_{y_1+\cdots+y_\ell}}{(aq/b)_{y_1+\cdots+y_\ell}} \prod_{k=1}^{\ell}\left[\frac{\left(\frac{x_k}{x_\ell}b\right)_{y_k}}{\left(\frac{x_k}{x_\ell}aq/c\right)_{y_k}}\right]$$

$$\times \left[\left(\frac{aq^{1+(N_1+\cdots+N_\ell)}}{bc}\right)^{y_1+\cdots+y_\ell} q^{y_2+2y_3+\cdots+(\ell-1)y_\ell}\right]\Bigg\}. \tag{2.2b}$$

Proof. First, rewrite Theorem 1.38 of [Mil87] by replacing n by $\ell+1$, making the substitutions

$$a_{\ell+1,\ell+1} = b/a, \quad z_\ell/z_{\ell+1} = a, \tag{2.3a}$$

and then taking $m = N$, and

$$a_{ii} = c_i \quad \text{and} \quad z_i = x_i, \qquad \text{for } i = 1,2,\dots,\ell. \tag{2.3b}$$

By the

$$c_i = q^{-N_i}, \qquad \text{for } i = 1,2,\dots,\ell, \tag{2.4}$$

case of this result and an elementary calculation involving its product side, it follows that the identity (2.2) holds for $c = q^{-N}$, with N any non-negative integer. However, (2.2) is a polynomial identity in c^{-1}, whose degree is a finite function of $\{N_1,\dots,N_\ell\}$. Hence, Theorem 2.1 is true in general. □

Remark. This is the proof of Theorems 2.1 and 2.4, respectively, in [Mil91a], with n replaced by ℓ. The paper [Mil91a] contains three additional A_ℓ terminating very well-poised ${}_6\phi_5$ summation theorems.

Remark. The $\ell = 1$ and $N_1 = n$ case of (2.2) is equation (II.21) of [GR90; pp. 238].

Next, Gustafson's C_ℓ ${}_6\psi_6$ summation theorem from [Gus89] leads in [LM91] to

Theorem 2.5. (The C_ℓ terminating ${}_6\phi_5$ summation theorem). *Let a, b and $x_1,\dots,x_\ell$ be indeterminate, let N_i be non-negative integers for $i = 1,2,\dots,\ell$ with $\ell \ge 1$, and suppose that none of the denominators in* (2.6) *vanishes. Then*

$$\sum_{\substack{0\le y_k\le N_k\\ k=1,2,\dots,\ell}} \Bigg\{ \prod_{k=1}^{\ell}\left[\frac{1-x_k^2q^{2y_k}}{1-x_k^2}\right] \prod_{1\le r<s\le\ell}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\,\frac{1-x_rx_sq^{y_r+y_s}}{1-x_rx_s}\right]$$

$$\times \prod_{r,s=1}^{\ell} \left[\frac{\left(\frac{x_r}{x_s} q^{-N_s}\right)_{y_r} (x_r x_s)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r} (q x_r x_s q^{N_s})_{y_r}} \right] \prod_{k=1}^{\ell} \left[\frac{(a x_k)_{y_k} (q x_k b^{-1})_{y_k}}{(b x_k)_{y_k} (q x_k a^{-1})_{y_k}} \right]$$

(2.6a)
$$\left. \times\, q^{(N_1+\cdots+N_\ell)(y_1+\cdots+y_\ell)}\, q^{y_2+2y_3+\cdots+(\ell-1)y_\ell} \left(\frac{b}{a}\right)^{y_1+\cdots+y_\ell} \right\}$$

$$= \prod_{k=1}^{\ell} \left[\frac{(q x_k^2)_{N_k}}{(b x_k)_{N_k} (q a^{-1} x_k)_{N_k}} \right] \prod_{1 \le r < s \le \ell} \left[\frac{(q x_r x_s)_{N_r}}{(q x_r x_s q^{N_s})_{N_r}} \right]$$

(2.6b)
$$\times \left(\frac{b}{a}\right)_{N_1+\cdots+N_\ell}.$$

Proof. We begin with Gustafson's C_ℓ ${}_6\psi_6$ summation theorem from [Gus89 ;Theorem 5.1]. Specializations serve to terminate this summation theorem from below and then from above. This yields the C_ℓ ${}_6\phi_5$ summation theorem , and then the C_ℓ terminating ${}_6\phi_5$ summation in Theorem 2.5, respectively.

Before carrying out the above specializations, we first make the following substitutions in Gustafson's C_ℓ ${}_6\psi_6$ summation theorem:

$$\begin{aligned} & a_i \mapsto a_i q^{-z_i}, && \text{for } i = 1,2,\ldots,\ell; \\ & a_{\ell+1} \mapsto a; \\ & b_i \mapsto b_i q^{-z_i}, && \text{for } i = 1,2,\ldots,\ell; \\ & b_{\ell+1} \mapsto b. \end{aligned} \tag{2.7}$$

Now set $b_1 = b_2 = \cdots = b_\ell = q$ in the resulting multiple Laurent series identity to terminate the sum side from below. Next, take $a_i = q^{-N_i}$ for $i = 1,2,\ldots \ell$, where each N_i is a non-negative integer. This terminates the sum side from above, and gives a summation theorem for a terminating multiple power series.

We then obtain Theorem 2.5 by first making the substitution $x_k = q^{z_k}$, for $k = 1,2,\ldots,\ell$, and then using $(a)_n = (a)_\infty / (aq^n)_\infty$ and $(a)_{-n} = (-q/a)^n q^{\binom{n}{2}} (q/a)_n^{-1}$ to simplify the product and sum side, respectively. □

Remark. A summary of the above substitutions that transform Gustafson's C_ℓ ${}_6\psi_6$ into Theorem 2.5 is given by:

$$\begin{aligned} & a_i \mapsto a_i q^{-z_i} \mapsto q^{-N_i} q^{-z_i} \mapsto q^{-N_i} x_i^{-1}, && \text{for } i = 1,2,\ldots,\ell; \\ & a_{\ell+1} \mapsto a; \\ & b_i \mapsto b_i q^{-z_i} \mapsto q^{1-z_i} \mapsto q x_i^{-1}, && \text{for } i = 1,2,\ldots,\ell; \\ & b_{\ell+1} \mapsto b; \\ & q^{z_i} \mapsto x_i. \end{aligned} \tag{2.8}$$

Remark. The $\ell = 1$ case of (2.6) is the classical terminating ${}_6\phi_5$ summation in equation (II.21) of [GR90; pp. 238] in which $a \mapsto x_1^2$, $n \mapsto N_1$, $b \mapsto a x_1$, $c \mapsto q x_1 b^{-1}$. That is, they are equivalent.

See §2 of [LM91] for the detailed proof of Theorem 2.5.

3. The G Bailey Transform

In this section we discuss the A_ℓ and C_ℓ multivariable extension of the classical A_1 Bailey Transform in Theorem 1.2. Motivated by Andrews [And79], Gessel and Stanton [GS83, GS86], and Agarwal, Andrews and Bressoud [AAB87] we generalize the matrix inversion formulation. This requires matrices M and M^* whose rows and columns are indexed by vectors of length ℓ of non-negative integers.

Throughout this talk, let $\boldsymbol{i} := (i_1,\ldots,i_\ell)$, $\boldsymbol{j} := (j_1,\ldots,j_\ell)$, $\boldsymbol{N} := (N_1,\ldots,N_\ell)$, and $\boldsymbol{y} := (y_1,\ldots,y_\ell)$ be vectors of length ℓ with non-negative integer components.

Define the Bailey transform matrices, M and M^*, as follows.

Definition 3.1 (M and M^* for A_ℓ). Let $a, x_1,\ldots,x_\ell$ be indeterminate. Suppose that none of the denominators in (3.2) vanishes. Then let

$$M(\boldsymbol{i};\, \boldsymbol{j};\, A_\ell) := \prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\ \prod_{k=1}^{\ell}\left(aq\frac{x_k}{x_\ell}\right)^{-1}_{i_k+(j_1+\cdots+j_\ell)}; \tag{3.2a}$$

and

$$\begin{aligned} M^*(\boldsymbol{i};\, \boldsymbol{j};\, A_\ell) := &\prod_{k=1}^{\ell}\left[1-a\frac{x_k}{x_\ell}q^{i_k+(i_1+\cdots+i_\ell)}\right]\prod_{k=1}^{\ell}\left(aq\frac{x_k}{x_\ell}\right)_{j_k+(i_1+\cdots+i_\ell)-1} \\ &\times\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\ (-1)^{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}\ q^{\binom{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}{2}}. \end{aligned} \tag{3.2b}$$

Definition 3.3 (M and M^* for C_ℓ). Let $x_1,\ldots,x_\ell$ be indeterminate. Suppose that none of the denominators in (3.4) vanishes. Then let

$$M(\boldsymbol{i};\, \boldsymbol{j};\, C_\ell) := \prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\ (qx_rx_sq^{j_r+j_s})^{-1}_{i_r-j_r}\right]; \tag{3.4a}$$

and

$$\begin{aligned} M^*(\boldsymbol{i};\, \boldsymbol{j};\, C_\ell) := &\prod_{r,s=1}^{\ell}\left[\left(q\frac{x_r}{x_s}q^{j_r-j_s}\right)^{-1}_{i_r-j_r}\ (x_rx_sq^{j_r+i_s})^{-1}_{i_r-j_r}\right] \\ &\times\prod_{1\le r<s\le\ell}\left[\frac{1-x_rx_sq^{j_r+j_s}}{1-x_rx_sq^{i_r+i_s}}\right]\ (-1)^{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}\ q^{\binom{(i_1+\cdots+i_\ell)-(j_1+\cdots+j_\ell)}{2}}. \end{aligned} \tag{3.4b}$$

Remark. The $\ell = 1$ case of (3.2) is the matrices in (1.3), and the $\ell = 1$ case of (3.4) is entrywise different than (1.3), but equivalent to it.

As in the classical case [AAB87], termwise transformations of a suitably modified A_ℓ or C_ℓ terminating very well-poised ${}_4\phi_3$ summation theorem lead to

Theorem 3.5 (Bailey Transform for A_ℓ and C_ℓ). *Let $G = A_\ell$ or C_ℓ. Let M and M^* be defined as in (3.2) and (3.4), with rows and columns ordered lexicographically. Then M and M^* are inverse, infinite, lower-triangular matrices. That is,*

$$\prod_{k=1}^{\ell} \delta(i_k, j_k) = \sum_{\substack{j_k \leq y_k \leq i_k \\ k=1,2,\ldots,\ell}} M(\mathbf{i};\, \mathbf{y};\, G)\, M^*(\mathbf{y};\, \mathbf{j};\, G), \tag{3.6}$$

where $\delta(r, s) = 1$ if $r = s$, and 0 otherwise.

Proof. In each case, A_ℓ and C_ℓ, we begin with a terminating very well-poised ${}_4\phi_3$ summation theorem. The A_ℓ ${}_4\phi_3$ summation follows immediately from the $b = aq/c$ case of Theorem 2.1 and the C_ℓ ${}_4\phi_3$ summation is similarly the $a = b$ case of Theorem 2.5.

We then multiply both the sum and product sides of the suitably specialized A_ℓ and C_ℓ terminating ${}_4\phi_3$ summations by some additional factors.

For A_ℓ, we multiply each side of the $N_k \mapsto i_k - j_k$, $\quad x_k \mapsto x_k q^{j_k}$, $\quad a \mapsto a q^{j_n + (j_1 + \cdots + j_\ell)}$ case of the A_ℓ terminating ${}_4\phi_3$ summation by the product

$$\prod_{r,s=1}^{\ell} \left(q\frac{x_r}{x_s} q^{j_r - j_s}\right)^{-1}_{i_r - j_r} \prod_{k=1}^{\ell} \left[\frac{\left(\frac{x_k}{x_\ell} aq\right)_{j_k + (j_1 + \cdots + j_\ell)}}{\left(\frac{x_k}{x_\ell} aq\right)_{i_k + (j_1 + \cdots + j_\ell)}}\right]. \tag{3.7}$$

For C_ℓ, we multiply each side of the $N_k \mapsto i_k - j_k$, $\quad x_k \mapsto x_k q^{j_k}$ case of the C_ℓ terminating ${}_4\phi_3$ summation by the product

$$\prod_{r,s=1}^{\ell} \left[\left(q\frac{x_r}{x_s} q^{j_r - j_s}\right)^{-1}_{i_r - j_r} \left(q x_r x_s q^{j_r + j_s}\right)^{-1}_{i_r - j_r}\right]. \tag{3.8}$$

In either case, the modified product side is seen to be the product of delta functions in the left-hand side of (3.6). The modified sum side is transformed term-by-term to yield the sum side of (3.6). The analysis here for the sum side consists of a lengthly series of elementary calculations. □

Remark. The detailed proof of the A_ℓ case of Theorem 3.5 is in §3 of [Mil91a], with ℓ replaced by n, and the above steps reversed into a verification proof. See §3 of [LM91] for the detailed analysis in the proof of the C_ℓ case.

Equations (3.2) and (3.4) motivate the definition of the A_ℓ and C_ℓ Bailey pair.

Definition 3.9 (G Bailey Pair). Let $G = A_\ell$ or C_ℓ. Let $N_k \geq 0$ be integers for $k = 1, 2, \ldots, \ell$. Let $A = \{A_{(\mathbf{y};\, G)}\}$ and $B = \{B_{(\mathbf{y};\, G)}\}$ be sequences. Let M and M^* be as above. Then we say that A and B form a G Bailey Pair if

$$B_{(\mathbf{N};\, G)} = \sum_{\substack{0 \leq y_k \leq N_k \\ k=1,2,\ldots,\ell}} M(\mathbf{N};\, \mathbf{y};\, G)\, A_{(\mathbf{y};\, G)}. \tag{3.10}$$

As a consequence of Theorem 3.5 and Definition 3.9 we have the following result.

Corollary 3.11 (G Bailey Pair Inversion). *A and B satisfy equation* (3.10) *if and only if*

$$A_{(N;\,G)} = \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} M^*(N;\,y;\,G)\,B_{(y;\,G)}. \tag{3.12}$$

We study an important application of Corollary 3.11 in the next section.

4. G Balanced ${}_3\phi_2$ Summation Theorems

Corollary 3.11 applied to the G Bailey Pairs $(A_{(y;\,G)}, B_{(y;\,G)})$ determined by Theorems 2.1 and 2.5 from §2 yields the corresponding G terminating balanced ${}_3\phi_2$ summations, and vice-versa. These calculations provide a G generalization of Andrews' application in [And79] of Corollary 1.7. The A_ℓ results here are contained in [Mil91a]. The $\ell = 1$ case of the summation theorems in this section are the corresponding classical results in [GR90].

In §4 of [Mil91a] we apply Corollary 3.11 to Theorem 2.1 to obtain

Theorem 4.1 (An A_ℓ generalization of the terminating balanced ${}_3\phi_2$ summation theorem). *Let a, b, c and $x_1, \ldots, x_\ell$ be indeterminate, let N_i be non-negative integers for $i = 1, 2, \ldots, \ell$ with $\ell \ge 1$, and suppose that none of the denominators in* (4.2) *vanishes. Then*

$$\left\{\frac{(c/a)_{N_1+\cdots+N_\ell}}{(c/ab)_{N_1+\cdots+N_\ell}} \prod_{k=1}^{\ell} \frac{\left(\frac{x_k}{x_\ell}c/b\right)_{N_k}}{\left(\frac{x_k}{x_\ell}c\right)_{N_k}}\right\} \tag{4.2a}$$

$$= \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} \left\{ \prod_{1\le r<s\le \ell} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \right.$$

$$\times \prod_{r,s=1}^{\ell} \left[\frac{\left(\frac{x_r}{x_s}q^{-N_s}\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\right] \prod_{k=1}^{\ell}\left[\frac{\left(\frac{x_k}{x_\ell}a\right)_{y_k}}{\left(\frac{x_k}{x_\ell}c\right)_{y_k}}\right]$$

$$\left. \times \left[\frac{(b)_{y_1+\cdots+y_\ell}}{\left((ab/c)q^{1-(N_1+\cdots+N_\ell)}\right)_{y_1+\cdots+y_\ell}}\, q^{y_1+2y_2+\cdots+\ell y_\ell}\right]\right\}. \tag{4.2b}$$

Proof. We begin by multiplying both sides of (2.2) by

$$\left\{\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}\right)_{N_r}^{-1} \prod_{k=1}^{\ell}\left(\frac{x_k}{x_\ell}aq\right)_{N_k}^{-1}\right\}, \tag{4.3}$$

and simplifying. By Definition 3.9, the product and sum sides of the resulting identity determine $B_{(N;\,G)}$ and $A_{(y;\,G)}$, respectively. Substitute this A_ℓ Bailey Pair into (3.12), simplify the resulting sum side termwise, and apply the relation $(a)_n = (-a)^n q^{\binom{n}{2}}\left(a^{-1}q^{1-n}\right)_n$

to suitable factors on the product side. Theorem 4.1 then follows once we make the substitutions $a \mapsto aq^{-(N_1+\cdots+N_\ell)}$, $c \mapsto (aq/c)q^{-(N_1+\cdots+N_\ell)}$, $b \mapsto c/b$, with x_i, N_i, q unchanged. □

Remark. The $\ell = 1$ and $N_1 = n$ case of (4.2) is equation (II.12) of [GR90; pp. 237].

We then went on in Theorem 4.15 of [Mil91a] to show that Theorem 4.1 and a polynomial argument lead to a summation theorem equivalent to (4.2) in which $b = q^{-N}$, $a = b$, and q^{-N_i} is replaced by a_i, for $i = 1, 2, \ldots, \ell$. The multiple sum in the second identity is taken over $y_1, \ldots, y_\ell \geq 0$ and $0 \leq y_1 + \cdots + y_\ell \leq N$, where N is a non-negative integer. The two identities are equivalent since the second one is a polynomial identity in each of a_i^{-1}, whose degree is a finite function of N, and (4.2) implies that the second holds for $a_i = q^{-N_i}$. Letting $N \to \infty$ in this second A_ℓ terminating balanced ${}_3\phi_2$ summation theorem then led in Theorem 5.1 of [Mil91a] to the A_ℓ Gauss summation theorem. This, in turn, yielded an A_ℓ q-Chu-Vandermonde summation and the non-terminating A_ℓ refinement of the q-binomial theorem. Many more analogous special limiting cases of additional A_ℓ terminating balanced ${}_3\phi_2$ summations can be found in §5 of [Mil91a].

We now consider the C_ℓ case. Applying Corollary 3.11 to Theorem 2.5 yields

Theorem 4.4.(A C_ℓ generalization of the terminating balanced ${}_3\phi_2$ summation theorem). *Let a, b and $x_1, \ldots, x_\ell$ be indeterminate, let N_i be non-negative integers for $i = 1, 2, \ldots, \ell$ with $\ell \geq 1$, and suppose that none of the denominators in (4.5) vanishes. Then*

$$\text{(4.5a)} \qquad \left\{\prod_{k=1}^{\ell}\left[\frac{(ax_k)_{N_k}\,(qx_kb^{-1})_{N_k}}{(bx_k)_{N_k}\,(qx_ka^{-1})_{N_k}}\right]\left(\frac{b}{a}\right)^{N_1+\cdots+N_\ell}\right\}$$

$$= \sum_{\substack{0\leq y_k\leq N_k\\ k=1,2,\ldots,\ell}}\left\{\prod_{1\leq r<s\leq\ell}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\,\frac{1-x_rx_sq^{y_r+y_s}}{1-x_rx_s}\right]\right.$$

$$\times\prod_{r,s=1}^{\ell}\left[\frac{\left(\frac{x_r}{x_s}q^{-N_s}\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}}\,\frac{\left(x_rx_sq^{N_s}\right)_{y_r}}{(qx_rx_s)_{y_r}}\right]$$

$$\times\prod_{1\leq r<s\leq\ell}\left[\frac{(qx_rx_s)_{y_r}}{(qx_rx_sq^{y_s})_{y_r}}\right]\prod_{k=1}^{\ell}\left[\frac{(qx_k^2)_{y_k}}{(bx_k)_{y_k}\,(qa^{-1}x_k)_{y_k}}\right]$$

$$\text{(4.5b)} \qquad \left.\times\left[\left(\frac{b}{a}\right)_{y_1+\cdots+y_\ell} q^{y_1+2y_2+\cdots+\ell y_\ell}\right]\right\}.$$

Proof. We begin by multiplying both sides of (2.6) by

$$\text{(4.6)} \qquad \left\{\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}\right)_{N_r}^{-1}\,(qx_rx_s)_{N_r}^{-1}\right\},$$

and simplifying. By Definition 3.9, the product and sum sides of the resulting identity determine $B_{(N;\,G)}$ and $A_{(y;\,G)}$, respectively. Substitute this C_ℓ Bailey Pair into (3.12), simplify the resulting sum side termwise, rewrite the product side, and then Theorem 4.4 follows. □

Remark. Note that there is some cancellation of factors in (4.5b). This allows us to write (4.5b) more compactly. In particular, the diagonal ($r = s$) factors in

$$\prod_{r,s=1}^{\ell} (qx_r x_s)_{y_r}^{-1} \qquad \text{cancell the factors} \qquad \prod_{k=1}^{\ell} (qx_k^2)_{y_k}.$$

Remark. The $\ell = 1$ case of (4.5) is the classical terminating balanced ${}_3\phi_2$ summation in equation (II.12) of [GR90; pp. 237] in which $n \mapsto N_1$, $a \mapsto x_1^2 q^{N_1}$, $b \mapsto b/a$, $c \mapsto bx_1$. That is, they are equivalent.

Just as in the A_ℓ case, Theorem 4.4 and a polynomial argument lead to a summation theorem equivalent to (4.5) in which $b = aq^{-N}$, $a = b$, and q^{-N_i} is replaced by a_i, for $i = 1, 2, \ldots, \ell$. The two identities are equivalent since the second one is a polynomial identity in each of a_i^{-1}, whose degree is a finite function of N, and (4.5) implies that the second holds for $a_i = q^{-N_i}$. That is, we have

Theorem 4.7.(Second C_ℓ generalization of the terminating balanced ${}_3\phi_2$ summation theorem). *Let $a_1, \ldots, a_\ell, b$ and $x_1, \ldots, x_\ell$ be indeterminate, let N be a nonnegative integer, let $\ell \geq 1$, and suppose that none of the denominators in* (4.8) *vanishes. Then*

$$\text{(4.8a)} \qquad \left\{ \prod_{k=1}^{\ell} \left[\frac{\left(qa_k x_k^{-1} b^{-1}\right)_N \left(qx_k a_k^{-1} b^{-1}\right)_N}{\left(qx_k^{-1} b^{-1}\right)_N \left(qx_k b^{-1}\right)_N} \right] \right\}$$

$$= \sum_{\substack{y_1, \ldots, y_\ell \geq 0 \\ 0 \leq y_1 + \cdots + y_\ell \leq N}} \left\{ \prod_{1 \leq r < s \leq \ell} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \, \frac{1 - x_r x_s q^{y_r + y_s}}{1 - x_r x_s} \right] \right.$$

$$\times \prod_{r,s=1}^{\ell} \left[\frac{\left(\frac{x_r}{x_s} a_s\right)_{y_r}}{\left(q\frac{x_r}{x_s}\right)_{y_r}} \, \frac{\left(x_r x_s a_s^{-1}\right)_{y_r}}{\left(qx_r x_s\right)_{y_r}} \right]$$

$$\times \prod_{1 \leq r < s \leq \ell} \left[\frac{(qx_r x_s)_{y_r}}{(qx_r x_s q^{y_s})_{y_r}} \right] \prod_{k=1}^{\ell} \left[\frac{(qx_k^2)_{y_k}}{(bx_k q^{-N})_{y_k} (qb^{-1} x_k)_{y_k}} \right]$$

$$\text{(4.8b)} \qquad \left. \times \left[\left(q^{-N}\right)_{y_1 + \cdots + y_\ell} q^{y_1 + 2y_2 + \cdots + \ell y_\ell} \right] \right\}.$$

Remark. Note that there is the same cancellation of factors in (4.8b) as there was in (4.5b).

Remark. The $\ell = 1$ case of (4.8) is the classical terminating balanced ${}_3\phi_2$ summation in equation (II.12) of [GR90; pp. 237] in which $n \mapsto N$, $a \mapsto a_1$, $b \mapsto x_1^2 a_1^{-1}$, $c \mapsto qx_1 b^{-1}$. That is, they are equivalent.

Letting $N \to \infty$ in Theorem 4.7 leads to the C_ℓ Gauss summation theorem. This, in turn, yields a C_ℓ q-Chu-Vandermonde summation and the non-terminating C_ℓ refinement of the q-binomial theorem. We include these results in our paper based on the longer version of this talk.

5. The G Bailey Lemma

In this section we motivate and then state the A_ℓ and C_ℓ generalization of the classical A_1 Bailey Lemma in Theorem 1.9. It shows how to construct another G Bailey Pair from an arbitrary G Bailey Pair.

Consider the sequence $A' = \{A'_{(N;G)}\}$ defined by

$$A'_{(N;G)} := C_N \; A_{(N;G)}, \tag{5.1}$$

where the sequence $C = \{C_y\}$ is as of yet unchosen, and A and B form a G Bailey Pair. We want to find a sequence $B' = \{B'_{(y;G)}\}$ so that A' and B' also form a G Bailey Pair. That is, we need

$$B'_{(N;G)} = \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\ldots,\ell}} M(N;\, y;\, G)\, A'_{(y;G)}. \tag{5.2}$$

Assume that (3.10), (3.12), (5.1), and (5.2) hold, and that $M(i;\, j;\, G) \equiv M(i;\, j)$ and $M^*(i;\, j;\, G) \equiv M^*(i;\, j)$. Then

$$B'_{(N;G)} = \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\ldots,\ell}} \{M(N;\, y)\, C_y\, A_{(y;G)}\} \tag{5.3a}$$

$$= \sum_{\substack{0 \le y_k \le N_k \\ k=1,2,\ldots,\ell}} \Big\{M(N;\, y)\, C_y \sum_{\substack{0 \le m_i \le y_i \\ i=1,2,\ldots,\ell}} [M^*(y;\, m)\, B_{(m;G)}]\Big\} \tag{5.3b}$$

$$= \sum_{\substack{0 \le m_k \le N_k \\ k=1,2,\ldots,\ell}} \Big\{B_{(m;G)} \sum_{\substack{m_i \le y_i \le N_i \\ i=1,2,\ldots,\ell}} [M(N;\, y)\, M^*(y;\, m)\, C_y]\Big\} \tag{5.3c}$$

$$= \sum_{\substack{0 \le m_k \le N_k \\ k=1,2,\ldots,\ell}} \Big\{B_{(m;G)} \sum_{\substack{0 \le y_i \le N_i - m_i \\ i=1,2,\ldots,\ell}} [M(N;\, y+m)\, M^*(y+m;\, m)\, C_{y+m}]\Big\}. \tag{5.3d}$$

We want to choose $C = \{C_y\}$ so that each C_{y+m} can be factored into a function that is independent of y times a function of m and y. The expression that is independent of y will then be pulled outside the sum. We also desire that the remaining terms combine with those in the inner sum of (5.3d) to form an easily summable expression. In effect, C allows us to pass from a G ${}_4\phi_3$ to a G ${}_6\phi_5$ which is summable by either Theorem 2.1 or 2.5. Such a choice of C allows us to sum the inner sum in (5.3d) and derive a more compact expression for $B'_{(N;G)}$.

Keeping in mind the above discussion, we first define the sequences $A' = \{A'_{(y;\,A_\ell)}\}$ and $B' = \{B'_{(y;\,A_\ell)}\}$ by

$$A'_{(N;\,A_\ell)} := \prod_{k=1}^{\ell}\left(\frac{aq}{\rho}\frac{x_k}{x_\ell}\right)^{-1}_{N_k} \prod_{k=1}^{\ell}\left(\sigma\frac{x_k}{x_\ell}\right)_{N_k} \times \frac{(\rho)_{N_1+\cdots+N_\ell}}{(aq/\sigma)_{N_1+\cdots+N_\ell}}\,(aq/\rho\sigma)^{N_1+\cdots+N_\ell}\,A_{(N;\,A_\ell)} \tag{5.4a}$$

and

$$B'_{(N;\,A_\ell)} := \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} \left\{\prod_{k=1}^{\ell}\left[\left(\sigma\frac{x_k}{x_\ell}\right)_{y_k}\left(\frac{aq}{\rho}\frac{x_k}{x_\ell}\right)^{-1}_{N_k}\right]\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{y_r-y_s}\right)^{-1}_{N_r-y_r} \times \frac{(aq/\rho\sigma)_{(N_1+\cdots+N_\ell)-(y_1+\cdots+y_\ell)}\,(\rho)_{y_1+\cdots+y_\ell}}{(aq/\sigma)_{N_1+\cdots+N_\ell}}\,(aq/\rho\sigma)^{y_1+\cdots+y_\ell}\,B_{(y;\,A_\ell)}\right\} \tag{5.4b}$$

We next define the sequences $A' = \{A'_{(y;\,C_\ell)}\}$ and $B' = \{B'_{(y;\,C_\ell)}\}$ by

$$A'_{(N;\,C_\ell)} := \prod_{k=1}^{\ell}\left[\frac{(\alpha x_k)_{N_k}\,(qx_k\beta^{-1})_{N_k}}{(\beta x_k)_{N_k}\,(qx_k\alpha^{-1})_{N_k}}\right]\left(\frac{\beta}{\alpha}\right)^{N_1+\cdots+N_\ell} A_{(N;\,C_\ell)} \tag{5.5a}$$

and

$$B'_{(N;\,C_\ell)} := \sum_{\substack{0\le y_k\le N_k\\ k=1,2,\ldots,\ell}} \left\{\prod_{k=1}^{\ell}\left[\frac{(\alpha x_k)_{y_k}\,(qx_k\beta^{-1})_{y_k}}{(\beta x_k)_{N_k}\,(qx_k\alpha^{-1})_{N_k}}\right]\prod_{r,s=1}^{\ell}\left(q\frac{x_r}{x_s}q^{y_r-y_s}\right)^{-1}_{N_r-y_r} \times \prod_{1\le r<s\le\ell}\left[\left(qx_rx_sq^{y_r+y_s}\right)^{-1}_{N_s-y_s}\left(qx_rx_sq^{N_s-y_s}\right)^{-1}_{N_r-y_r}\right] \times \left(\frac{\beta}{\alpha}\right)_{(N_1+\cdots+N_\ell)-(y_1+\cdots+y_\ell)}\left(\frac{\beta}{\alpha}\right)^{y_1+\cdots+y_\ell} B_{(y;\,C_\ell)}\right\} \tag{5.5b}$$

These definitions lead to

Theorem 5.6 (The G generalization of the classical A_1 Bailey Lemma). *Let $G = A_\ell$ or C_ℓ. Suppose $A = \{A_{(N;\,G)}\}$ and $B = \{B_{(N;\,G)}\}$ form a G Bailey Pair. If $A' = \{A'_{(N;\,G)}\}$ and $B' = \{B'_{(N;\,G)}\}$ are as above, then A' and B' also form a G Bailey Pair.*

Proof. The definitions in (5.4) and (5.5) are substituted into (3.10). After an interchange of summation, the inner sum is seen to be a special case of the appropriate ${}_6\phi_5$. The ${}_6\phi_5$ is then summed, and the desired result follows. □

Corollary 5.7. *With $A' = \{A'_{(y;\,G)}\}$ and $B' = \{B'_{(y;\,G)}\}$ defined as in Theorem* 5.6, *A' and B' satisfy equation* (3.12).

Notice that we may apply the G Bailey Lemma to the new G Bailey Pair A' and B'. Call the resulting G Bailey Pair (A'', B''). We may continue applying the G Bailey Lemma and create a sequence of G Bailey Pairs

$$(A,B) \to (A',B') \to (A'',B'') \to \cdots .$$

We call this sequence the "G Bailey Chain." This definition is motivated by Andrews [And86a].

We may also move from (A',B') back to (A,B). Given a G Bailey Pair (A',B'), we may determine A from equation (5.4a) or (5.5a) and then B from equation (3.10). Thus, we can move from right to left in the G Bailey Chain. This gives us the "bilateral G Bailey Chain"

$$\cdots \to (A^{(-2)},B^{(-2)}) \to (A^{(-1)},B^{(-1)}) \to (A,B) \to (A',B') \to (A'',B'') \to \cdots .$$

Many of the classical applications mentioned just after Theorem 1.9 extend to the setting of the above G Bailey Chains.

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