# ON PERMUTATION REPRESENTATIONS OF WEYL GROUPS, DESCENT NUMBERS, AND THE FACE RING OF THE COXETER COMPLEX

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## Extended Abstract

Let $R$ be a (reduced, crystallographic) root system with Weyl group $W = W(R)$ and Coxeter complex $\Delta_R$. This talk will be concerned with a certain representation $\rho^R$ of $W$ that has been largely ignored until recently. It can be succinctly described as the representation carried by the cohomology of the toric variety of $\Delta_R$, although it can also be given a purely algebraic definition as the representation carried by a certain quotient of the face ring of $\Delta_R$. It should be emphasized that this is not the representation one obtains from the homology of the Coxeter complex itself; this latter representation has received considerably more attention, thanks to the work of Björner [B], Garsia-Stanton [GS], and Stanley [St1].

Our main result is the fact that $\rho^R$ is, for no easily explainable reason, a *permutation* representation; i.e., there exists a basis for $\rho^R$ such that the Weyl group acts by permuting this basis. Unfortunately however, there are two senses in which we regard our proof of this result as unsatisfying. First, it must be applied to each root system on a case-by-case basis. Second, it is non-constructive—we lack an explicit basis for $\rho^R$ that is permuted by $W$, even for the root systems of type $A$. Even more vexing is the fact that we can exhibit a number of beautiful properties that come close to characterizing $\rho^R$ as a permutation module, but we are unable to explicitly construct (except in particular cases) a simple set of combinatorial objects permuted naturally by $W$ in a manner isomorphic to $\rho^R$. In fact, since $\dim \rho^R = |W|$, this means that $W$ itself is an obvious choice for the set of objects. It is hard to imagine that there could exist a natural permutation action of $W$ upon itself

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that does not admit a simple description, such as left-multiplication or conjugation, but that is the state we are in.

On the positive side, in the course of proving this theorem we did make a number of interesting combinatorial discoveries, mostly involving descent numbers (i.e., the Weyl group generalization of the Eulerian numbers). Another byproduct of this work is a collection of *Maple* procedures we created for manipulating Weyl groups and root systems. Aside from the easy case $R = G_2$, our proofs for the cases involving the exceptional root systems rely on these procedures, and thus are computer-based.

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*Origins*

The original motivation for this work can be found in Stanley's discussion of some interesting combinatorial properties of the special case $R = A_n$ in [St2, pp. 524–529]. This provoked me into studying this particular case in more detail [Ste]. More recently, Dolgachev and Lunts proved a nice character formula for $\rho^R$ in the general case [DL]. This reawakened my interest in the subject, and led to the discoveries I will report on here.

## Some Details about the Representation

Let $V$ be an $n$-dimensional real Euclidean space and let $R$ be a root system in $V$. Associated with $R$ there is a hyperplane arrangement $\mathcal{H}_R = \{\alpha^\perp : \alpha \in R\}$. The Coxeter complex $\Delta_R$ can be defined geometrically as the simplicial decomposition of the unit sphere $S^{n-1}$ in $V$ induced by $\mathcal{H}_R$.

The Coxeter complex can also be regarded as a (complete, simplicial) fan in $V$ with respect to the weight lattice $P$. That is, it is a decomposition of $V$ into strongly convex (simplicial) cones, each generated by certain integral weights in $P$. As is the case with any fan, there is a toric variety $X_R$ naturally associated with $\Delta_R$ [O]. The Weyl group acts naturally on $\Delta_R$, and therefore also on $X_R$ and the cohomology ring $H^*(X_R, \mathbf{C})$.

Let $f_i$ denote the number of $i$-dimensional faces of $\Delta_R$ (with $f_{-1} = 1$), and define

$$P_R(q) = \sum_{i=0}^{n} f_{i-1}(q-1)^{n-i} = \sum_{i=0}^{n} h_i q^{n-i}.$$

The first equality serves to define the polynomial $P_R(q)$, and the second equality serves to define the $h$-vector $h(R) = (h_0, \ldots, h_n)$ of $\Delta_R$. By a theorem of Danilov and Jurkiewicz

[D], one knows that

$$\dim H^{2i}(X_R, \mathbf{C}) = h_i(R), \qquad \dim H^{2i+1}(X_R, \mathbf{C}) = 0,$$

so $P_R(q)$ is essentially the Poincaré polynomial of $X_R$.

In particular, it follows that $\rho^R$, the $W$-representation carried by $H^*(X_R)$, has $n+1$ (nonzero) graded components, and the total dimension of $\rho^R$ is $P_R(1) = f_{n-1}(R) = |W|$. Let $\chi_q^R$ denote the graded character of $\rho^R$; i.e., for $w \in W$,

$$\chi_q^R(w) = \sum_{i=0}^{n} \operatorname{tr}(H^{2i}(X_R), w) q^i,$$

where $\operatorname{tr}(U, w)$ denotes the trace of $w$ on $U$.

**Theorem** (Dolgachev-Lunts [DL]). *For $w \in W$, let $\delta(w) = \dim\{v \in V : wv = v\}$ and $\Delta_R^w = \{F \in \Delta_R : w(F) = F\}$ (a subcomplex of $\Delta_R$). We have*

$$\chi_q^R(w) = P_{R,w}(q) \frac{\det(1 - qw)}{(1-q)^{\delta(w)}},$$

*where (1) $P_{R,w}(q)$ denotes the Poincaré polynomial of $\Delta_R^w$, and (2) the determinant is evaluated with respect to the reflection representation.*

It is also possible to give a purely algebraic definition of the cohomology ring $H^*(X_R, \mathbf{C})$ and the representation $\rho^R$ it carries. To describe this, let $v_i \in V$ denote the set of vertices of $\Delta_R$, with $i$ ranging over some suitable index set $I$. Recall that the face ring (or Stanley-Reisner ring) $\mathcal{F}_R$ of $\Delta_R$ is the quotient $\mathbf{C}[x_i : i \in I]/\Psi$, where $\Psi$ is the ideal generated by monomials whose supports (i.e., subset of vertices with nonzero exponent) are not faces of $\Delta_R$. If we define

$$\theta_j = \sum_{i \in I} \langle v_i, \varepsilon_j \rangle x_i,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ is some basis for $V$, then $\Theta = (\theta_1, \ldots, \theta_n)$ forms a system of parameters for $\mathcal{F}_R$. By a theorem of Danilov [D], it is known that

$$H^*(X_R, \mathbf{C}) \cong \mathcal{F}_R / \Theta$$

is an isomorphism of graded rings (as well as $W$-modules). This isomorphism, together with the fact that $\mathcal{F}_R$ is Cohen-Macaulay, can be used to give an alternative proof of the Dolgachev-Lunts formula.

## The Main Result

First consider some generalities about permutation representations.

Suppose $G$ is some finite group acting by permutations on a finite set $X$. Let $X = X_1 \cup \cdots \cup X_l$ be the partition of $X$ into $G$-orbits. The action of $G$ on $X_i$ is isomorphic to left multiplication on the cosets of $H_i$ in $G$, where $H_i$ denotes the stabilizer of some $x \in X_i$. Thus the permutation module for $G$ on $X$ is determined up to isomorphism by the list of point-stabilizers $H_1, \ldots, H_l$. In general, there exist non-isomorphic permutation representations of finite groups $G$ that become isomorphic once they are linearized (i.e., once one permits linear changes of basis). Thus in the following, the assertion that the character of $\rho^R$ agrees with the character of some permutation representation does not necessarily imply that there is only one such isomorphism class of permutation representations (even after allowing for conjugacy).

To state the main result, let $S = \{s_1, \ldots, s_n\}$ denote the set of simple reflections for $W(R)$. Define $\chi^R$ to be the specialization of $\chi^R_q$ at $q = 1$; i.e., the $W$-character of $H^*(X_R, \mathbf{C})$, or equivalently of $\mathcal{F}_R/\Theta$, *with the grading ignored.*

**Theorem.**

(a) *$\chi^R$ is the character of some permutation representation $\pi^R$ of $W$.*

(b) *The degree of $\pi^R$ is $|W|$, and the number of orbits is $2^n$.*

(c) *The point-stabilizers of $\pi^R$ are generated by reflections, but not necessarily by simple reflections.*

(d) *If $R$ is reducible; say, $R = R_1 \oplus R_2$, then $\pi^R \cong \pi^{R_1} \otimes \pi^{R_2}$ (outer tensor product).*

From now on, assume $R$ is irreducible. In that case, $R$ has a unique highest root $\alpha_0$. Let $s_0$ denote the corresponding reflection across $\alpha_0^\perp$, and set $S' = S \cup \{s_0\}$. For any nonempty subset $J$ of $S'$, let $W(J)$ denote the subgroup of $W$ generated by $S' - J$. (If $J$ includes $s_0$, then $W$ will be a parabolic subgroup, but not otherwise.) Note that if $|J| = r + 1$, then $W(J)$ is a reflection group of rank $n - r$.

(e) *The point-stabilizers are all of the form $W(J)$ for various $J$ ($\varnothing \neq J \subset S'$).*

(f) *The number of point-stabilizers that are reflection groups of rank $n-r$ is $\binom{n+1}{2r+1}$.*

(g) *For each $(2r+1)$-subset $J$ of $S'$, it is possible to choose an $(r+1)$-subset $J'$ of $J$ so that the point-stabilizers of $\pi^R$ are precisely $\{W(J') : |J| \text{ odd}\}$; i.e.,*

$$\chi^R = \sum_{|J| \text{ odd}} 1^W_{W(J')}. \qquad (*)$$

REMARKS.

(1) The reflections $S'$ are the $W$-images of the simple reflections for the affine Weyl group $\tilde{W}$ attached to $W$.

(2) By (f), the smallest possible rank of any point-stabilizer is $n/2$.

(3) Unfortunately, the only rules we have for choosing $J'$ from $J$ are *ad hoc*. For example, for the root systems of types $A$ or $C$, it is possible to linearly order the reflections $S'$ so that if $J = \{\beta_1 < \cdots < \beta_{2r+1}\}$, then $J' = \{\beta_1, \beta_3, \ldots, \beta_{2r+1}\}$. This rule does not seem to generalize.

(4) The permutation representation $\pi^R$ depends on $R$ itself, not just $W(R)$. Indeed, even though $W(B_n) = W(C_n)$ and the corresponding representations $\rho^R$ are isomorphic, the highest roots of $B_n$ and $C_n$ are different, and the corresponding decompositions in (*) are not equivalent.

(5) For a given root system $R$, there may be several possible ways to choose $J'$ from $J$ so that (*) is satisfied. However, in the special case $r = 0$, the constraints of (g) are unambiguous—if $J$ is a singleton, then $J' = J$. In other words, the rank $n$ point-stabilizers that occur in $\pi^R$ are (with multiplicity) the $n+1$ subgroups of $W$ generated by the $n$-subsets of $S'$.

(6) For some root systems (notably types $A$ and $C$), one can show that the decomposition of $\chi^R$ implied by this result is consistent with the grading of $H^*(X_R, \mathbf{C})$ (or equivalently, $\mathcal{F}_R/\Theta$). That is, it is possible to assign a grading to the orbits of $\pi^R$ so that $\chi_q^R$ is the graded character of $\pi^R$. For other root systems, such as type $D$, one can show that $\chi_q^R$ is the character of a graded permutation representation, but not one whose point-stabilizers are all generated by reflections in $W(D_n)$ (in violation of (c)). For still other root systems, such as $G_2$, the grading of $H^*(X_R, \mathbf{C})$ is not consistent with any grading of any permutation representation of $W(G_2)$.

(7) If an explicit construction of a permutation representation $\pi^R$ satisfying all of (a)–(g) can be found, it is reasonable to expect that this should be accompanied by a combinatorial bijection explaining the evaluation of (*) at $w = 1$; i.e.,

$$|W| = \sum_{|J| \text{ odd}} |W|/|W(J')|.$$

This is non-trivial even for type $A$ (see [Ste]).

(8) The complex $\Delta_R$ is known to be shellable [B]. Therefore by a result of Garsia [B, Thm. 1.7], one can construct from the shelling a canonical basis for $\mathcal{F}_R/\Theta$, and hence for $\rho^R$. However, this basis is not permuted by $W(R)$.

## Descent Numbers

As in the previous section, let $S = \{s_1, \ldots, s_n\}$ denote the set of simple reflections. The descent set of any $w \in W$ is defined to be $D(w) = \{i : \ell(ws_i) < \ell(w)\}$, where $\ell(\cdot)$ denotes the length function with respect to $S$. The shellability of $\Delta_R$ leads to a nice combinatorial interpretation of the Poincaré polynomial $P_R(q)$, or equivalently, of the $h$-vector for $\Delta_R$.

**Theorem** (essentially [B, Thm. 2.1]). $P_R(q) = \sum_{w \in W} q^{|D(w)|}$.

In the case $R = A_{n-1}$, $h_i(R)$ is thus the number of permutations in $S_n$ with $i$ descents. These are the classical Eulerian numbers and $qP_{A_{n-1}}(q)$ is the classical Eulerian polynomial. Although these numbers do not have simple explicit formulas, there is a well-known closed formula for the exponential generating function:

$$\sum_{n \geq 0} P_{A_{n-1}}(q) \frac{t^n}{n!} = \frac{(1-q)e^{(1-q)t}}{1 - qe^{(1-q)t}}.$$

Likewise for the $B/C$-series, there is a similar formula that is widely known (but seemingly unpublished):

$$\sum_{n \geq 0} P_{B_n}(q) \frac{t^n}{n!} = \frac{(1-q)e^{(1-q)t}}{1 - qe^{2(1-q)t}}.$$

For the $D$-series, it turns out that there is also a closed formula for the exponential generating function. It is an easy consequence of the following surprising relationship.

**Proposition 1.** $P_{B_n}(q) = P_{D_n}(q) + 2^{n-1} nqP_{A_{n-2}}(q)$.

We have two proofs, including a bijective one.

In another direction, the following result applies to all root systems except types $D$ and $E$. In particular, it applies to the non-crystallographic cases $I_2(m)$, $H_3$, and $H_4$. We write $W_{[i,j]}$ for the parabolic subgroup of $W$ generated by $\{s_i, s_{i+1}, \ldots, s_j\}$.

**Proposition 2.** *If $S$ can be linearly ordered so that $s_i$ and $s_j$ commute for $|i-j| > 1$ (i.e., the Dynkin diagram of $R$ has no forks), then we have*

$$P_R(q) = |W| \cdot \det[a_{ij}]_{1 \leq i,j \leq n+1},$$

*where $a_{ij} = 0$ for $i - j > 1$, $a_{ij} = 1 - q$ for $i - j = 1$, and $a_{ij} = 1/|W_{[i,j-1]}|$ for $i \leq j$.*

In analyzing the character of $H^*(X_R, \mathbf{C})$, one needs to know the Poincaré polynomials of not just the Coxeter complexes $\Delta_R$, but also the various fixed-point subcomplexes $\Delta_R^w$ (cf. the Dolgachev-Lunts formula). Usually these restrictions turn out to be Coxeter complexes of smaller rank, but not always. Among the classical cases, the only new simplicial complexes that arise in this way are (the complexes associated with) the following simplicial hyperplane arrangements:

$$\mathcal{H}_{k,n} = \{\varepsilon_i^\perp : 1 \le i \le k\} \cup \{(\varepsilon_i \pm \varepsilon_j)^\perp : 1 \le i < j \le n\} \qquad (0 \le k \le n).$$

Note that we recover the Coxeter complexes for $D_n$ and $B_n$ at $k = 0$ and $n$, respectively. The Poincaré polynomials $P_{n,k}(q)$ for these complexes satisfy a simple relationship generalizing Proposition 1.

**Proposition 3.** $P_{n,k}(q) = P_{D_n}(q) + 2^{n-1}kqP_{A_{n-2}}(q)$.

In the general case, it is possible to give explicit (but more complicated) formulas for the Poincaré polynomials of the fixed-point subcomplexes. Up to isomorphism, these complexes are indexed by $J \subset S$. More precisely, any $\Delta_R^w$ is conjugate under the Weyl group to some $\Delta_R^{w_J}$, where $w_J$ denotes a Coxeter element for the parabolic subgroup $W_J$ generated by some $J \subset S$. Using $P_{R,J}(q)$ to denote the corresponding Poincaré polynomial, the following result gives a formula for $P_{R,J}(1)$, the number of maximal faces in $\Delta_R^{w_J}$.

**Proposition 4.** $P_{R,J}(1) = \dfrac{|N(W_J)|}{|W_J|} \cdot |\{K \subset S : W_K \sim W_J\}|$, *where* $N(W_J)$ *denotes the normalizer of* $W_J$ *in* $W$, *and* $\sim$ *denotes conjugacy of subgroups.*

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