The homology representation of the symmetric group on Cohen-Macaulay subposets of the partition lattice

Extended Abstract

Sheila Sundaram

Université du Québec à Montréal
Montréal H3C 3P8, Québec, Canada

0. Introduction

The action of the symmetric group S_n on the top homology $\tilde{H}_{n-3}(\Pi_n)$ of the partition lattice Π_n has been studied from various viewpoints in recent years, beginning with a character computation by Hanlon ([H1]). Following questions raised by Stanley ([St]), in this paper we consider the S_n-representation on the homology of certain Cohen-Macaulay subposets of Π_n. We present a general technique for manipulating these homology modules. The unique properties of the partition lattice allow further simplification of these formulas, culminating in plethystic generating functions which, by recursive computation, yield the Frobenius characteristic of the representation. We illustrate our technique by giving simple derivations of three known formulas:

1. a formula for the plethystic inverse of the sum of the cycle indicators of the symmetric groups; this is essentially equivalent to Cadogan's formula ([C]);
2. a plethystic formula which determines the characteristic of the homology representation on the lattice Π_n^d of partitions of a set of size nd all of whose blocks are multiples of d; this formula was originally derived in a paper of Calderbank, Hanlon and Robinson ([CHR]);
3. a plethystic generating function identity, also due to these authors, which determines the homology representation on the subposet $\Pi_n^{(i,d)}$ of partitions of n elements into blocks of size congruent to i mod d.

The idea behind our basic observation, Lemma 2.1, is implicitly used to calculate the Möbius functions of fixed-point posets by Calderbank, Hanlon and Robinson ([CHR], Theorem 2.2). Our approach has the advantage of being more conceptual from the representation-theoretic point of view: we avoid intricate Möbius function computations by manipulating virtual homology modules, and by exploiting the considerable machinery of symmetric function theory as presented by Macdonald.

Our main result (Theorem 1.12) is a recursive formula for the characteristic of the homology representation on an arbitrary rank-selected subposet of Π_n. This formula is easily adapted to handle rank-selection in the posets Π_n^d and $\Pi_n^{(1,d)}$. In particular, our methods give elegant plethystic formulas for the characteristic of the homology representation on the following rank-selected subposets of Π_n :

1. when the ranks are selected in arithmetic progression: in particular for initial or final segments of consecutive ranks;
2. when the subposet is obtained by deleting a single rank from Π_n;
3. when the subposet consists of arbitrarily placed segments of consecutive ranks in Π_n;
4. when the subposet is as in 3. but with one rank deleted.

Using these formulas we can extract a considerable amount of concrete information about the representation.

The same idea allows us to give recursive formulas for the permutation representation on the rank-selected chains of Π_n. In particular we determine completely the orbit decomposition of the representation on the maximal chains of Π_n. The multiplicities we obtain turn out to enumerate a known refinement of the Euler numbers. In fact we show that the action of S_n on the maximal chains is closely related to an action of S_2^{n-1} on the set of $n!$ permutations, defined by Foata and Strehl ([FS1], [FS2]) in their enumerative work on the Euler numbers.

A result of Stanley ([St]) states that when all the rank-selected homologies of the partition lattice are combined into one S_n-module, the multiplicity of the trivial representation in this direct sum is the Euler number E_{n-1} of alternating permutations in S_{n-1}. Our computations enable us to describe combinatorially the multiplicity of the trivial representation in the homology module in some specific cases of rank-selection, although it is as yet unclear how these multiplicities enumerate a refinement of the alternating permutations. By examining the restrictions of these S_n-homology modules to S_{n-1}, we obtain a second (different) refinement of the Euler number E_n into nonnegative integers also indexed by subsets.

Our methods apply to the posets Π_n^d and $\Pi_n^{(i,d)}$ as well. As a by-product of our study we show that one can associate to these posets a vector-space structure which strikingly resembles that of the Orlik-Solomon algebra for Π_n.

Since we are interested primarily in group actions, all homology will be taken over the complex field.

1. Results

Our results are described most conveniently by using the theory of symmetric functions (see [M]). Recall that the complete homogeneous symmetric function h_n is the Frobenius characteristic of the trivial representation of the symmetric group S_n, while the elementary symmetric function e_n is the characteristic of the sign representation. We shall write π_n for the characteristic of the top homology module $\tilde{H}_{n-3}(\Pi_n)$ of the partition lattice. Following [M], denote by ω the involution on the ring of symmetric functions which takes h_n to e_n. A well known result on the homology of Π_n, which follows from Hanlon's computation of the character values of π_n and earlier results of Witt and Brandt, states that $\pi_n = \omega(\ell_n)$, where ℓ_n is the characteristic of the S_n-representation on the nth graded component of the free Lie algebra. Joyal gives a direct functorial proof of this result in [J].

The plethysm operation occurs naturally in connection with the partition lattice; for symmetric functions f and g we write $f[g]$ for the plethysm of f with g.

We begin by looking at particular cases of rank-selection. The partition lattice is known to be Cohen-Macaulay, and this property is preserved by rank-selection. The problem is therefore to determine the S_n-module structure of the unique non-vanishing reduced homology (in the highest degree) of the rank-selected subposet.

Write $\Pi_n(\underline{r})$ for the subposet consisting of the first r ranks of Π_n, excluding the 0-element at rank 0. Also, for any Cohen-Macaulay poset P, we generally suppress the homology degree and write $\tilde{H}(P)$ for the (reduced) top homology of P. Denote by $V_n(\underline{r})$ the S_n-representation on the top homology of $\Pi_n(\underline{r})$. (Define $V_n(\underline{0}) = h_n$; if $r > n-2$ or $r < 0$, set $V_n(\underline{r}) = 0$.)

Theorem 1.1

1. The characteristic of the representation $V_n(\underline{r})$ is given by the degree n term in the plethysm

$$(-1)^r(h_{n-r} + h_{n-r+1} + \ldots + h_n)[\pi_1 - \pi_2 + \ldots + (-1)^{n-1}\pi_n].$$

2. Using down and up arrows to indicate respectively restriction and induction of modules, we have the following recursive property relating the S_n-module $V_n(\underline{r})$ to its structure as an S_{n-1}-module:

$$V_n(\underline{r})\downarrow_{S_{n-1}} \simeq V_{n-1}(\underline{r}) \oplus V_{n-1}(\underline{r-1})\downarrow_{S_{n-2}}\uparrow^{S_{n-1}},$$

for all $r = 1, \ldots, n-2$.

3. Let σ be a permutation in S_n of type $\prod_i i^{m_i}$, that is, with m_i cycles of length i. The generating function for the character values $tr(\sigma)$ of σ on $V_n(\underline{r})$, $0 \le r \le n-2$, is

$$\sum_{r=0}^{n-2} tr(\sigma)|_{V_n(\underline{r})}\, u^r = \frac{1}{1+u} \prod_i \prod_{k=0}^{m_i-1} \left(\sum_{d|i} \mu(d) u^{i-i/d} - ki \right).$$

The expression $(1+u)\sum_{r=0}^{n-2} tr(\sigma)|_{V_n(\underline{r})}\, u^r$ is computed in [OS] (p. 183, Example 4.10) for small values of n. Note that this is essentially a kind of "characteristic polynomial "of the subposet of Π_n fixed by σ. Also note that in general the factors in the numerator of the right hand side are not linear.

Next let $V_n(\bar{r})$ denote the subposet obtained by selecting the top r ranks of Π_n, excluding the top element of rank $(n-1)$. The ranks chosen are thus $n-1-r, n-r, \ldots, n-3, n-2$. We have

Theorem 1.2

1. The characteristic of the homology representation of $V_n(\bar{r})$ is given by the degree n term in the plethysm

$$(\pi_{r+1} - \pi_r + \ldots + (-1)^r \pi_1)[h_1 + h_2 + \ldots + h_n].$$

2. The restriction of $V_n(\bar{r})$ to S_{n-1} is a permutation module, whose orbit decomposition is given by the characteristic

$$\sum_{\substack{\lambda \vdash (n-1), \ell(\lambda) = r+1 \\ \lambda = 1^{m_1} 2^{m_2} \ldots}} \binom{r+1}{m_1, m_2, \ldots} h_\lambda,$$

where the sum runs over all partitions λ of $n-1$ of length $r+1$, and m_i denotes the multiplicity of the part i in λ.

Now let $d \ge 2$, and let R_k denote the characteristic of the S_k-representation on the top homology of the subposet of Π_k obtained by selecting all ranks $i \ge 1$ such that $k-i$ is a multiple of d. (Recall that the rank of a partition x in Π_n is n minus the number of blocks in x; the subposet therefore consists of all partitions with *number* of blocks divisible by d). For $1 \le i \le d-1$, define $R_i = h_i$. Note that $R_d = h_d$, the poset in this case being empty.

The following plethystic identity completely determines the characteristics R_k :

Theorem 1.3 Assume $d \ge 2$. Then

$$\sum_{n \ge 0} (-1)^n \sum_{i=1}^{d-1} R_{nd+i} = (h_1 - R_d + R_{2d} - \ldots)[h_1 + h_2 + \ldots].$$

Let $Q_k(i)$ be the characteristic of the S_k action on the top homology of the subposet of Π_k obtained by selecting those ranks corresponding to partitions such that the number of blocks is congruent to i mod d. Define $Q_i(i) = h_i$ and for $1 \le j \le d-1$, set $Q_j(i) = h_j$.

One has the following plethystic generating functions:

Theorem 1.4 Let $d \geq 2$.

1. For $i = 2, \ldots, d-1$:

$$(h_1 - Q_i(i) + Q_{i+d}(i) - Q_{i+2d}(i) + \ldots + (-1)^n Q_{i+d(n-1)}(i) + \ldots)[\sum_{i\geq 1} h_i]$$
$$= \sum_{n\geq 0}(-1)^n \sum_{0\leq j\leq d-1, j\neq i} Q_{nd+j}(i)$$

2. For $i = 1$:

$$(h_1 - Q_{1+d}(1) + Q_{1+2d}(1) - \ldots + (-1)^{n-1} Q_{1+(n-1)d} + \ldots)[\sum_{i\geq 1} h_i]$$
$$= \sum_{n\geq 1}(-1)^{n-1} \sum_{0\leq j\leq d-1, j\neq 1} Q_{nd+j}(1).$$

Theorem 1.5 Let Q_k denote the poset obtained from Π_n by deleting all elements at rank k. The characteristic of S_n acting on the top homology of Q_k is given by the degree n term in the symmetric function

$$(-1)^k \pi_{n-k}[\pi_1 - \pi_2 + \ldots + (-1)^{n-1}\pi_n] - \pi_n.$$

For instance, one easily concludes from this formula that the homology of the subposet obtained by deleting all the atoms is given by the symmetric function $h_2 h_1^{n-2} - \pi_n$.

It is not hard to describe how to compute the representation when two ranks are selected. Let $a < b$ be two ranks chosen among the $n-2$ nontrivial ranks $\{1, 2, \ldots, n-2\}$ of Π_n. Denote the corresponding rank-selected subposet by $\Pi_n\{a, b\}$, and the characteristic of the homology representation by $\beta_n(\{a, b\})$. We will give two equivalent formulas for $\beta_n(\{a, b\})$.

For each integer partition λ of n with $n-b$ parts, we define the symmetric function

$$G_a^\lambda(n) = \sum_{(\alpha^{(1)}, \ldots, \alpha^{(s)}, \ldots)} \prod_s \prod_{k=1}^{s} h_{m_k(\alpha^{(s)})}[\, h_k[\sum_{i\geq 1} h_i]|_{deg\ s}\,],$$

where the sum runs over all sequences of integer partitions $(\alpha^{(1)}, \ldots, \alpha^{(s)}, \ldots)$ such that

1. $\alpha^{(i)}$ is a nonempty partition iff the part i appears in λ with positive multiplicity $m_i(\lambda) > 0$;
2. every part of $\alpha^{(i)}$ is of size at most i;
3. $\alpha^{(i)}$ has exactly $m_i(\lambda)$ parts;
4. writing $|\alpha^{(i)}|$ for the sum of the parts of $\alpha^{(i)}$, we must have $\sum_i |\alpha^{(i)}| = n - a$.

Note that the degree of $G_a^\lambda(n)$ is $\sum_{\{s: m_s(\lambda)>0\}} s \sum_{k=1}^{s} m_k(\alpha^{(s)}) = \sum_{\{s: m_s(\lambda)>0\}} s\ m_s(\lambda) = n$. Now define $G_a^b(n) = \sum_{\{\lambda\vdash n,\ \ell(\lambda)=n-b\}} G_a^\lambda(n)$. When $a = b$, this expression is still meaningful, and simplifies considerably to give $G_a^a(n) = h_{n-a}[\sum_{i\geq 1} h_i]|_{deg\ n}$.

Theorem 1.6

0. The characteristic $\beta_n(\{a\})$ of the homology of $\Pi_n\{a\} = \{x \in \Pi_n : x \text{ is of rank } a\}$ is

$$G_a^a(n) - h_n.$$

1. The characteristic $\beta_n(\{a,b\})$ of the homology representation of $\Pi_n\{a,b\}$ is

$$G_a^b(n) - G_a^a(n) - G_b^b(n) + h_n.$$

2. We also have the simpler recurrence

$$\beta_n(\{a,b\}) + \beta_n(\{b\}) = \beta_{n-a}(\{b-a\})[\sum_{i\geq 1} h_i]|_{deg\, n}.$$

We turn next to the more general case of segments of consecutive ranks. For $1 \leq k \leq r \leq n-2$ define symmetric functions g_{n-k}^r by the formula

$$g_{n-k}^r = (-1)^{k-r}(h_{n-r} + h_{n-r+1} + \ldots + h_{n-k})[\pi_1 - \pi_2 + \ldots + (-1)^{n-k-1}\pi_{n-k}]|_{deg\,(n-k)}.$$

Comparison with Theorem 1.1 shows that g_{n-k}^r is in fact the characteristic of the S_{n-k}-module $V_{n-k}(\underline{r-k})$. Thus $g_n^r = ch(V_n(\underline{r})) = ch(V_n([1,r]))$, while $g_{n-r}^r = h_{n-r}$. For $0 \leq r < s \leq n-2$, denote by $V_n([r+1,s])$ the reduced top homology of the subposet of Π_n obtained by selecting all elements in ranks $r+1, r+2, \ldots, s$. By convention $V_n([0,s]) = 0$, while $V_n([s+1,s])$ is the trivial module. We are now ready to give a recursive formula for computing $V_n([r+1,s])$ as an S_n-module.

Theorem 1.7 With the preceding definitions, writing $ch(V)$ for the Frobenius characteristic of the S_n-module V, one has, for $0 \leq a \leq r \leq n-2$:

$$ch(V_n[a+1,r]) + ch(V_n[a,r]) = g_{n-a}^r[\sum_{i\geq 1} h_i]|_{deg\, n}.$$

Consequently one immediately has the formulas

$$\begin{aligned} &ch(V_n([a,r])) \\ &= (g_{n-a}^r - g_{n-a-1}^r + \ldots + (-1)^{r-a} g_{n-r}^r)[\sum_{i\geq 1} h_i]|_{deg\, n} - (-1)^{r-a} h_n \\ &= (g_{n-a+1}^r - g_{n-a+2}^r + \ldots + (-1)^{k-1} g_{n-a+k}^r + \ldots + (-1)^{a-1} g_{n-r}^r)[\sum_{i\geq 1} h_i]|_{deg\, n}. \end{aligned}$$

Theorem 1.8 The homology representation of the subposet of Π_n consisting of the interval $[1,r]$ with the rank k removed, is given by

$$(-1)^k g_{n-k}^r[\pi_1 - \pi_2 + \ldots]|_{deg\, n} - V_n([1,r]).$$

At the end of the next section we give an alternative computation for $V_n([2,n-3])$.

Using these results and the underlying methods, one can also obtain information about the multiplicity of the trivial representation (and other irreducibles in specific instances) in the homology. As an example, either directly from Part (1) or by recursively applying Part (2) of Theorem 1.1, one concludes that the trivial representation does not appear in $V_n(\underline{r})$, for any $r \geq 2$. This was first proved by Hanlon ([H2]) by more intricate calculations. We now present two theorems which strengthen this result:

Theorem 1.9 Consider the rank-selected subposet $\Pi_n(S)$, where the set S of ranks chosen is of the form $[1,r] \cup \{a\}$, for $a \geq r \geq 2$. Then the trivial representation does not occur in $\tilde{H}(\Pi_n(S))$, provided one of the following conditions is satisfied:
(1) $n \leq 3r+2$;
(2) $a < 2(r+1)$, or $a \geq n-r$; or $r \geq 3$ and $a \geq n-r-1$;
(3) $r < n-a < \binom{r+1}{2}$.

One is tempted to conjecture that the trivial representation does not appear for large subsets S of ranks which contain an interval of the form $[1,r]$, of length at least 2. That this is false is seen by using Theorems 1.5 and 1.8:

Theorem 1.10

1. Let Q_k denote the subposet $\Pi_n - \{\text{elements at rank } k\}$. The trivial representation does not occur in the homology of Q_k if $n \leq 2k$. Otherwise it appears with nonzero multiplicity equal to

$$\frac{1}{n-k} \sum_{d|(k,n-k)} \mu(d)(-1)^{n-\frac{n}{d}} \binom{\frac{n-k}{d}}{\frac{k}{d}}.$$

This is also the number of standard Young tableaux of hook shape $(k, 1^{n-2k})$ or $(k+1, 1^{n-2k-1})$ whose major index is congruent to 1 modulo $n-k$. (See [KW]). For some special values of n and k, this expression has a more elegant combinatorial interpretation which follows from work of Gessel and Reutenauer ([GR], Theorem 9.4). Let $C_{m,i}$ denote the number of m cycles in S_m with a unique descent, in position i. Then (for $n > 2k$) the above multiplicity equals $C_{n-k,k} + C_{\frac{n-k}{2},\frac{k}{2}}$ if n and k are both twice an odd number, and it equals $C_{n-k,k}$ otherwise. (It follows for instance that if $k = 1$ or $n = 2k+1$, the multiplicity is always 1).

2. The trivial representation does not occur in the homology of the subposets obtained by taking a segment of consecutive ranks $[1,r]$, and then deleting a rank k, provided $n \leq 2k (\leq 2r)$, or if $n = 2k+1$ and $r \leq n-3$.

Similarly, using Theorem 1.7, we obtain

Theorem 1.11

1. The multiplicity of the trivial representation in the homology of the subposet consisting of a segment of consecutive ranks, beginning with rank 2, is always 1;
2. The multiplicity of the trivial representation in the homology of the subposet consisting of a segment of consecutive ranks, beginning with rank 3, that is, in $V_n([3,r])$, is given below. First let $r = n-2$. Then the multiplicity is

$$\begin{cases} \lfloor \frac{n}{2} \rfloor - 1 & \text{if } n \text{ is odd or if } n \equiv 0 \bmod 4; \\ \frac{n}{2} - 2 & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Now assume $r \leq n-3$. Then the multiplicity is

$$\begin{cases} r-1 & \text{if } r \equiv 2 \bmod 4 \text{ or } r \equiv 3 \bmod 4; \\ r-2 & \text{if } n \equiv 0 \bmod 4 \text{ or } r \equiv 1 \bmod 4. \end{cases}$$

The preceding formulas were obtained by technical calculations with symmetric functions. The proof of Theorem 1.11 also required knowledge of the behaviour of the S_n-modules $V_n(\underline{r})$

upon restriction to $S_2 \times S_{n-2}$. The ultimate simplicity of these formulas is surprising, given the relative intricacy of our plethystic expressions in Theorems 1.7 and 1.8.

The homology computations discussed above are particular cases of the following recurrence for arbitrary rank-selection:

Theorem 1.12 Let $S = \{s_1 < s_2 < \ldots < s_r\}$ be a subset of the ranks $\{1, \ldots, n-2\}$ of Π_n, and denote by $\beta_n(S)$ the Frobenius characteristic of the homology of the rank-selected subposet of Π_n corresponding to the set S. We have the recurrence

$$\beta_n(S) + \beta_n(S \backslash \{s_1\}) = \beta_{n-s_1}(S - s_1)[\sum_{i \geq 1} h_i]|_{deg\ n},$$

where $S - s_1$ denotes the subset of ranks $\{s_2 - s_1 < s_3 - s_1 < \ldots < s_r - s_1\}$ in Π_{n-s_1}.

Denote by $\alpha_n(S)$ the permutation representation of S_n acting on the maximal chains of the rank-selected subposet $(\Pi_n)_S$ of Π_n corresponding to the subset S. A standard Hopf trace formula computation shows that

$$\beta_n(T) = \sum_{S \subseteq T} \alpha_n(S)(-1)^{|T|-|S|},$$

thereby expressing the homology module as an alternating sum of the permutation modules of chains. This is valid for any Cohen-Macaulay poset. Theorem 1.12 in effect gives an expression for the homology modules β_n as an alternating sum of *homology* modules of lower degree. An analogue of Theorem 1.12 holds for arbitrary Cohen-Macaulay posets; see Section 2, Theorem 2.8.

The permutation representation $\alpha_n(S)$ is consequently the sum of the homology modules $\beta_n(T)$, as T ranges over subsets of S. That is, the homology modules $\beta_n(T)$, $T \subseteq S$, *refine* the permutation module $\alpha_n(S)$ of maximal chains. This permutation representation is frequently interesting in its own right. In the case of the partition lattice, Theorem 1.12 gives

Corollary 1.13 $\alpha_n(S) = \alpha_{n-s_1}(S - s_1)[\sum_{i \geq 1} h_i]|_{deg\ n}$.

In particular, we obtain an elegant orbit decomposition for the action α_n of S_n on the maximal chains of Π_n :

Theorem 1.14

$$\alpha_n = \sum_{i=1}^{\lfloor n/2 \rfloor} a_i(n) \ 1\uparrow_{S_2^i \times S_1^{n-2i}}^{S_n},$$

where the $a_i(n)$ are determined by the recurrence

$$a_i(n+1) = i a_i(n) + (n - 2i + 2) a_{i-1}(n),$$

with initial conditions $a_0(1) = 1 = a_1(2)$, $a_0(n) = 0$, $n > 1$, and $a_i(n) = 0$ if $2i > n$. (Observe that the character values are supported on the set of involutions).

The proof consists of translating Corollary 1.13 into the following recursive description of the α_n :

$$\alpha_n = \left(1_{S_2} \otimes \alpha_{n-1}\downarrow_{S_{n-2}}\right)\uparrow^{S_n}.$$

The statement of the theorem is then obtained by an inductive computation. Note that the point stabilisers of this action are all Young subgroups of the form $S_2^i \times S_1^{n-2i}$.

The recurrence for the $a_i(n)$ is remarkably similar to the one satisfied by the Eulerian numbers $A(n,i)$ (which count the number of permutations in S_n with i descents). Based on this observation, R. Simion found the following combinatorial description for the multiplicity $a_i(n)$: It is the number of permutations in S_{n-2} with exactly $i-1$ descents, none consecutive, and with the "hereditary"property that when the letters $n-2$, $n-3, \ldots 3, 2, 1$ are erased in succession, the property of non-consecutive descents is preserved after each erasure. A slightly different but equinumerous set of permutations appears in work of Foata and Schützenberger ([FSch]), who call them André permutations. In fact in two subsequent papers ([FS1], [FS2]), Foata and Strehl construct an action of S_2^{n-1} on the set of $n!$ permutations whose orbit decomposition is remarkably reminiscent of ours. The precise connection between the Foata-Strehl representation $W_{FS}(n)$ of S_2^{n-1} and the action α_n of S_n on the maximal chains of Π_n is as follows:

Proposition 1.15

$$\alpha_{n+1}\uparrow_{S_{n+1}}^{S_{2n}} = (W_{FS}(n) \otimes 1_{S_2})\uparrow_{S_2{}^{n-1}\times S_2}^{S_{2n}}.$$

In [St], Stanley shows that the multiplicity of the trivial representation in α_n is the Euler number E_{n-1}. It follows from Theorem 1.14 that the André permutations in S_{n-1} counted by $a_i(n)$ refine the number of alternating permutations E_{n-1} in S_{n-1}. Such refinements are the subject of the papers [FSch], [FS1], [FS2]. In particular, in [FSch] the authors give explicit bijections to explain these refinements. The recurrences of [FSch] for the numbers $a_i(n)$ can all be derived by counting relative orbits of maximal chains in Π_n.

Since the rank-selected homologies $\beta_{n+1}(S)$ (in Π_{n+1}) refine the S_{n+1}-module α_{n+1}, the multiplicities $b_S(n+1)$ of the trivial representation in the homology $\beta_{n+1}(S)$ provide another refinement of the Euler number E_n into nonnegative integers indexed by subsets $S \subseteq \{1, 2, \ldots, n-1\}$. More generally, if we write $b_S(\lambda)$ for the multiplicity of the irreducible indexed by the partition λ of n in the homology of the rank-selected subposet $(\Pi_n)_S$, and $A_n(\lambda)$ for the multiplicity of the irreducible λ in α_n, then it follows that $\sum_S b_S(\lambda) = A_n(\lambda)$, where the sum runs over all subsets of $\{1, 2, \ldots, n-2\}$.

We use Theorem 1.14 to compute the multiplicities of certain irreducibles $A_n(\lambda)$ in the permutation representation α_n.

Proposition 1.16

0. ([St]) $A_n((n)) = E_{n-1}$.
1. The multiplicity of the trivial representation in the *restriction* of α_n to S_{n-1} is the Euler number E_n. Equivalently, $A_n((n-1,1)) = E_n - E_{n-1}$.
2. $A_n((3, 1^{n-3})) = 2^{n-2} - 1$;
3. $A_n((2^2, 1^{n-4})) = 2^{n-2} - 2$;
4. $A_n((2, 1^{n-2})) = 1$ and $A_n((1^n)) = 0$ for all n.

The preceding proposition suggests that the irreducible $(3, 1^{n-3})$ appears exactly once in each rank-selected homology corresponding to a nonempty subset S. This is, however, false, as can be seen by examining the top homology π_n itself: in π_6, this irreducible appears twice. There is consequently a different refinement of 2^{n-2}, also indexed by subsets. We also conclude that the irreducible $(2, 1^{n-2})$ appears exactly once in π_n, and never in any $\beta_n(S)$ for $S \neq \{1, 2, \ldots, n-2\}$.

In view of Part 1. of Propostion 1.16, it is also of enumerative interest to determine the restrictions of the rank-selected homology modules of Π_n to S_{n-1}. For, denoting by $b'_S(n)$ the multiplicity of the trivial representation in this restricted rank-selected homology module $\beta_n(S)\downarrow_{S_{n-1}}^{S_n}$, one has a *second* refinement of the Euler number E_n into nonnegative integers, now indexed by subsets $S \subseteq \{1, 2, \ldots, n-2\}$. Note that $b'_S(n)$ is always greater than or equal to $b_S(n)$.

Finally we remark that from Theorems 1.1 and 1.2 we can compute $b'_n(S)$ when S is an initial or final segment of consecutive ranks. We record these and other assorted results in

Proposition 1.17

0. Let $1 \le r \le n-2$. Then $b_{\{r\}}(n) = p(n, n-r) - 1$, and $b'_{\{r\}}(n) = \sum_{i=n-r}^{n-1} p(i, n-r-1)$, (where $p(m,k)$ denotes the number of integer partitions of m into k parts).
1. Let $S = \{1, r\}$, $1 < r \le n-2$. Then $b_S(n) = b'_{\{r-1\}}(n-1) - b_{\{r\}}(n)$.
2. Let S be the interval $[1, r]$ of ranks in Π_n. Then $b_S(n) = 0$, and $b'_S(n) = 1$ for all $r = 1, \ldots, n-2$.
3. Let S be the interval $[n-1-r, n-2]$ of ranks in Π_n. Then for all $r = 1, \ldots, n-2$,
$$b'_S(n) = \binom{n-2}{r}.$$
4. Let $S = [1, n-2] \backslash \{k\}$. Then
$$b'_S(n) = \sum_{r=0}^{min(n-k-1,k)} \binom{n-k-1}{r} - 1.$$
In particular, if $2k > n-1$, then $b'_S(n) = 2^{n-k-1} - 1$ (while $b_S(n) = 0$ by Theorem 1.10).
5. Let $n \ge 6$, $S = [2, n-3]$. Then $b'_S(n) = 2n - 7$.

2. Methods and further results

Let P be a poset with least element 0. Following Baclawski, let $D_i(P)$ be the free abelian group generated by all $i-1$-chains $a_1 < \ldots < a_i$ of elements of $P - \{0\}$. Set $D_{-1}(P) = 0$. For $i > 0$ and $D_i(P) \neq 0$, define a differential $d_i^W : D_i(P) \longrightarrow D_{i-1}(P)$ by
$$d_i^W(a_1, \ldots, a_i) = \sum_{j=1}^{i-1} (-1)^i (a_1, \ldots, \hat{a}_j, \ldots, a_i);$$
define $d_i^W = 0$ otherwise. (As usual the hat over an element in the chain denotes suppression of that element).

The homology of the algebraic complex $(D(P), d^W)$ is the *Whitney* homology of P. It was defined and studied first by Baclawski ([Bal]) and then by Björner ([Bj]), who related it to the usual order homology of the poset P. We write $WH_i(P)$ for the ith Whitney homology group, that is, $WH_i(P) = ker\ d_i^W / im\ d_{i+1}^W$. Note that if P has a least element 0 and a greatest element 1, the Whitney homology in the highest degree coincides with the top homology of the order complex of $P - \{0, 1\}$.

A fundamental theorem is that, when P is Cohen-Macaulay, the homology groups of the complex $(D(P), d^W)$ are free. This was proved by Baclawski for geometric lattices ([Bal]), and follows from work of Björner ([Bj]) in the general case.

Our first basic result is:

Lemma 2.1 Let P be a poset with least element 0 and greatest element 1; let G be the automorphism group of P. Assume that the Whitney homology groups are free in all degrees. Then each Whitney homology is a G-module. If r is the length of the longest chain in $P - \{0\}$, then as a virtual sum of G-modules, one has
$$WH_r(P) - WH_{r-1}(P) + \ldots + (-1)^{r-1} WH_1(P) + (-1)^r WH_0(P) = 0.$$

Since $WH_r(P) = \tilde{H}_{r-2}(P)$, ($P$ has a greatest element 1), one has, equivalently, the equality of G-modules

$$\tilde{H}_{r-2}(P) = WH_{r-1}(P) - WH_{r-2}(P) + \ldots + (-1)^{r-1} WH_0(P).$$

The proof follows from a routine application of the Hopf trace formula to the Whitney complex $(D(P), d^W)$. By combining this lemma with work of Björner, we obtain a powerful tool for computing the top homology of the order complex of P in the case when the poset P is Cohen-Macaulay.

To apply this lemma to obtain our results, we need to compute the Whitney homology of the partition lattice, as an S_n-module. For a finite poset P with 0, write $WH(P)$ for the direct sum of all the Whitney homology groups of P. A theorem of Orlik and Solomon states that $WH(\Pi_n)$ is isomorphic to the cohomology ring of the complement of the thick diagonal in $\mathbf{C}_n$, and consequently has the structure of a graded anti-commutative algebra, in this case the Orlik-Solomon algebra for the root system A_{n-1} (See [OS], Corollary 5.6). In fact Orlik and Solomon show that for any geometric lattice, the associated graded anti-commutative algebra satisfies the acyclicity property of Lemma 2.1. (See [OS], Lemma 2.18). Our Lemma 2.1 points out that this is true in the more general context of the Whitney homology of any poset. Orlik and Solomon also show that for any geometric lattice L, the Whitney homology $W(L)$ does indeed have the structure of a graded anti-commutative algebra.

The S_n-module structure of $WH(\Pi_n)$ was determined by Lehrer and Solomon, (and in somewhat more general terms in [OS]), although this information can also be extracted from previous calculations of Hanlon. We state Lehrer and Solomon's result below, in a form more suited to our requirements:

Theorem 2.2 ([LS], Theorem 4.5) For $i = 0, 1, \ldots n-1$, the characteristic of the ith Whitney homology of Π_n, is given by

$$\sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=n-i \\ \lambda=(1^{m_1}, 2^{m_2}, \ldots)}} h_{m_1}[\pi_1] e_{m_2}[\pi_2] \ldots e_{m_{2i}}[\pi_{2i}] h_{m_{2i+1}}[\pi_{2i+1}] \ldots,$$

the sum ranging over all (integer) partitions λ of n with exactly $n-i$ parts; m_i denotes the multiplicity of the part i.

We can give a direct proof of a more general result on the module structure of any interval in Π_n.

The results presented in Theorems 1.1 and 1.2 depend on the following preliminary observation:

Proposition 2.3 Assume P is a Cohen-Macaulay poset of rank n; denote by $P(\underline{r})$ the subposet of P obtained by selecting ranks $\{1, \ldots r\}$ in P. Let G be the automorphism group of P. Then the Whitney homology of $P(\underline{r})$ coincides with that of P in degrees $0, 1, \ldots r$. In particular, the order homology of $P(\underline{r})$ is completely determined by the Whitney homology of P, by means of the G-equivariant recurrence

$$\tilde{H}(P(\underline{r})) \oplus \tilde{H}(P(\underline{r-1})) = WH_r(P).$$

Here $1 \le r \le n-1$, and by definition $\tilde{H}(P(\underline{0})) = WH_0(P)$, while $\tilde{H}(P(\underline{r})) = 0$ if $r < 0$ or $r > n-1$. Using Björner's characterisation of $WH_r(P)$ in terms of the order homology of P ([Bj]), we can construct a surjective group-equivariant map from $WH_r(P)$ onto $\tilde{H}(P(\underline{r-1}))$, whose kernel is $\tilde{H}(P(\underline{r}))$.

Part (1) of Theorem 1.1 is proved using Lehrer and Solomon's result in conjunction with the basic Lemma 2.1. The second part follows by suitable manipulation of the symmetric functions in Part (1). Alternatively, noting that the subposet $\Pi_n(\underline{r})$ is also a geometric lattice, we use Björner's theory of NBC bases to construct a basis for the homology which makes the statement transparent for all values of r except $r = n-2$. In the latter case we are dealing with the top homology of the partition lattice, and the statement, which is equivalent to the fact that S_{n-1} simply permutes the $(n-1)!$ homology spheres in Π_n, is then due to Stanley ([St]). Curiously enough, this fact is not obvious from Björner's NBC basis for homology. In [Wa], Wachs constructs a new basis which does reflect this property.

To prove Theorem 1.2, we compute the Whitney homology of the *dual* of the partition lattice. (Although the dual is no longer a geometric lattice, it is still Cohen-Macaulay). Write $WH_r^*(\Pi_n)$ for the rth Whitney homology of the dual poset of Π_n. We have

Theorem 2.4 The characteristic of the S_n-module $WH_r^*(\Pi_n)$ is given by the degree n term in the plethysm

$$\pi_{r+1}[h_1 + h_2 + \ldots],\ r = 0, 1, \ldots, n-1.$$

The general technique therefore consists of first determining the Whitney homology of the rank-selected subposet, and then applying Lemma 2.1.

Examining Lemma 2.1 more carefully, we find that, when combined with Theorems 2.2 and 2.4, we obtain the following plethystic inverse identity:

Corollary 2.5 The plethystic inverse of the series $T = \sum_{i\geq 1} h_i$ is the alternating sum $I = \sum_{i\geq 1}(-1)^{i-1}\pi_i$. More specifically, Theorem 2.2 corresponds to the identity $T[I] = h_1$, while Theorem 2.4 corresponds to $I[T] = h_1$.

On the other hand, it follows from a computation of Hanlon ([H1]) on the fixed-point partition lattices that $\pi_n = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}}(-1)^{n-\frac{n}{d}}$, where p_d denotes the dth power-sum. Consequently we recover Cadogan's formula ([C]), which says that the plethystic inverse of the series $T = \sum_{i\geq 1} h_i$, is the series $I = \sum_{d\geq 1}\mu(d)/d\ log(1+p_d)$.

This is the basis for our derivation of the generating functions of Calderbank, Hanlon and Robinson for the other subposets of partitions described in the introduction. We mention briefly a particular result for the lattice Π_n^d of partitions of a set of size nd whose block sizes are all multiples of d. A nice basis for the top homology of this lattice was found recently by Wachs ([Wa]). Her basis suggests that the S_{nd}-representation on $\tilde{H}(\Pi_n^d)$ is in fact a submodule of the wreath product of $\tilde{H}(\Pi_n)$ with the trivial representation of S_d. Using the techniques described above we can easily prove this conjecture, and identify the complementary module as follows:

Proposition 2.6 Write π_n^d for the characteristic of $\tilde{H}(\Pi_n^d)$, and let $q_n^d(1)$ denote the characteristic of the top homology of the subposet of Π_n^d obtained by deleting the atoms. Then one has

$$\pi_n[h_d] = \pi_n^d + q_n^d(1).$$

Wachs has observed that a constructive proof of this observation follows from the results in [Wa], by writing down bases for all the modules involved.

Our next theorem shows that the S_{nd}-module structure of the Whitney homology of the lattice Π_n^d is remarkably similar to that of the partition lattice.

Theorem 2.7 The rth Whitney homology of Π_n^d, as an S_{nd}-module, for $0 \le r \le n$, is given by

$$WH_r(\Pi_n^d) = \sum_{\substack{\lambda \vdash n,\ \ell(\lambda)=n+1-r \\ \lambda=(1^{m_1},2^{m_2},\ldots)}} h_{m_1}[\pi_1^d]e_{m_2}[\pi_2^d]\ldots e_{m_{2i}}[\pi_{2i}^d]h_{m_{2i+1}}[\pi_{2i+1}^d]\cdots,$$

with all notation as in Theorem 2.2.

Our second basic technique, which is most useful for describing the homology of the rank-selected subposet when the rank set is large, is the following:

Theorem 2.8 Let P be a Cohen-Macaulay poset of rank r with automorphism group G, and S a subset of the ranks $\{1,\ldots,r-1\}$. Let P_S be the rank-selected subposet of P, consisting of the ranks in S. Then as G-modules:

$$\begin{aligned}&(-1)^{r-|S|}\tilde{H}(P_S) - \tilde{H}(P) \\ &= \bigoplus_{\substack{0<x_1<\ldots<x_k<1 \\ rank(x_i)\notin S}} (-1)^k \left(\tilde{H}(0,x_1)_P \otimes \tilde{H}(x_1,x_2)_P \otimes \ldots \otimes \tilde{H}(x_k,1)_P\right)\Big\uparrow_{stab(x_1,\ldots,x_k)}^{G};\end{aligned}$$

where

1. the sum runs over all chains of elements *not* in S;
2. the homologies involved are all top homologies of open intervals $(x_i, x_{i+1})_P$ in the poset P;
3. $stab(x_1,\ldots,x_k)$ is the intersection of the stabilisers of all $(k+1)$ intervals $(0,x_1)_P$, $(x_i,x_{i+1})_P$, $(x_k,1)_P$, in P;
4. the tensor product is an *internal* tensor product of representations of $stab(x_1,\ldots,x_k)$; this representation is then induced up to G.

The starting point for the proof is a lemma of Baclawski ([Ba2]), which is precisely the Möbius function version of the formula above. Baclawski's lemma applies to arbitrary posets; our proof of the group-equivariant result requires that the poset be Cohen-Macaulay.

In the particular case when exactly one rank is removed, this theorem when applied to the partition lattice, yields the formula of Theorem 1.5. One obtains less extensive simplification when two ranks are deleted, but ultimately one can write down an explicit closed form for the homology representation. We mention one particular example:

Let $V_n([2,n-3])$ be as in Theorem 1.7. Since the subposet in question is obtained by deleting ranks 1 and $n-2$ from Π_n, we may use the preceding result to calculate the homology representation. We have

Proposition 2.9

$$\begin{aligned}&ch(V_n([2,n-3]) \\ &= \pi_n + h_2(\pi_{n-2} + h_1\pi_{n-3} + h_1^2\pi_{n-4} + \ldots + h_1^{n-5}\pi_3 + h_1^{n-4}\pi_2) \\ &- \sum_{1\le k<n-k} \pi_k\pi_{n-k} - f_n;\end{aligned}$$

where the last term f_n is nonzero only if n is even, in which case it is $h_2[\pi_{\frac{n}{2}}]$ if $\frac{n}{2}$ is odd, and equals $e_2[\pi_{\frac{n}{2}}]$ if $\frac{n}{2}$ is even.

It is clear from this expression that the trivial representation appears exactly once (since it appears in π_n iff $n \le 2$).

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