Weyl Group Symmetric Functions and the Representation Theory of Lie Algebras

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0. Introduction

In view of the many applications of the theory of symmetric functions to representation theory it seems desirable to have a theory of symmetric functions in the spirit of Macdonald's book [Mac] for Weyl groups other than the symmetric group. In spite of the feeling that finding suitable generalizations may seem to be an intractable problem I remain optimistic and want to study symmetric functions from the point of view of representation theory of Lie algebras whenever possible. We must keep an open mind and we must not expect our generalizations to have *all* the magical properties that abound in the classical case. The idea is to find an algebraic structure which motivates each statement in the classical symmetric function theory. If this algebraic notion occurs across the board then this should indicate what the proper generalization is for other types. Note that from this point of view there may be several useful generalizations of a given concept depending on what symmetric function are desirable.

The goal of this paper is to offer a suggestion for the analogue of the basis of homogeneous symmetric functions for Weyl group symmetric functions. In this case the definition is motivated by the theory of centralizer algebras. The idea motivating the generalization is that it is really the Frobenius image of the homogeneous symmetric function that is the useful object. It is clear from the double centralizer theory that an analogue of the Frobenius characteristic map is a feature of the double centralizer mechanism, see [R]. With this point of view one finds an analogue of the "Jacobi-Trudi" formula in the work of Verma [V], Zelevinsky [Z] and Goodman-Wallach [GW]. In this paper I simply offer a mechanism by which to transfer their results to Weyl group symmetric functions.

I would like to thank D.-N. Verma for so patiently explaining the many many things about representations of Lie algebras and their representations which he considered necessary for me to know. In particular, he showed me the representation theoretic result, Theorem (4.6) in this paper, which motivates the definition of the homogeneous symmetric functions. I would like to thank N. Wallach for further useful discussions with me on this topic. I would like to thank the Tata Institute of Fundamental Research for their hospitality during my visit.

I have tried to organize this paper to motivate the concepts of symmetric functions by facts from representation theory. My hope is that this may serve as an introduction to representation theory for algebraic combinatorists who do not already know the subject.

I begin with a brief resume of the classical symmetric function theory. Then in section 2 I try to copy this theory except in the context of a general Weyl group. The remaining sections are an attempt to explain the representation theoretic motivations behind these generalizations.

1. Classical symmetric functions

This section gives a brief summary of the classical symmetric function theory. See [Mac] Chapter 1 for a complete treatment.

Fix a positive integer n. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ of nonnegative integers. Let $\mathcal{P}$ denote the set of partitions. We have the following sequence of inclusions

$$\mathcal{P} \subset \mathbb{Z}^n \subset \mathbb{C}^n . \tag{1.1}$$

There is a partial ordering, the *dominance ordering*, on elements of $\mathbb{Z}^n$ given by

$$\gamma \geq \kappa \quad \text{if} \quad \gamma_1 + \gamma_2 + \cdots + \gamma_i \geq \kappa_1 + \kappa_2 + \cdots \kappa_i, \text{ for all } i. \tag{1.2}$$

Let S_n denote the *symmetric group.* The *sign* $\varepsilon(w)$ of a permutation $w \in S_n$ is the determinant of the corresponding permutation matrix. S_n acts on elements of $\mathbb{Z}^n$ by permuting the positions.

For each $1 \leq i < j \leq n$ the *raising operator* R_{ij} is the operator which acts on elements of $\mathbb{Z}^n$ by

$$R_{ij}(\gamma_1, \gamma_2, \ldots, \gamma_n) = (\gamma_1, \gamma_2, \ldots, \gamma_i + 1, \ldots, \gamma_j - 1, \ldots, \gamma_n). \tag{1.4}$$

Let $x_1, x_2, \ldots, x_n$ be commuting variables and for each $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^n$ define

$$x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}. \tag{1.5}$$

Define an action of S_n on monomials by

$$w x^\gamma = x^{w\gamma}. \tag{1.6}$$

The ring

$$\Lambda_n = \mathbb{Q}\,[x_1, x_2, \ldots, x_n]^{S_n}$$

is the ring of symmetric functions.

Bases of symmetric functions

For each partition λ define the *monomial symmetric function* by

$$m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma, \tag{1.7}$$

where the sum runs over all $\gamma \in \mathbb{Z}^n$ in the S_n orbit of λ, i.e., over all distinct permutations of λ.

The *Schur functions* are given by

$$s_\lambda = \frac{\sum_{w\in S_n} \varepsilon(w)x^{w(\lambda+\rho)}}{\sum_{w\in S_n} \varepsilon(w)x^{w\rho}}, \tag{1.8}$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$.

The *elementary symmetric functions* are given by defining

$$e_0 = 1,$$
$$e_r = \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}x_{i_2}\cdots x_{i_r},$$

for each positive integer r, and

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_n}, \tag{1.9}$$

for all partitions λ.

The *homogeneous symmetric functions* are given by defining

$$h_0 = 1,$$
$$h_r = \sum_{1\le i_1\le\cdots\le i_r\le n} x_{i_1}x_{i_2}\cdots x_{i_r},$$

for each positive integer r, and

$$h_\gamma = h_{\gamma_1}h_{\gamma_2}\cdots h_{\gamma_n},$$

for all sequences $\gamma \in \mathbb{Z}^n$.

Define integers $K_{\lambda\mu}$ by

$$s_\lambda = \sum_{\mu\in\mathcal{P}} K_{\lambda\mu}m_\mu. \tag{1.10}$$

One has the following (nontrivial) facts:
(a) The $K_{\lambda\mu}$ are nonnegative integers.
(b) $K_{\lambda\lambda} = 1$ for all $\lambda \in \mathcal{P}$.
(c) $K_{\lambda\mu} = 0$ if $\mu \not\le \lambda$.

Each of the sets $\{m_\lambda\}_{\lambda\in\mathcal{P}}$, $\{s_\lambda\}_{\lambda\in\mathcal{P}}$, $\{e_\lambda\}_{\lambda\in\mathcal{P}}$, $\{h_\lambda\}_{\lambda\in\mathcal{P}}$, forms a $\mathbb{Z}$-basis of Λ_n.

Inner product

There is an inner product on the ring of symmetric functions given by making the Schur functions orthonormal,

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}. \tag{1.11}$$

Further facts

The homogeneous symmetric functions are the dual basis to the basis of monomial symmetric functions,

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}. \tag{1.12}$$

A consequence of this is that

$$h_\mu = \sum_\lambda s_\lambda K_{\lambda\mu}. \tag{1.13}$$

One has the following "Jacobi-Trudi" formula for the Schur functions in terms of the homogeneous symmetric functions,

$$s_\lambda = \sum_{w \in S_n} \varepsilon(w) h_{\lambda+\rho-w\rho}. \tag{1.14}$$

There is also a formula for the Schur function in terms of raising operators and the homogeneous symmetric function.

$$s_\lambda = \prod_{i<j}(1 - R_{ij}) h_\lambda. \tag{1.15}$$

2. Weyl group symmetric functions

Each of the simple root systems is determined by a *Cartan matrix* C. A list of the Cartan matrices for simple root systems can be found in [Bou] p. 250-258. We shall denote the (i,j) entry of the Cartan matrix by $\langle \alpha_i, \alpha_j \rangle$ so that

$$C = (\langle \alpha_i, \alpha_j \rangle).$$

Let n be the dimension of the Cartan matrix.

Let $\omega_1, \omega_2, \ldots, \omega_n$ be basis vectors in a vector space. Define

$$\mathfrak{H}^* = \sum_i \mathbb{C}\, \omega_i, \quad P = \sum_i \mathbb{Z}\, \omega_i, \quad P^+ = \sum_i \mathbb{N}\, \omega_i,$$

where $\mathbb{N}$ denotes the nonnegative integers. The elements of $\mathfrak{H}^*, P$, and P^+ are called the *weights*, the *integral weights*, and the *dominant integral weights*, respectively. The ω_i are called the *fundamental weights*. We have the following sequence of inclusions

$$P^+ \subseteq P \subseteq \mathfrak{H}^*. \tag{2.1}$$

Let $\gamma = \sum_i \gamma_i \omega_i$ be an element of P. We shall use the notation $\langle \gamma, \alpha_i \rangle$ for the integer γ_i. The *simple roots* α_i are given in terms of the entries of the Cartan matrix,

$$\alpha_i = \sum_j \langle \alpha_i, \alpha_j \rangle \alpha_j.$$

There is a partial ordering on the weight lattice given by

$$\gamma \geq \kappa \quad \text{if} \quad \kappa = \gamma - \sum k_i \alpha_i, \tag{2.2}$$

for nonnegative integers k_i. We say that $\gamma \geq \kappa$ in *dominance.*

Define linear operators $s_i \colon P \to P$ by

$$s_i \gamma = \gamma - \langle \gamma, \alpha_i \rangle \alpha_i.$$

The *Weyl group* is the group generated by the s_i: $W = < s_1, s_2, \ldots, s_n >$. The *sign* of an element $w \in W$ is $\varepsilon(w) = (-1)^p$, where p is the smallest nonnegative integer such that there exists an expression $s_{i_1} s_{i_2} \cdots s_{i_p} = w$. We will need the following proposition, see [Bou] Ch. 6, §1 Thm. 2.

(2.3) Proposition.
(a) Every Weyl group orbit $W\gamma$, $\gamma \in P^+$ contains a unique element in P^+.
(b) If $\lambda, \mu \in P^+$ and $\rho = \sum_i \omega_i$ then, for $v, w \in W$,

$$w(\lambda + \rho) = v(\mu + \rho) \iff v = w.$$

$\alpha \in P$ is a root if $\alpha = w\alpha_i$ for some $w \in W$ and simple root α_i. Let Φ be the set of roots and let $\Phi^+ = \{\alpha \in \Phi | \alpha > 0\}$ and $\Phi^- = \{\alpha \in \Phi | \alpha < 0\}$ where the ordering is as in (2.2). It is true that $\Phi = \Phi^+ \cup \Phi^-$. The elements of Φ^+ and Φ^- are called positive and negative roots respectively. The *raising operator* R_α associated to a positive root α is the operator which acts on elements of P by

$$R_\alpha \gamma = \gamma + \alpha. \tag{2.4}$$

Corresponding to each $\lambda \in P$ we write, formally, e^λ so that

$$e^\lambda e^\mu = e^{\lambda + \mu}.$$

In particular if $\lambda = \sum_i \lambda_i \omega_i$ then

$$\begin{aligned} e^\lambda &= e^{\lambda_1 \omega_1} e^{\lambda_2 \omega_2} \cdots e^{\lambda_n \omega_n} \\ &= (e^{\omega_1})^{\lambda_1} (e^{\omega_2})^{\lambda_2} \cdots (e^{\omega_n})^{\lambda_n}. \end{aligned} \tag{2.5}$$

(If one finds this "exponential" notation unsettling one can substitute z_i for e^{ω_i} and write $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}$ instead of e^λ.) Define an action of the Weyl group by

$$w e^\lambda = e^{w\lambda}, \tag{2.6}$$

for each $w \in W$ and $\lambda \in P$. Define

$$A^W = \mathbb{Z}\, [e^{\omega_1}, e^{-\omega_1}, \cdots, e^{\omega_n}, e^{-\omega_n}]^W.$$

Bases of A^W

For each $\lambda \in P^+$ define the *orbit sum*, or *monomial symmetric function*, by

$$m_\lambda = \sum_{\nu \in W\lambda} e^\nu. \tag{2.7}$$

For each $\lambda \in P^+$ define the *Weyl character* by

$$\chi^\lambda = \frac{\sum_{w \in W} \varepsilon(w) x^{w(\lambda+\rho)}}{\sum_{w \in W} \varepsilon(w) x^{w\rho}}, \tag{2.8}$$

where $\rho = \sum_i \omega_i$.

The *elementary, or fundamental, symmetric functions* are given by defining

$$e_0 = 1,$$
$$e_r = \chi^{\omega_r},$$

for each positive integer r, and

$$e_\lambda = e_1^{\lambda_1} e_2^{\lambda_2} \cdots e_n^{\lambda_n}, \tag{2.9}$$

for all elements $\lambda = \sum_i \lambda_i \omega_i$ in P^+.

Define integers $K_{\lambda\mu}$ by the identity

$$\chi^\lambda = \sum_{\mu \in P^+} K_{\lambda\mu} m_\mu. \tag{2.10}$$

It is true that

(a) The $K_{\lambda\mu}$ are nonnegative integers.
(b) $K_{\lambda\lambda} = 1$ for all $\lambda \in P^+$.
(c) $K_{\lambda\mu} = 0$ if $\mu \not\leq \lambda$.

All of these facts follow from representation theory see §3 (3.5). I know of no easy way to prove these results without using representation theory.

Each of the sets

$$\{m_\lambda\}_{\lambda \in P^+},$$
$$\{\chi_\lambda\}_{\lambda \in P^+},$$
$$\{e_\lambda\}_{\lambda \in P^+},$$

forms a $\mathbb{Z}$-basis of A^W. The fact that the m_λ form a $\mathbb{Z}$ basis of A^W follows immediately from (2.3). One can show by elementary techniques and without using representation theory, see [Bou] Ch. 6, §3, that the χ^λ, $\lambda \in P^+$ form a $\mathbb{Z}$ basis of A^W. This fact also follows from the facts about the numbers $K_{\lambda\mu}$ above. Assuming that the χ^λ form a $\mathbb{Z}$-basis of A^W it follows that the e_λ, $\lambda \in P^+$ form a $\mathbb{Z}$-basis of A^W simply by expanding e_λ in terms of e^μ, $\mu \in P^+$. One can also obtain this result in a different fashion by using representation theory.

Inner product

Let

$$d = \sum_{w \in W} \varepsilon(w) e^{w\rho},$$

where $\rho = \sum_i \omega_i$. If $f = \sum_{\nu \in P} f_\nu e^\nu$ then define $\bar{f} = \sum_\nu f_\nu e^{-\nu}$. Let $[f]_1$ denote taking the coefficient of the identity, e^0, in f. Then define

$$\langle f, g \rangle = \frac{1}{|W|}[f d \bar{g} \bar{d}]_1,$$

where $|W|$ is the order of the Weyl group.

(2.11) Proposition. ([Mac2]) *The inner product defined above satisfies*

$$\langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu}.$$

Proof. Since $\chi^\lambda = d^{-1} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}$,

$$\langle \chi^\lambda, \chi^\mu \rangle = \frac{1}{|W|} \sum_{v,w \in W} \varepsilon(v)\varepsilon(w)[e^{v(\lambda+\rho)} e^{-w(\mu+\rho)}]_1.$$

This is zero if $\lambda \neq \mu$, because, (2.3), the orbits $W(\lambda+\rho)$ and $W(\mu+\rho)$ do not intersect. If $\lambda = \mu$, then $v(\lambda+\rho) = w(\lambda+\rho) \Leftrightarrow v = w$. Thus $\langle \chi^\lambda, \chi^\lambda \rangle = \frac{1}{|W|} \sum_{w \in W} 1 = 1$. □

If one prefers one may simply *define* the inner product by making the Weyl characters orthonormal.

Homogeneous symmetric functions

Let $\kappa \in P^+$ and define

$$\Gamma_\kappa = \{\mu \in P^+ | \mu \leq \kappa\}, \text{ and } \Lambda_\kappa = \text{ span}\{\chi^\mu | \mu \in \Gamma_\kappa\}.$$

Since Γ_κ is always finite Λ_κ is always finite dimensional.

Define an inner product on Λ_κ by defining

$$\langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu}$$

for all $\lambda, \mu \in \Gamma_\kappa$. Then define the homogeneous symmetric functions h_λ, $\lambda \in \Gamma_\kappa$ to be the dual basis to the monomial symmetric functions,

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}. \tag{2.12}$$

Using the integers $K_{\lambda\mu}$ defined in (2.10), the homogeneous symmetric functions are given in terms of the Weyl characters by

$$h_\mu = \sum_{\lambda \in \Gamma_\kappa} \chi^\lambda K_{\lambda\mu}, \tag{2.13}$$

for all $\mu \in \Gamma_\kappa$. The h_μ, $\mu \in \Gamma_\kappa$ form a basis of Λ_κ.

Each of the sets

$$\{m_\lambda\}_{\lambda\in P^+},$$
$$\{\chi_\lambda\}_{\lambda\in P^+},$$
$$\{e_\lambda\}_{\lambda\in P^+},$$
$$\{h_\lambda\}_{\lambda\in P^+},$$

forms a $\mathbb{Z}$-basis of Λ_κ. To see this choose some total ordering of the elements of Γ_κ which is a refinement of the dominance partial order. Then, by (2.10a-c), the matrix, with rows and columns indexed by elements of Γ_κ, having $K_{\lambda\nu}$ as the λ, ν entry is upper unitriangular with nonnegative integer entries. This implies that it is invertible as a matrix with integer entries. The fact that the χ^μ are a basis of Λ_κ is by definition. The other two statements now follow from (2.10) and (2.13).

"Jacobi-Trudi" formulas

Fix $\kappa \in P^+$. Define

$$h_{w\lambda} = h_\lambda$$

for all $\lambda \in \Gamma_\kappa$ and all $w \in W$ so that h_λ is defined for all $\lambda \in W\Gamma_\kappa$. One has the following "Jacobi-Trudi" type identity for the Weyl characters in terms of the h_λ.

(2.14) Theorem. *Let $\rho = \sum_i \omega_i$. Then for each $\lambda \in \Gamma_\kappa$*

$$\chi^\lambda = \sum_{w\in W} \varepsilon(w) h_{\lambda+\rho-w\rho}.$$

Proof. We show that elements $\sum_{w\in W} \varepsilon(w) h_{\lambda+\rho-w\rho}$ are the dual basis to the basis χ^μ, $\mu \in \Gamma_\kappa$.

$$\chi^\lambda = \sum_{\mu\in W\Gamma_\kappa} \langle \chi^\lambda, h_\mu\rangle e^\mu.$$

Expanding χ^λ by (2.8) and clearing denominators we have that

$$\begin{aligned}\sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} &= \Big(\sum_{w\in W} \varepsilon(w) e^{w\rho-\rho}\Big)\Big(\sum_{\mu\in W\Gamma_\kappa} \langle \chi^\lambda, h_\mu\rangle e^\mu\Big)\\ &= \sum_{\mu\in W\Gamma_\kappa}\sum_{w\in W} \varepsilon(w)\langle \chi^\lambda, h_\mu\rangle e^{\mu+w\rho-\rho}\end{aligned}$$

Substitute $\gamma = \mu + w\rho - \rho$ to get

$$\sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} = \sum_{\mu\in W\Gamma_\kappa} \langle \chi^\lambda, \sum_{w\in W} \varepsilon(w) h_{\gamma+\rho-w\rho}\rangle e^\gamma.$$

Compare coefficients of e^γ for $\gamma \in P^+$ on each side of this equation. Since $\lambda \in P^+$ we know by (2.3) that $w(\lambda+\rho)-\rho$ is not an element of P^+ for any $w \in W$ except the identity. Thus we know that if $\mu \in P^+$ then

$$\langle \chi^\lambda, \sum_{w\in W} \varepsilon(w) h_{\mu+\rho-w\rho}\rangle = \delta_{\lambda\mu}. \qquad \square$$

Recall that raising operators act on elements of P. We allow the raising operators to act upon the h_λ by defining

$$R(h_\lambda) = h_{R(\lambda)},$$

for each sequence $R = R_{\beta_1} R_{\beta_2} \cdots R_{\beta_k}$. (Note: It is important to keep in mind that raising operators act on elements of P and not on symmetric functions.)

(2.15) Corollary. *For all $\lambda \in \Gamma_\kappa$,*

$$\chi^\lambda = \prod_{\alpha>0} (1 - R_\alpha) h_\lambda.$$

Proof. A sketch of the proof is as follows. Evaluating the right hand side of the above we get

$$\prod_{\alpha>0} (1 - R_\alpha) h_{\lambda+\rho-\rho} = \sum_{E \subseteq \Phi+} (-1)^{|E|} h_{\lambda+\rho+(-\rho+\sigma_E)},$$

where $\sigma_E = \sum_{\alpha \in E} \alpha$. An element $\gamma = \sum_i \gamma_i \omega_i$ in P^+ is called regular if $\gamma_i > 0$ for all i. The sets $\{-\rho + \sigma_E | E \subseteq \Phi^+, \rho + \sigma_E \text{ regular}\}$ and $\{-w\rho | w \in W\}$ are equal. This is proved by expressing ρ in the form $\rho = \sum_{\alpha \in \Phi^+} \alpha$ and using [Bou] Ch. 6, §1 Cor. 2. Under this bijection $(-1)^{|E|} = \varepsilon(w)$. The terms arising from the subsets E for which $-\rho + \sigma_E$ is not regular cancel with each other. This can be shown by showing that $\prod_{\alpha>0}(1 - R_\alpha)(-\rho)$ is skew-symmetric with respect to W and that if $\gamma \in P^+$ is not regular then $\sum_{w \in W} \varepsilon(w) w\gamma = 0$. These arguments show that

$$\prod_{\alpha>0} (1 - R_\alpha) h_\lambda = \sum_{w \in W} \varepsilon(w) h_{\lambda+\rho-w\rho}. \qquad \square$$

The proof of Cor. (2.15) was motivated by the proof of the Weyl denominator formula given in [Mac2].

Direct limits

The above definition defines an analogue of homogeneous symmetric functions for the spaces Λ_κ. One would like to say that in some sense the h_λ are well defined on all of A^W. With this in mind we introduce the following.

For each pair $\beta, \kappa \in P^+$ such that $\beta \leq \kappa$ define a linear map $f_{\beta\kappa} : \Lambda_\kappa \to \Lambda_\beta$ by

$$s_\lambda \mapsto \begin{cases} s_\lambda, & \text{if } \lambda \leq \beta; \\ 0, & \text{if } \lambda \not\leq \beta. \end{cases}$$

It is clear that

1) If $\beta \leq \gamma \leq \kappa$ then $f_{\beta\kappa} = f_{\beta\gamma} \circ f_{\gamma\kappa}$,

2) For each $\beta \in P^+$, $f_{\beta\beta}$ is the identity on Λ_β.

Thus $(\Lambda_\beta, f_{\beta\gamma})$ form an inverse system of vector spaces, see Bourbaki Theory of Sets I §7, and Bourbaki Algebra I §10. Define

$$\Lambda = \varprojlim(\Lambda_\beta, f_{\beta\gamma}).$$

Then the homogeneous symmetric function h_λ is a well defined element of Λ for all $\lambda \in P^+$ and is equal to

$$h_\mu = \sum_{\lambda \in P+} s_\lambda K_{\lambda\mu}.$$

An alternate option is to view the homogeneous symmetric function as an element in the direct product of vector spaces

$$\prod_{\lambda \in P+} \mathbb{Z}\, \chi^\lambda.$$

Depending on what one would like to compute this can create problems with infinite sums. The direct limit approach allows one to control these problems by fixing an ordering on infinite sums.

3. Representation theory

Fix a Cartan matrix $C = (\langle \alpha_i, \alpha_j \rangle)$ and define $\mathfrak{U}$ to be the associative algebra (over $\mathbb{C}$) with 1 generated by $x_i, y_i, h_i, 1 \le i \le n$ with relations (Serre relations)

$$[h_i, h_j] = 0 \qquad (1 \le i, j \le n), \tag{$S1$}$$

$$[x_i, y_i] = h_i, \qquad [x_i, y_j] = 0 \text{ if } i \neq j, \tag{$S2$}$$

$$[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j, \qquad [h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j, \tag{$S3$}$$

$$(\text{ad } x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0 \qquad (i \neq j), \tag{S_{ij}^+}$$

$$(\text{ad } y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0 \qquad (i \neq j). \tag{S_{ij}^-}$$

Here $[a, b] = ab - ba$ and $(\text{ad } a)^k(b) = [a, [a, [a, \cdots, [a, b]] \cdots]$.

Let $\mathfrak{H}^* = \sum_i \mathbb{C}\, \omega_i$. Let V be a $\mathfrak{U}$ module. A vector $v \in V$ is called a weight vector if, for each i,

$$h_i v = \lambda_i v,$$

for some constant $\lambda_i \in \mathbb{C}$. We associate to v the weight $\lambda = \sum_i \lambda_i \omega_i$.

If v is a weight vector of weight γ then $x_i v$ is a weight vector of weight $\gamma + \alpha_i$. This is the motivation for the definition of the raising operators R_α. This also gives the motivation for the definition of the dominance partial order as $\lambda \le \mu$ if and only if $\mu = R\lambda$ for some sequence of raising operators $R = R_{\beta_1} R_{\beta_2} \cdots R_{\beta_k}$.

The dominant weights appear as a result of the following fact.

(3.1) Theorem. (see [Hu]) *There is a unique finite dimensional irreducible representation V^λ of $\mathfrak{U}$ corresponding to each dominant weight $\lambda \in P^+$. This irreducible representation is characterized by the fact that it contains a unique vector, up to scalar multiples, which is a weight vector of weight λ.*

(3.2) Theorem. (Weyl) *Every finite dimensional representation V of $\mathfrak{U}$ (corresponding to a simple Cartan matrix) is completely decomposable as a direct sum of irreducible representations V^λ, $\lambda \in P^+$.*

The motivation for Weyl group symmetric functions is that they are the generating functions of the weights in these representations. Let V be a finite dimensional $\mathfrak{U}$-module. Let B be a basis of $\mathfrak{U}$ such that each element of B is a weight vector. Then the character of V is the generating function of the weights of B,

$$\chi_V = \sum_{b \in B} e^{wt(b)}.$$

The irreducibles V^λ, combinatorially.

In this section we shall define the vector spaces V^λ affording the irreducible $\mathfrak{U}$ representations in a combinatorial fashion to motivate the Weyl group, the Weyl characters and the monomial symmetric functions.

Recall the notations $[a,b] = ab - ba$ and $(\text{ad } a)^k(b) = [a,[a,[a,\cdots,[a,b]]\cdots]$. Let $y_1, y_2, \cdots, y_n$ be letters and suppose that they satisfy the relations

$$(\text{ad } y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0, \qquad (i \neq j).$$

(We should not be surprised by the appearance of the values $\langle \alpha_i, \alpha_j \rangle$ as it is clear that our basic object, the Cartan matrix, must come into the picture in a crucial way.) We shall study words in the letters y_i. Let v^+ be a dummy letter marking the end of a word. Fix a dominant integral weight $\lambda \in P^+$. Define V^λ to be the vector space which is a linear span of words in the y_i ending with v^+,

$$V^\lambda = \text{span}\{y_{i_1} y_{i_2} \cdots y_{i_k} v^+\},$$

with the *additional relations*

$$y_j^{\lambda_j + 1} v^+ = 0.$$

Given that the vectors of the form $y_{i_1} \cdots y_{i_k} v^+$ span V^λ it is possible to choose a basis B of V^λ of vectors of this form. Define the weight of a word $v = y_{i_1} \cdots y_{i_k} v^+$ to be

$$wt(v) = \lambda - \sum_{j=1}^{k} \alpha_{i_j},$$

where the α_{i_j} are simple roots. Define the Weyl character χ^λ to be the generating function of the weights of the basis B,

$$\chi^\lambda = \sum_{b \in B} e^{wt(b)}. \tag{3.3}$$

(This is analogous to the combinatorial definition of the Schur functions in terms of column strict tableaux.) The definition of the Weyl characters is motivated by the following theorem.

(3.4) Theorem. (Weyl character formula) *For each* $\lambda \in P^+$,

$$\chi^\lambda = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)}}{\sum_{w\in W} \varepsilon(w) e^{w\rho}},$$

where $\rho = \sum_i \omega_i$.

If we define

$$h_i v^+ = \lambda_i v^+,$$
$$x_i v^+ = 0,$$

for the generators $x_i, h_i, 1 \leq i \leq n$ of $\mathfrak{U}$ then we can use the defining relations for $\mathfrak{U}$ to rewrite the expressions of the form $x_i y_{i_1} y_{i_2} \cdots y_{i_k} v^+$ and $h_i y_{i_1} \cdots y_{i_k} v^+$ as elements of V^+. In this way we get an action of $\mathfrak{U}$ on V^λ. V^λ is an irreducible $\mathfrak{U}$-module.

Weight multiplicities

Let $\mu \in P$ and define

$$(V^\lambda)_\mu = \operatorname{span}\{v = y_{i_1} \cdots y_{i_k} v^+ | wt(v) = \mu\}.$$

Define

$$K_{\lambda\mu} = \dim((V^\lambda)_\mu). \tag{3.5}$$

It is clear from the definitions that

(a) The $K_{\lambda\mu}$ are nonnegative integers.
(b) $K_{\lambda\lambda} = 1$ for all $\lambda \in P^+$.
(c) $K_{\lambda\mu} = 0$ if $\mu \not\leq \lambda$.

The key fact motivating the Weyl group is the following.

(3.6) Proposition.

$$K_{\lambda\mu} = K_{\lambda, w\mu},$$

for all $w \in W$.

To show this one uses the $\mathfrak{U}$ action on V^λ to define a map $\tilde{s}_i$ on V^λ by

$$\tilde{s}_i v = exp(x_i) exp(-y_i) exp(x_i) v.$$

One can show that $\tilde{s}_i$ is a bijection from $(V^\lambda)_\mu$ to $(V^\lambda)_{s_i\mu}$ for each $\mu \in P$. Then if $w = s_{i_1} \cdots s_{i_p}$ is an element of the Weyl group, $\tilde{s}_{i_1} \cdots \tilde{s}_{i_p}$ is a bijection between $(V^\lambda)_\mu$ and $(V^\lambda)_{w\mu}$. This gives that $K_{\lambda\mu} = K_{\lambda, w\mu}$.

From (3.3) and (2.3) we have that

$$\begin{aligned} \chi^\lambda &= \sum_{\mu\in P} K_{\lambda\mu} e^\mu \\ &= \sum_{\nu\in P^+} K_{\lambda\nu} \sum_{\mu\in W\nu} e^\mu \\ &= \sum_{\nu\in P^+} K_{\lambda\nu} m_\nu. \end{aligned} \tag{3.7}$$

Note that
$$\dim V^\lambda = \sum_{\mu\in P+}\sum_{W\mu} K_{\lambda\mu}.$$

Since the orbits $W\mu$ are finite and the integers $K_{\lambda\mu}$ are finite we see that V^λ is finite dimensional.

Tensor products

Let $\lambda, \mu \in P^+$ and let V^λ and V^μ be as given above so that $V^\lambda = \operatorname{span}\{y_{i_1}\cdots y_{i_k}v^+\}$, and $V^\mu = \operatorname{span}\{y_{i_1}\cdots y_{i_k}\bar{v}^+\}$. Then the vector space $V^\lambda \otimes V^\mu$ is

$$V^\lambda \otimes V^\mu = \operatorname{span}\{y_{i_1}\cdots y_{i_r}v^+ \otimes y_{j_1}\cdots y_{j_s}\bar{v}^+\}.$$

The weight of a composite word is given by

$$wt(y_{i_1}\cdots y_{i_r}v^+ \otimes y_{j_1}\cdots y_{j_s}\bar{v}^+) = wt(y_{i_1}\cdots y_{i_r}v^+) + wt(y_{j_1}\cdots y_{j_s}\bar{v}^+).$$

Now let B be a basis of V^λ of vectors of the form $y_{i_1}\cdots y_{i_r}v^+$ and let $\bar{B}$ be a basis of V^μ of vectors of the form $y_{j_1}\cdots y_{j_s}\bar{v}^+$. Then the words $b\otimes\bar{b}$, $b\in B$, $\bar{b}\in\bar{B}$ form a basis of $V^\lambda\otimes V^\mu$. Then the character of $V^\lambda\otimes V^\mu$, the generating function of the basis $B\otimes\bar{B}$, is

$$\sum_{b\otimes\bar{b}} e^{wt(b\otimes\bar{b})} = \sum_{b\in B}\sum_{\bar{b}\in\bar{B}} e^{wt(b)}e^{wt(\bar{b})} = \chi^\lambda\chi^\mu.$$

So even the multiplication of elements of A^W is "coming from" representation theory!

$V^\lambda\otimes V^\mu$ is also a $\mathfrak{U}$-module. If g is one of the generators x_i, y_i or h_i then we define

$$g(y_{i_1}\cdots y_{i_r}v^+ \otimes y_{j_1}\cdots y_{j_s}\bar{v}^+) = gy_{i_1}\cdots y_{i_r}v^+ \otimes y_{j_1}\cdots y_{j_s}\bar{v}^+ + y_{i_1}\cdots y_{i_r}v^+ \otimes gy_{j_1}\cdots y_{j_s}\bar{v}^+.$$

This defines a $\mathfrak{U}$ action on $V^\lambda\otimes V^\mu$.

The elementary symmetric functions e_λ are the characters of the tensor product $(V^{\omega_1})^{\otimes\lambda_1}\otimes\cdots\otimes(V^{\omega_n})^{\otimes\lambda_n}$. By Weyl's theorem (3.2) we know that this tensor product can be decomposed as a direct sum of irreducible representations. In fact, for each λ,

$$e_\lambda = \chi^\lambda + \sum_{\mu<\lambda}\tilde{K}_{\lambda\mu}\chi^\mu, \tag{3.8}$$

for nonnegative integers $\tilde{K}_{\lambda\mu}$. This is the motivation for the definition of the elementary symmetric functions. Since the ω_r are the basic generators of the weight lattice P, the $e_r = \chi^{\omega_r}$ are the most fundamental Weyl characters. (3.8) shows that these do indeed generate A^W.

4. Centralizer algebras

Let M be a finite dimensional module for a semisimple Lie algebra $\mathfrak{U}$. By Weyl's complete reduciblity theorem we know that M decomposes as a direct sum of irreducible $\mathfrak{U}$ modules with certain multiplicities c_λ,

$$M \cong \bigoplus_{\lambda \in \hat{M}} c_\lambda V^\lambda.$$

$\hat{M}$ denotes the set of $\lambda \in P^+$ such that V^λ appears in the decomposition of M. Let C be the centralizer of the action of $\mathfrak{U}$ on M, i.e. $C = \mathrm{End}_{\mathfrak{U}}(M)$. The following theorem is the basic result of the double centralizer theory.

(4.1) Theorem. *M is a bimodule for $C \times \mathfrak{U}$ and*

$$M \cong \bigoplus_{\lambda \in \hat{M}} C_\lambda \otimes V^\lambda,$$

where C_λ is an irreducible module for C.

This decomposition induces a pairing between the irreducible modules C_λ of C and irreducible $\mathfrak{U}$ modules V^λ, $\lambda \in \hat{M}$. It is true that the set of C_λ, $\lambda \in \hat{M}$, is a complete set of irreducible modules of C. The pairing between irreducible C modules and the irreducible $\mathfrak{U}$ modules appearing as factors in M can be given in terms of characters. By definition, a character of C is a linear functional $\xi: C \to \mathbb{C}$ such that

$$\xi(c_1c_2) = \xi(c_2c_1),$$

for all $c_1, c_2 \in \mathbb{C}$. Let $\hat{C}$ be the vector space of characters of C. Given a C-module M the character ξ of C corresponding to M is given by defining $\xi(c)$, $c \in C$, to be the trace of the action of c on M. Let ξ_λ denote the irreducible character of C corresponding to the irreducible C-module C_λ. For each $\lambda \in P^+$ let χ^λ denote the corresponding Weyl character for $\mathfrak{U}$. Let $\Lambda_{\hat{M}}$ be the vector space which is the span of the χ^λ, $\lambda \in \hat{M}$. Define the characteristic map ch to be the map given by

$$\begin{array}{cccc} ch: & \hat{C} & \to & \Lambda_{\hat{M}} \\ & \xi_\lambda & \mapsto & \chi^\lambda \end{array}$$

For each $\mu = \sum_i \mu_i\omega_i \in P$ define the μ weight space of M to be the vector space

$$M_\mu = \{m \in M \mid \text{for each } 1 \le i \le n,\ h_i m = \mu_i m\}.$$

Let $\mathfrak{U}(\mathfrak{H})$ be the subalgebra of $\mathfrak{U}$ generated by the h_i. M decomposes as a direct sum of weight spaces under the action of $\mathfrak{U}(\mathfrak{H})$,

$$M \cong \bigoplus_{\mu \in P} M_\mu. \tag{4.2}$$

Since $\mathfrak{U}(\mathfrak{H})$ is a subalgebra of $\mathfrak{U}$ each M_μ is a C module. The bicharacter of M is defined to be

$$\mathrm{bichar} M = \sum_{\mu} \eta_\mu e^\mu, \tag{4.3}$$

where η_μ denotes the character of M_μ as a C-module. In view of the decomposition in (4.1) we also have

$$\mathrm{bichar} M = \sum_{\lambda \in \hat{M}} \xi_\lambda \chi^\lambda. \tag{4.4}$$

Construction of irreducible modules for centralizer algebras

Consider the action of the generators $x_i \in \mathfrak{U}$ on the weight spaces M_μ of M. For each $\mu \in P$

$$x_i \colon M_\mu \to M_{\mu+\alpha_i}.$$

Let $\ker_\mu(x_i)$ denote the kernel of the map x_i acting on M_μ. Let

$$\bar{C}_\mu = \bigcap_i \ker_\mu(x_i).$$

(4.5) Theorem.

1) *$\bar{C}_\mu$ is either 0 or an irreducible C module. Furthermore all irreducible C modules can be obtained in this fashion.*
2) *$\bar{C}_\mu \neq 0 \iff \mu \in \hat{M}$.*

Proof. Using (4.1) we have that $M_\mu = \oplus_{\lambda \in \hat{M}} C_\lambda \otimes (V^\lambda)_\mu$. But the only weight in V^λ killed by *all* x_i is λ. So there are no elements of $(V^\lambda)_\mu$ in the $\cap_i \ker_\mu(x_i)$ unless $\lambda = \mu$. When $\lambda = \mu$ we have $(V^\mu)_\mu \subseteq \ker_\mu(x_i)$. Thus

$$\begin{aligned} \bar{C}_\mu &= \bigcap_i \ker_\mu(x_i) \\ &\cong \bigoplus_{\lambda \in \hat{M}} C_\lambda \otimes (V^\lambda)_\mu \delta_{\lambda\mu} \\ &\cong C_\mu, \end{aligned}$$

as C-modules. □

A character formula for C_λ

(4.6) Theorem. *The character of the irreducible C-module C_λ can be given by*

$$\xi_\lambda = \sum_{w \in W} \varepsilon(w) \eta_{\lambda+\rho-w\rho},$$

where η_μ is the character of M_μ as a C-module.

Proof. Equating the expressions (4.3) and (4.4) and rewriting χ^λ by using (3.4) we have

$$\sum_\lambda \xi_\lambda \sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} = \sum_{\mu\in P}\sum_{w\in W} \eta_\mu \varepsilon(w) e^{\mu+w\rho-\rho}.$$

Substitute $\gamma = \mu + w\rho - \rho$ to get

$$\sum_\lambda \xi_\lambda \sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} = \sum_{\mu\in P}\Big(\sum_{w\in W} \varepsilon(w)\eta_{\gamma+\rho-w\rho}\Big) e^{\gamma}.$$

Compare coefficients of e^γ for $\gamma \in P^+$ on each side of this equation. Since $\lambda \in P^+$ we know by (2.3) that $w(\lambda+\rho)-\rho$ is not an element of P^+ for any $w \in W$ except the identity. Thus

$$\xi_\lambda = \sum_{w\in W} \varepsilon(w)\eta_{\lambda+\rho-w\rho}. \qquad \square$$

(4.7) Corollary. *The dimension c_λ of the irreducible C-module C_λ is given by*

$$c_\lambda = \sum_{w\in W} \varepsilon(w) d_{\lambda+\rho-w\rho},$$

where d_μ is the dimension of the weight space M_μ.

Proof. Evaluate the identity in Theorem (4.6) at the identity of C. $\square$

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