Hadamard invertibility of linearly recursive sequences in several variables

Earl J. Taft
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
U.S.A.

1. Introduction:

Several authors have considered the algebra of linearly recursive sequences over a field k under the Hadamard (pointwise) product. Recently, R. Larson and the author, [L–T], gave a Hopf algebraic proof of the result determining the invertible elements of this algebra as the set of sequences with non–zero initial data which, starting from some coordinate, become interlacings of a finite–number of non–zero geometric sequences. This result had been obtained earlier in [B] by analytic methods for characteristic k zero, and in [R] using algebraic methods. Our Hopf algebra methods are also effective, in the sense of finite algorithms (see Section 2). In this communication, we extend the discussion to the case of $n>1$ variables, i.e., to multisequences $(f_{i_1 i_2 \cdots i_n})$, $i_j \geq 0$ for $1 \leq j \leq n$, which for each such j, and each choice of i_t for $t \neq j$, each row $(f_{i_1 \cdots i_{j-1} s i_{j+1} \cdots i_n})_{s \geq 0}$ satisfies a linearly recursive relation (independent of the choice of i_t for $t \neq j$). The criterion for Hadamard invertibility will still be that a finite set of initial entries should be non–zero, and that each row should eventually be the interlacing of geometric series. Moreover, the procedure is effective.

See also [C–P] for a discussion of the coalgebraic aspects of linearly recursive sequences in several variables.

2. The case of $n=1$ variable

We recall here the ideas and methods of [L–T]. Let $A = k[x]$ be the polynomial algebra in one variable x, with bialgebra structure given by $\Delta x = x \otimes x$ and $\epsilon(x) = 1$, i.e., Δ and ϵ are algebra homomorphisms from A to $A \otimes A$ and k respectively such that $\Delta(x^n) = x^n \otimes x^n$ and $\epsilon(x^n) = 1$ for all $n \geq 0$. We consider the dual space A^* of linear functions on A as all sequences $(f_n)_{n \geq 0}$ for f_n in k, where such a linear function f is identified with (f_n) for $f_n = f(x^n)$ for $n \geq 0$. A^* has a subspace A^o, called the continuous linear dual of A, consisting of those f which vanish on some ideal of finite codimension in A

(see [S]). Since each such ideal in A is generated by a monic polynomial, one sees that A^o consists precisely of the linearly recursive sequences $f = (f_n)$, i.e., there is a polynomial $h(x) = x^r - h_1x^{r-1}. -h_r$ such that $f_n = h_1f_{n-1} + h_2f_{n-2}+\cdots+h_rf_{n-r}$ for all $n \geq r$ (see [P–T]). We say that (f_n) satisfies the relation $h(x)$. If r is minimal, we call $h(x)$ the minimal recursive polynomial of f. A^o is a bialgebra, whose algebra structure depends on the coalgebra structure of A. In [P–T], we chose $\Delta x = 1 \otimes x + x \otimes 1$ and $\epsilon(x) = 0$, yielding the Hurwitz (or divided power) product (see also [T]). In this paper, $\Delta x = x \otimes x$ and $\epsilon(x) = 1$, which yields the Hadamard (or pointwise) product $(f_n)(f'_n) = (f''_n)$, where $f''_n = f_nf'_n$ for all $n \geq 0$.

A geometric sequence (ar^n) for $a,r \neq 0$ in k satisfies the relation $x-r$, and its inverse $(a^{-1}r^{-n})$ satisfies $x-r^{-1}$. We note that we do not assume f in A^o satisfies $h(x)$ from the beginning, i.e., $h(0) \neq 0$. For example, we allow $(\pi,\sqrt{6},1,2,4,8,16,\cdots)$ which satisfies $x^3 - 2x^2$. It's inverse $(\frac{1}{\pi}, \frac{1}{\sqrt{6}}, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},\cdots)$ satisfies $x^3 - \frac{1}{2}x^2$. More generally, if $(e_n = ar^n)$, $(f_n = bs^n),\cdots,(g_n = ct^n)$ are a finite number m of non-zero geometric sequences, then their interlacing $(e_0,f_0,\cdots,g_0,e_1,f_1,\cdots,g_1,e_2,f_2,\cdots,g_2,e_3,\cdots)$ satisfies $(x^m - a)(x^m - b)\cdots(x^m - c)$, so is linearly recursive. It's inverse is of the same form, so is also linearly recursive. In [L–T], we prove that f in A^o is Hadamard invertible if and only if all $f_n \neq 0$ and, except for a finite number of terms, f is an interlacing of geometric sequences.

In order to use Hopf algebra ideas, one needs to extend $A = k[x]$ to $H = k[x,x^{-1}]$, Laurent polynomials in x. H is a Hopf algebra, and so is its continuous dual H^o. H^o consists of doubly–infinite sequences (f_n) for n in the integers Z, satisfying a recursive relation $f_n = h_1f_{n-1}+ h_2f_{n-2}+\cdots+ h_rf_{n-r}$ for all $n \in Z$, i.e., the polynomial $h(x)$ satisfies $h(0) \neq 0$. The restriction map H^o to A^o is a bialgebra homomorphism. There is an important bialgebra homomorphism from A^o to H^o called backsolving. One writes the minimal recursive relation for f as $h(x) = p(x)x^k$ with $p(0) \neq 0$. Then we delete $f_0,\cdots,f_{k-1}$, and starting with f_k, we use $p(x)$ to backsolve to the left. For example, $f = (2,5,1,1,2,3,5,8,\cdots)$ satisfies $x^4-x^3-x^2 = x^2(x^6-x-1)$. We delete $f_0=2$ and $f_1=5$, and obtain a doubly–infinite sequence $g = (\cdots,g_2=2,g_{-1}=-1,g_0=1,g_1=0,g_2=1,g_3=1,g_4=2,g_5=3,g_6=5,g_7=8,\cdots)$. This enables us to transfer the determination of the invertible elements of A^o to those of H^o. The group of group-like elements of H^o, i.e., those $f = (f_n)_{n\in Z}$ with $\Delta f = f\otimes f$ and $f\neq 0$ is precisely the group of geometric sequences $e(r) = (r^n)$, with $e(r)e(s) = e(rs)$. We then use the Hopf algebra structure of H^o to show that each invertible element is an interlacing of non-zero geometric sequences.

The procedure is effective, provided one knows some information about the non-zero roots of a recursive polynomial $h(x)$ for a sequence $(f_n)_{n\geq 0}$. We write $h(x) = x^k p(x)$ where $p(0)\neq 0$. Then we back-solve to consider a doubly-infinite sequence (F_n) for n in Z. We must check that $f_0,\cdots,f_{k-1}$ are non-zero, and then test (F_n) for being an interlacing of geometric sequences. We consider the subgroup of the multiplicative group k^* of non-zero elements of k generated by the roots of $p(x)$. It is a finitely generated abelian group, with a torsion subgroup of finte order m. If k has characteristic $p>0$, let p^r be the smallest power of p not less than the largest multiplicity of the roots of $p(x)$. Then F is invertible in H^o if and only if F is the interlacing of m geometric sequences (or mp^r such at characteristic $p>0$). So the finite algorithm for f is to determine if $f_0,\cdots,f_{k-1}$ are non-zero, and then to examine the first $2m$ (or $2mp^r$) terms if F to predict the m ratios $r_1,\cdots,r_m$ and the polynomial $(x^m-r_1),(x^m-r_2)\cdots(x^m-r_m)$ that F would satisfy if it were the interlacing of m (or mp^r) geometric sequences. Then we see if $p(x)$ divides this polynomial. Clearly this is a finite procedure, once m has been determined.

3. The case of $n>1$ variables

We consider the bialgebra $A = k[x_1,\cdots,x_n]$ with each x_i (and hence each mononomial $x_1^{i_1}\cdots x_n^{i_n}$) group-like. $A = k[x_1]\otimes k[x_2]\otimes\cdots\otimes k[x_n]$ as an algebra (also as a coalgebra), and thus $A^o \cong k[x_1]^o\otimes\cdots\otimes k[x_n]^o$ as a bialgebra.

We identify each f in A^* as a multisequence $(f_{i_1 i_2\cdots i_n})$ for all $i_1,\cdots,i_n \geq 0$, where $f_{i_1 i_2\cdots i_n} = f(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n})$. A "row" of such a multisequence is a sequence $\{f_{i_1\cdots i_{\ell-1} j i_{\ell+1}\cdots i_n} \mid j\geq 0\}$ for a fixed $i_1,\cdots,i_{\ell-1},i_{\ell+1},\cdots,i_n\geq 0$, which we say is parallel to the x_ℓ – axis. The product in A^* (and in A^o) is the Hadamard product.

Let f be in A^o, $f(J) = 0$ for a cofinite ideal J of A. For each $1 \leq i \leq n$, the powers of x_i span a finite-dimensional space in A/J, so there is a minimal monic $h_i(x)$ in k[x] such that each row of f parallel to the x_i-axis satisfies $h_i(x)$. Thus J contains the cofinite elementary ideal Γ generated by $h_1(x_1),\cdots,h_n(x_n)$.

We introduce $H = k[x_1,x_1^{-1},x_2,x_2^{-1},\cdots,x_n,x_n^{-1}]$, a Hopf algebra isomorphic to $h[x_1,x_1^{-1}]\otimes\cdots\otimes h[x_n,x_n^{-1}]$, so $H^o \cong k[x_1,x_1^{-1}]^0\otimes\cdots\otimes k[x_n,x_n^{-1}]^0$. We think of each f in H^o as a scalar field attached to the integral lattice points in n - space. So $f = (f_{i_1,\cdots,i_n})$ for $i_1,i_2,\cdots,i_n$ in Z, $f_{i_1\cdots i_n} = f(x_1^{i_1}\cdots x_n^{i_n})$. The rows of f are doubly-infinite sequences with all rows parallel to the x_i - axis satisfying a $p_i(x_i)$ with $p_i(0) \neq 0$. The restriction map $H^o \longrightarrow A^o$ is a bialgebra homomorphism. We use the elementary ideal Γ to backsolve from A^o to H^o. We write $h_i(x_i) = p_i(x_i)x_i^{k_i}$ with $p_i(0) \neq 0$. In each row parallel to the x_i-axis, we backsolve starting from the k_i-th coordinate. Let α: $A^o \longrightarrow H^o$ be this backsolving map. If $\alpha(f)$ is not invertible, i.e., if some row of $\alpha(f)$ is not an interlacing of geometric sequences, then f is not invertible. f is invertible if and only if all $f_{i_1\cdots i_n} \neq 0$ and each row of f is eventually an interlacing of geometric sequences. This is because all rows parallel to a fixed axis must satisfy the same recursive relation, and be themselves invertible. This transfers the problem to showing that in H^o, each row of $\alpha(f)$ is an interlacing of geometric sequences.

We claim the procedure is finite, i.e., we only have to check a finite number of rows in $\alpha(f)$ for interlacing of geometric sequences, as well as checking only a finite number of coordinates of f for being non-zero. Starting at the position $k_1k_2\cdots k_n$, we consider the finite hypercube having $f_{k_i\cdots k_n}$ in the upper left corner, and which includes all the initial data beyond this point, i.e., in each row parallel to the x_i-th axis, include positions k_i to degree $h_i(x) - 1$. Then, we backsolve along each of the (finite number of) rows through and in front of the hypercube, which involves deleting only a finite number of coordinates of f. If some backsolved row is not an interlacing of geometric sequences, then f is not invertible. If each of these backsolved rows is an interlacing of geometric sequences, then $\alpha(f)$ is invertible, because outside of these rows, the recursive relation will guarantee the interlacing of geometric sequences in the other rows. Then we have only to check that the entries deleted in the back-solving were not zero, and then f is invertible, since the reciprocal sequence f^{-1} will satisfy $x_i^{k_i}$ times an appropriate interlacing relation for each $1 \le i \le n$. Thus we have an effective procedure to accompany the following theorem.

<u>Theorem</u>: f in $k[x_1,\cdots,x_n]^o$ is Hadamard invertible if and only if there is a finite hypercube in the multisequence representation of f, such that the rows through and in front of the hypercube have non-zero coordinates before the hypercube, and are eventually interlacing of non-zero geometric sequences.

We illustrate with $n=2$, and $f \in k[x,y]^o$ in tableau form

	1	x	x^2	x^3	x^4	x^5	x^6
1	$*$	$*$	$*$	.....	.....	.....	.....
y	$*$	$*$	$*$	.....	.....	.....	.....
y^2	$*$	1	5	2	10	4	20···
y^3	$*$	7	9	14	18	28	36···
y^4	⋮	3	15	6	30	12	60···
y^5	⋮	21	27	42	54	84	108···
	⋮	⋮	⋮	⋮	⋮	⋮	⋮

satisfying $x^3 - 2x = x(x^2-2)$ and $y^4 - 3y^2 = y^2(y^2 - 3)$. We must first check that the $*$ entries are non-zero. Then we test separately the first two rows and the first column (here we use row and column in the usual 2 by 2 sense) by the $n=1$ method. Then, starting with the hypercube $\boxed{\begin{matrix} 1 & 5 \\ 7 & 9 \end{matrix}}$, we test the 2 rows to the right, and the two columns going down. This will certainly be invertible if all the $*$ entries are non-zero.

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