Combinatorics of special functions : facets of Brock's identity

Volker Strehl
Computer Science Department - Informatik I
University of Erlangen-Nürnberg, Germany

Abstract

These notes are intended to give an overview of the relation between an old binomial identity, originally proposed as a problem by P. Brock, its various extensions and, in particular, its relation to combinatorial models for special functions that have been studied in more recent years. A comprehensive treatment of the results mentioned here is given in [33]. The list of references is selective and contains only articles mentioned in this overview.

1 Introduction

In these notes and in my talk I will try to explain the intimate link between two combinatorial results which, at first sight, appear quite unrelated.

- *Brock's identity* : in 1960 the following was presented by P. Brock [2] to the readers of the SIAM Review as problem 60-2:
  for integers $A, B \geq 0$ let

$$H(A,B) := \sum_{i=0}^{A}\sum_{j=0}^{B} \binom{i+j}{j}\binom{A-i+j}{j}\binom{B+i-j}{B-j}\binom{A-i+B-j}{B-j}. \tag{1}$$

  Then show that

$$H(A,B) - H(A-1,B) - H(A,B-1) = \binom{A+B}{A}^2, \tag{2}$$

where $H(-1,B) = H(A,-1) = 0$.

This problem arose in the study of a sorting problem, or, to be a bit more precise, it reflects a particular way of counting permutations having exactly one increasing and one decreasing subsequence of maximal length. A rather involved inductive proof of the above identity is given in an article by R.M. Baer and P. Brock [1] on 'natural sorting'.

More on solutions and extensions of Brock's problem will be said in sec. 3 below.

- *PLI-endofunctions* : for any integer $p \geq 1$ and for any p-tuple $\mathbf{U} = (U_1, \ldots, U_p)$ of finite sets let ${}^{[p]}PLI[\mathbf{U}]$ denote the set of all functions $f : |\mathbf{U}| \to |\mathbf{U}|$, where $|\mathbf{U}| = U_1 \cup \ldots \cup U_p$, such that $f(U_i) \subseteq U_i \cup U_{i+1}$ and such that all the restrictions $f|_{U_i} : U_i \to U_i \cup U_{i+1}$ are injective maps ($1 \leq i \leq p$, reading indices modulo p).

  Let a_i ($1 \leq i \leq p$) and β be parameters and put a weight
$$\prod_{1\leq i\leq p} (1+\alpha_i)^{cyc_i(f)} \cdot (1+\beta)^{cyc_m(f)} \tag{3}$$
  on each $f \in {}^{[p]}PLI[\mathbf{U}]$, where (for $1 \leq i \leq p$) $cyc_i(f)$ denotes the number of ('pure') f-cycles contained in U_i, and $cyc_m(f)$ denotes the number of the remaining ('mixed') f-cycles.

  The exponential generating function associated to this p-sorted species of *periodic, locally injective endofunctions* ${}^{[p]}PLI(X_1, \ldots, X_p)$ under this weight can then be written in two different ways:
  First we have
$$\frac{\prod_{1\leq i\leq p} (1+\xi_i(\mathbf{x}))^{1+\alpha_i}}{\left(1-\prod_{1\leq i\leq p}\xi_i(\mathbf{x})\right)^{1+\beta}} \quad , \tag{4}$$
  where $\mathbf{x} = (x_1 \ldots x_p)$ and where the functions $\xi_j = \xi_j(\mathbf{x})$ are determined by the 'cyclic' implicit system
$$\xi_j = x_j \cdot (1+\xi_{j-1}) \cdot (1+\xi_j) \qquad (1 \leq j \leq p) \ , \tag{5}$$
  again reading indices modulo p.
  On the other hand we get
$$\sum_{\mathbf{n}} \frac{\mathbf{x}^{\mathbf{n}}}{\mathbf{n}!} \prod_{1<i<p} (1+\alpha_i+n_{i+1})_{n_i} \cdot {}_{p+1}F_p \left[\begin{matrix} \beta \ , \ -n_1 \ldots -n_p \\ \ldots 1+\alpha_j+n_{j+1} \ldots \end{matrix} ; (-1)^p \right] =$$

$$= \sum_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \prod_{1 \le i \le p} \binom{\alpha_i + n_i + n_{i+1}}{n_i} \cdot {}_{p+1}F_p[\cdots] \ , \tag{6}$$

where the summation runs over all $\mathbf{n} = (n_1, \ldots, n_p) \in \mathrm{N}^p$ and $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \ldots x_p^{n_p}$, $\mathbf{n}! = n_1! \ldots n_p!$.

This result, which reflects on the generating function level two different combinatorial views of the underlying structures is, in fact, the specialization of something quite more general. Comments on the general situation will be made in sec. 4. Some consequences of this result will be presented in the next section. The relation between this kind of result and Brock's identity and its various extensions will be described in sec. 5.

2 Some consequences

In this section I will briefly discuss several simple, yet interesting particular cases of the result on generating functions for PLI-endofunctions just mentioned.

- in the case $p = 1$ and $\alpha = \beta$ we find

$$\sum_{n \ge 0} (1 + \beta + n)_n \; {}_2F_1 \left[\begin{matrix} \beta \ , \ -n \\ 1 + \beta + n \end{matrix} ; -1 \right] \frac{x^n}{n!} = \left[\frac{1 + \xi}{1 - \xi} \right]^{1+\beta} \ , \tag{7}$$

where $1 + \xi = (1 - \sqrt{1 - 4x})/(2x)$, and hence

$$\frac{1 + \xi}{1 - \xi} = (1 - 4x)^{-1/2}$$

from solving the implicit equation (5). Comparison of coefficients on both sides of (7) then shows

$${}_2F_1 \left[\begin{matrix} \beta \ , \ -n \\ 1 + \beta + n \end{matrix} ; -1 \right] = \frac{\Gamma(1 + \beta + n)\Gamma(1 + \frac{\beta}{2})}{\Gamma(1 + \frac{\beta}{2} + n)\Gamma(1 + \beta)} \ ,$$

which is Kummer's formula, cf. [24].

- in the case $p = 2, \beta = 0$ we get the classical generating function for the Jacobi polynomials (writing now (α, β) in place of (α_1, α_2)) :

$$\begin{aligned}\sum_{k,m\geq 0} \frac{x^k}{k!}\frac{y^m}{m!}(1+\alpha+m)_k(1+\beta+k)_m &= \\ &= \frac{(1+\xi_1(x,y))^{1+\alpha}\,(1+\xi_2(x,y))^{1+\beta}}{1-\xi_1(x,y)\,\xi_2(x,y)} \\ &= \left(\frac{1-x+y-\mathcal{R}}{2y}\right)^{\alpha}\left(\frac{1+x-y-\mathcal{R}}{2x}\right)^{\beta}\mathcal{R}^{-1} \quad ,\end{aligned} \tag{8}$$

where

$$\xi_1(x,y) = \frac{1-x-y-\mathcal{R}(x,y)}{2y} \quad , \quad \xi_2(x,y) = \frac{1-x-y-\mathcal{R}(x,y)}{2x} \quad ,$$

and

$$\mathcal{R} = \sqrt{[1-x-y]^2 - 4xy} \quad .$$

Note that

$$P_n^{(\alpha,\beta)}(x) = \sum_{k+m=n} \frac{(1+\alpha+m)_k}{k!}\frac{(1+\beta+k)_m}{m!}\left(\frac{x+1}{2}\right)^k\left(\frac{x-1}{2}\right)^m$$

is one of the various ways of writing the Jacobi polynomials, see [24]. Note furthermore that the combinatorial model of PLI-endofunctions reduces in this case to the model introduced by D. Foata and P. Leroux in [15], a model which has subsequently turned out to be very fruitful for the study of combinatorial properties of the Jacobi polynomials, see e.g. [23].

- in the case $p = 2, \alpha_1 = \alpha_2 = \beta$, we can evaluate the ${}_3F_2$-term in (6) via Dixons formula [12], [24]:

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\beta\ ,\ -n_1\ ,\ -n_2 \\ 1+\beta+n_1\ ,\ 1+\beta+n_2\end{matrix};1\right] = \\ &= \frac{\Gamma(1+\frac{\beta}{2})\Gamma(1+\beta+n_1)\Gamma(1+\beta+n_2)\Gamma(1+\frac{\beta}{2}+n_1+n_2)}{\Gamma(1+\beta)\Gamma(1+\frac{\beta}{2}+n_1)\Gamma(1+\frac{\beta}{2}+n_2)\Gamma(1+\beta+n_1+n_2)}\end{aligned}$$

which after simplification leads to the classical generating function for the Gegenbauer (ultrasherical) polynomials

$$P_n^{\nu}(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})} \quad ,$$

namely:

$$\sum_{n\geq 0} P_n^{\nu}(x)\, z^n = (1-2xz+z^2)^{-\nu} \quad .$$

This generating function cannot be obtained from the Foata-Leroux model mentioned above. Instead, a completely different approach for that purpose, using the model of 'complete oriented matchings', was presented by myself at the Montreal colloquium in 1985, see [29]. Putting this latter approach together with the combinatorial (!) proof of the result on PLI-endofunctions mentioned above, one obtains - as a by-product - another combinatorial proof of Dixon's formula.

3 On proofs and extensions of Brock's identity

It has been mentioned above that P. Brock himself (together with R.M. Baer) in [1] gave a rather involved inductive proof of (2). A better way of attacking the problem combinatorially has been pointed out independently by C.A. Church, jr. in [8] and myself in [32], where an appropriate class of configurations (pairs of crossing lattice paths or marked permutations) is examined. Both approaches turn out to be 'bijectively equivalent' via the Robinson-Schensted correspondence. This 'geometric' way of proving Brock's identity can also be used to obtain almost 'visually' various extensions that appear in the literature together with laborious analytic proofs (e.g. [9], [10]).

More interesting for us, however, is the kind of proof that has been initiated by D. Slepian's original analytic solution [27] of Brock's problem, using generating functions and complex integration. This kind of approach was later simplified and largely extended by L. Carlitz in a series of articles, beginning with [3].

L. Carlitz first proves

$$\sum_{i,j,k,l\geq 0} \binom{i+j}{j}\binom{j+k}{k}\binom{k+l}{l}\binom{l+i}{i} u^i v^j w^k x^l =$$
$$\left([(1-v)(1-x)-w+u(1-w)]^2 - 4u(1-v-w)(1-w-x)\right)^{-1/2} ,$$

from which he gets

$$\sum_{m,n\geq 0} H(m,n)\; u^m v^n = \frac{1}{1-u-v} \sum_{m,n\geq 0} \binom{m+n}{n}^2 u^m v^n \tag{9}$$

by identification of variables $u = w, v = x$. Note that (9) is obviously the generating function equivalent of Brock's original assertion (2).

Carlitz then proceeds to consider generating functions for 'cyclic products of binomial coefficients'

$$\binom{n_1+n_2}{n_2}\binom{n_2+n_3}{n_3}\cdots\binom{n_r+n_1}{n_r}$$

in general, and he obtains various identities of Brock-type for the numbers

$$H(n_1,n_2,\ldots,n_r) := \sum_{\substack{i_s+i_{r+s}=n_s\\ 1\leq s\leq r}} \binom{i_1+i_2}{i_2}\binom{i_2+i_3}{i_3}\cdots\binom{i_{2r}+i_1}{i_1} . \tag{10}$$

To give an idea of his results, I just state the two simplest ones:
case $r = 3$:

$$H(m,n,p) - H(m-1,n,p) - H(m,n-1,p) - H(m,n,p-1) = \binom{m+n}{n}\binom{n+p}{p}\binom{p+m}{m} \tag{11}$$

case $r = 4$:

$$\begin{aligned} &H(n_1,n_2,n_3,n_4) - \\ &H(n_1-1,n_2,n_3,n_4) - H(n_1,n_2-1,n_3,n_4) - \\ &H(n_1,n_2,n_3-1,n_4) - H(n_1,n_2,n_3,n_4-1) + \\ &H(n_1-1,n_2,n_3-1,n_4) + H(n_1,n_2-1,n_3,n_4-1) \\ &= \binom{n_1+n_2}{n_2}\binom{n_2+n_3}{n_3}\binom{n_3+n_4}{n_4}\binom{n_4+n_1}{n_1} \end{aligned} \tag{12}$$

Another line of generalization is opened by Carlitz in [4] by introducing extra parameters, i.e. by considering

$$H(n_1,\dots,n_r|\alpha_1,\dots,\alpha_{2r}) = \sum_{\substack{i_s+i_{r+s}=n_s\\ 1\le s\le r}} \binom{\alpha_1+i_1+i_2}{i_1}\binom{\alpha_2+i_2+i_3}{i_2}\cdots\binom{\alpha_{2r}+i_{2r}+i_1}{i_{2r}}\,. \tag{13}$$

Here (in the case $r = 2$) Brock's identity generalizes to

$$\begin{aligned} H(m,n|\alpha,\beta,\gamma,\delta) - H(m-1,n|\alpha,\beta,\gamma,\delta) - H(m,n-1|\alpha,\beta,\gamma,\delta) \\ = \binom{\alpha+\gamma+m+n}{m}\binom{\beta+\delta+m+n}{n} \end{aligned} \tag{14}$$

(As an aside: this is one of the extensions of Brock's identity which can be readily obtained from the 'geometric' approach mentioned above.) It is interesting to note that that for this generalization Carlitz starts from the classical generating function for the Jacobi polynomials (8).

A final word in this direction has been said by J.P. Singhal, who in [26] gives generating functions for cyclic product of parametrized binomial coefficients, which in turn can be used to obtain Brock-Carlitz type identites for the numbers $H(n_1,\dots,n_r|\alpha_1,\dots,\alpha_{2r})$.

One should note that all these results just mentioned, which are also (at least partially) treated in Riordan's [25] and Egorychev's [13] books, were obtained by rather involved calculations on formal series, without reference to any kind of combinatorial structures. This makes it difficult to digest the results in their general form and to notice the common pattern underlying these complicated looking generating functions and coefficient identities. A combinatorial view explains much of the mystery and opens the way for further extensions.

4 PLI-endofunctios and locally structured endofunctions

The result on the generating functions for periodic, locally injective endofunctions mentioned at the beginning of this note is indeed a specialization of a much more general result about the generating functions of combinatorial structures which I call *locally structured endofunctions*. Loosely speaking, given any p-tuple $\mathbf{A} = (A^{(1)}, \ldots, A^{(p)})$ of p-sorted species we may define the p-sorted species of $\mathbf{A}$-endofunctions by associating with each p-tuple $\mathbf{U} = (U_1, \ldots, U_p)$ of finite sets the set of all $\langle f, (a_u)_{u \in |\mathbf{U}|} \rangle$, where $f : |\mathbf{U}| \to |\mathbf{U}|$ is a mapping, and for each vertex $u \in U_i$ we require that

$$a_u \in A^{(i)}[f^{-1}(u) \cap U_1, \ldots, f^{-1}(u) \cap U_p] \quad (1 \leq i \leq p) \quad ,$$

i.e. there is an $A^{(i)}$-structure associated with the f-preimage of each element $u \in |\mathbf{U}|$, depending on the sort i of vertex u.

A cycle weight may be associated with such objects, as has been done in (3) for the particular case of PLI-endofunctions. In that particular situation

$$A^{(i)} = (1 + X_i) \cdot (1 + X_{i-1}) \quad (1 \leq i \leq p) \quad ,$$

where X_i is simply the species 'vertices of sort i' (again reading indices modulo p). The concept of locally structured endofunctions allows for the specification of a wide variety of combinatorial structures which can be defined characterized by 'local' conditions. In particular, most of the models that were introduced for the study of classical orthogonal polynomials from a combinatorial point of view belong to this class, see [14], [21], [22], [23], for example.

For any species of $\mathbf{A}$-endofunctions the generating function (taking cycle weights into account) may be presented in two different ways

- by viewing $\mathbf{A}$-endofunctions as permutations of $\mathbf{A}$-contractions - which yields an expression in terms of the generating function of the implicitly defined species of $\mathbf{A}$-trees. See (4,5) for this type of result in the particular case of PLI-endofunctions.

- by constructing $\mathbf{A}$-endofunctions (or more generally: partial $\mathbf{A}$-endofunctions) by using a combinatorial differential operator. This leads to an expression which contains no implicitly defined functions - at the expense of using a certain diagonalization operator. In the particular case of PLI-endofunctions this version of the generating function can be made explicit and leads to (6) above.

It should be mentioned that the first way of presenting the generating function owes much to and indeed generalizes the combinatorial approach to multivariable Lagrange inversion as proposed by I. Gessel [16] and further extended and simplified by J. Zeng [34]. The second way comes from a combinatorial interpretation of Hurwitz' parametrized Lagrange inversion formula [19], appropriately generalized to a multivariable and cycle-weighted situation.

The comparison of both ways of writing the generating function for A-endofunctions gives a result (not reproduced here) which contains many interesting special cases of varying degree of generality, e.g

- I. Gessel's multivariate Lagrange inversion formula and J. Zeng's β-generalization of it, as already mentioned;
- Joni's 'general formula' [20];
- the series expansions given by L. Carlitz e.g. in [5],[6] and [7];
- various generalizations and variants of the Pfaff-Saalschütz formula, due (with different proofs, analytical and combinatorial) to I. Gessel, D. Stanton, D. Sturtevant, I. Constantineau and myself ([17], [18], [11]);
- generating functions for the Jacobi polynomials with a linear shift in the parameters, due to H.M. Srivastava and J.P. Singhal in [28], and already treated from a combinatorial perspective by myself in [31].

All these results can be further generalized by making use of the weight function put on f-cycles in full generality.

5 PLI-endofunctions and Brock-Carlitz type identities

The most interesting application (from my point of view) of the general theory just sketched deals with the situation of PLI-endofunctions. In this case both versions of the generating function can be made completely explicit. Indeed, the implicit system (5) for the p tree generating functions $\xi(\mathbf{x}) = \xi_i(x_1, \ldots, x_p)$ $(1 \leq i \leq p)$ can be solved in terms of matching polynomials (which are, to be a bit more specific, multivariable analogues of the Chebychev polynomials). The appearance of matching polynomials in this context appears quite natural from the kind of combinatorial objects considered.

In order to state the central result, let us put for $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbf{N}^p$, parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$ and β

$$^{[p]}H(\mathrm{n}|\alpha;\beta) = \qquad (15)$$
$$\prod_{1\le i\le p} \binom{\alpha_i + \beta + n_i + n_{i-1}}{n_i} \; {}_{p+1}F_p \left[\begin{matrix} \beta \; - n_1 \ldots - n_p \\ \ldots 1 + \alpha_j + \beta + n_{j+1} \ldots \end{matrix} ; (-1)^p \right] .$$

Then the generating function for the species of p-sorted PLI-endofunctions $^{[p]}PLI(X_1, \ldots, X_p)$ (with a slight variation of the weight function) can be written as

$$^{[p]}PLI^{(\boldsymbol{\alpha};1+\beta)}(\mathrm{x}) = \sum_{\mathbf{n}} {}^{[p]}H(\mathrm{n}|\alpha;\beta)\; \mathrm{x}^{\mathbf{n}} \quad .$$

On the other hand

$$\begin{aligned} ^{[p]}PLI^{(\boldsymbol{\alpha};\beta)}(\mathrm{x}) &= \frac{\prod_{1\le i\le p}(1+\xi_i)^{\alpha_i+\beta}}{\left[1 - \prod_{1\le i\le p} \xi_i(\mathrm{x})\right]^{\beta}} \qquad (16) \\ &= \frac{\prod_{1\le i\le p}(1+\xi_i)^{\alpha_i}}{\left[c_p(\mathrm{x}) - 4\prod_{1\le i\le p} x_i\right]^{\beta/2}} \; , \end{aligned}$$

where the $\xi_i = \xi_i(\mathrm{x})$ are as in (5) above and $c_p(\mathrm{x}) = c_p(x_1, \ldots, x_p)$ is a matching polynomial for cycles of length p with weights $x_1, \ldots x_p$ put on the edges.

Here I will not give the explicit form of the functions ξ_i (which can be expressed in terms of $c_p(\mathrm{x})$ and a similar matching polynomial $l_{p-1}(\mathrm{x})$ for a line of length $p-2$). The precise knowledge is not even necessary because in order to obtain Brock-Carlitz type identities knowledge of the denominator in the above presentation is sufficient.

Now let $r = p \cdot q$, take parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_r)$ and β. Denote by $^{[p,q]}PLI(X_1, \ldots, X_p)$ the p-sorted species obtained from the r-sorted species $^{[r]}PLI(X_1, \ldots, X_r)$ by identifying sorts of vertices modulo p, i.e.

$$X_i \equiv X_j \;\text{ iff }\; i \equiv j \bmod p \quad .$$

Accordingly, let $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p)$ with

$$\tilde{\alpha}_j = \sum_{0\le i<q} \alpha_{j+i\cdot p} \quad (1 \le j \le p) \quad ,$$

and similarly, for each $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbf{N}^r$ let $\widetilde{\mathbf{m}} = (\widetilde{m}_1, \ldots, \widetilde{m}_p) \in \mathbf{N}^p$ with

$$\widetilde{m}_j = \sum_{0 \le i < q} m_{j+i \cdot q} \quad (1 \le j \le p) \ .$$

We know that the generating function for ${}^{[p,q]}PLI(X_1, \ldots, X_p)$ can be written as

$$ {}^{[p,q]}PLI^{(\alpha, 1+\beta)}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbf{N}^p} {}^{[p,q]}H(\mathbf{n}|\alpha;\beta)\, \mathbf{x}^{\mathbf{n}} \ , \tag{17}$$

where

$$ {}^{[p,q]}H(\mathbf{n}|\alpha;\beta) = \sum_{\widetilde{\mathbf{m}}=\mathbf{n}} {}^{[r]}H(\mathbf{m}|\alpha;\beta) \ ,$$

the sum on the right running over all $\mathbf{m} \in \mathbf{N}^r$ such that $\widetilde{\mathbf{m}} = \mathbf{n}$.

Note that for $q = 2, \beta = 0$ we obtain the Brock numbers $H(m, n)$ in (1) as well as Carlitz' generalizations (10),(13) mentioned in sec. 3.

With all the combinatorial machinery behind our structures properly put into action (in particular, duplication and composition properties of the matching polynomials play a rôle), one obtains from (15) the following identity for generating functions:

$$ {}^{[p,q]}PLI^{(\alpha,\beta)}(\mathbf{x}) = \frac{1}{\lambda_{q-1}(\prod_i x_i, c_p(\mathbf{x}))^{\beta}}\ {}^{[p]}PLI^{(\widetilde{\alpha},\beta)}(\mathbf{x}) \ , \tag{18}$$

where $\lambda_{q-1}(u, v)$ is another matching polynomial; indeed,

$$\lambda_n(u^2, v) = u^n \cdot U_n(v/2u) \ ,$$

where U_n is the classical Chebychev polynomial.

In order to see how this extends the Brock-Carlitz identities, consider now the case $q = 2$ and put $\beta = 1$ in (18) (which means taking $\beta = 0$ in (15) and (17)). We have $\lambda_1(u, v) = v$, so that the denominator in (18) (for $\beta = 1$) is nothing but $c_p(x_1, \ldots, x_p)$. ¿From the combinatorial signification it is obvious that

$$\begin{aligned} c_2(x_1, x_2) &= 1 - x_1 - x_2 \ , \\ c_3(x_1, x_2, x_3) &= 1 - x_1 - x_2 - x_3 \ , \\ c_4(x_1, x_2, x_3, x_4) &= 1 - x_1 - x_2 - x_3 - x_4 + x_1x_3 + x_2x_4 \ , \end{aligned}$$

which brings us back to (2) (or (9) resp.), (11) and (12).

It can be shown that the general 'rule' for obtaining Brock type identities in the case $q = 2$, established by Carlitz and extended by Singhal in the parametrized situation, is just a cryptic way of stating that the matching polynomial $c_p(x_1, \ldots, x_p)$ appears as denominator polynomial in an identity like (18) relating the appropriate generating functions.

To mention just the simplest case of a binomial identity not covered by previous results, let us put $q = 3$ and $p = 2$. We have $\lambda_2(u, v) = v^2 - u$ and $c_2(x_1, x_2)$ as above. This leads to

$$\lambda_2(x_1x_2, c_2(x_1, x_2)) = 1 - 2x_1 - 2x_2 + x_1^2 + x_1x_2 + x_2^2 \ ,$$

which shows that the numbers

$$h(m,n) = \sum_{\substack{i_1+i_3+i_5=m \\ i_2+i_4+i_6=n}} \binom{i_1+i_2}{i_1}\binom{i_2+i_3}{i_2}\cdots\binom{i_6+i_1}{i_6}$$

satisfy

$$\begin{aligned} &h(m,n) - 2h(m-1,n) - 2h(m,n-1) \\ &+h(m-2,n) + h(m-1,n-1) + h(m,n-2) = \binom{m+n}{n}^2 \ . \end{aligned}$$

Obviously, many more results can be pulled out of this general approach. The 'opposite' situation to the Brock-Carlitz case $p = 2$, namely $q = 2$ and p arbitrary is of particular interest due to its intimate relation to the Jacobi polynomials. In particular, the classical generating function for the Jacobi polynomials can be presented as the limit of a rapidly converging (in the sense of formal series) sequence of rational generating functions which are, in fact, quotients of matching polynomials, see [30] for a direct approach.

References

[1] R.M. Baer and P. Brock. Natural sorting. *SIAM Journal*, 10:284–304, 1962.

[2] P. Brock. On a binomial identity arising from a sorting problem. *SIAM Review*, 2:40, 1960. problem 60-2.

[3] L. Carlitz. A binomial identity arising from a sorting problem. *SIAM Review*, 6:20–30, 1964.

[4] L. Carlitz. Some generating functions and a binomial identity. *Duke Math. J.*, 35:541–548, 1968.

[5] L. Carlitz. An application of MacMahon's master theorem. *SIAM J. Appl. Math.*, 26:431–436, 1974.

[6] L. Carlitz. Some expansion and convolution formulas related to MacMahon's master theorem. *SIAM J. Math. Anal.*, 8:320–336, 1977.

[7] L. Carlitz. Multiple binomial and power sums. *Houston J. Math.*, 6:331–354, 1980.

[8] C.A. Church, jr. Combinatorial proof for a sorting problem identity. *Fibonacci Quart.*, 33:366–368, 1985.

[9] M.E. Cohen. Some extensions of Carlitz on cyclic sums and generating functions. *J. Math. Anal. Appl.*, 60:493–501, 1977.

[10] M.E. Cohen and H.S. Sun. Some extensions of the Brock-Carlitz identity. *Proc. Amer. Math. Soc.*, 76:178–185, 1979.

[11] I. Constantineau. A generalization of the Pfaff-Saalschütz theorem. *Studies in Appl. Math.*, 85:243–248, 1991.

[12] A.C. Dixon. Summation of a certain series. *Proc. London Math. Soc.*, 35:284–289, 1903.

[13] G.P. Egorychev. *Integral Representations and the Computation of Combinatorial Sums*, volume 59 of *Translations of Mathematical Monographs*. Amer. Math. Soc., Providence, R.I., 1984.

[14] D. Foata and J. Labelle. Modèles combinatoires pour les polynômes de Meixner. *Europ. J. Combinatorics*, 4:305–311, 1983.

[15] D. Foata and P. Leroux. Polynômes de Jacobi, interprétation combinatoire et fonction génératrice. *Proc. Amer. Math. Soc.*, 87:47–53, 1983.

[16] I. Gessel. A combinatorial proof of the multivariable Lagrange inversion formula. *J. Combin. Theory, series A*, 45:178–195, 1987.

[17] I. Gessel and D. Stanton. Short proofs of Saalschütz and Dixon's theorem. *J. Combin. Theory, series A*, 38:87–90, 1985.

[18] I. Gessel and D. Sturtevant. A combinatorial proof of Saalschütz' theorem. preprint, 1989.

[19] A. Hurwitz. Über einige Verallgemeinerungen der Leibniz'schen Differentiationsformel und des polynomischen Lehrsatzes. *Z. Math. Phys.*, 35:56–58, 1890.

[20] S.A. Joni. Expansion formulas I. A general method. *J. Math. Anal. Appl.*, 81:364–377, 1981.

[21] J. Labelle and Y.-N. Yeh. A combinatorial model for Hahn polynomials. In D. Foata, editor, *Actes Séminaire Lotharingien de Combinatoire , 16 session*, pages 97–107. Publications de l' IRMA, Strasbourg, 341/S–16, 1987.

[22] J. Labelle and Y.-N. Yeh. The combinatorics of Laguerre, Charlier and Hermite polynomials. *Studies in Appl. Math.*, 80:25–36, 1989.

[23] P. Leroux and V. Strehl. Jacobi polynomials: combinatorics of the basic identities. *Discrete Mathematics*, 57:167–187, 1985.

[24] E. Rainville. *Special Functions.* Chelsea, Bronx, N.Y., 1971.

[25] J. Riordan. *Combinatorial Identities.* J. Wiley, New York, 1968.

[26] J.P. Singhal. A generating function and a binomial identity. *Duke Math. J.*, 39:25–30, 1972.

[27] D. Slepian. Solution to problem 60-2. *SIAM Review*, 4:396–398, 1962.

[28] H.M. Srivastava and J.P. Singhal. New generating functions for Jacobi and related polynomials. *J. Math. Anal. Appl.*, 41:725–748, 1972.

[29] V. Strehl. Combinatorics of Jacobi-configurations I : Complete oriented matchings. In G. Labelle et al., editor, *Combinatoire Énumérative, Montréal 1985*, pages 294–307, Berlin, 1986. Springer-Verlag. (*Lecture Notes in Math., vol. 1234*).

[30] V. Strehl. Combinatorics of Jacobi-configurations II : A rational approximation via matching polynomials. In G. Nicoletti, editor, *Actes Séminaire Lotharingien de Combinatoire , 13e session*, pages 112–133. Publ. IRMA Strasbourg, 316/S–13, 1986.

[31] V. Strehl. Combinatorics of Jacobi-configurations III : The Srivastava-Singhal generating relation revisited. *Discrete Math.*, 73:221–232, 1988.

[32] V. Strehl. Endliche Funktionen und spezielle Funktionen : eine kombinatorische Theorie zu einer Identität von P. Brock und deren Erweiterungen. In D. Foata, editor, *Actes Séminaire Lotharingien de Combinatoire , 17e session*, pages 23–67. Publ. IRMA Strasbourg, 348/S–17, 1988.

[33] V. Strehl. *Zykel-Enumeration bei lokal-strukturierten Funktionen.* Universität Erlangen-Nürnberg, 1990.

[34] J. Zeng. La β-extension de la formule d'inversion de Lagrange à plusieurs variables. In L. Cerlienco and D. Foata, editors, *Actes Séminaire Lotharingien de Combinatoire, 20e session*, pages 23–38, Strasbourg, 1988. Publications de l' I.R.M.A., 372/S-20.