A SURVEY OF POLYOMINO ENUMERATION

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Abstract. In this talk I will give a survey of the state of the art in the enumeration of polyominoes and lattice animals. A visual walk is given through the zoo of families of polyominoes which can be enumerated, among the garden of related bijections, with some glimpses on various related topics.

Keywords: Classical analytic methods (recurrence relations, continued fractions, q-series, ...), transition matrix, Temperley methodology, DSV and q-DSV methodology, theoretical computer science (automata, algebraic languages, Dyck words, attribute grammars), bijections, heaps of pieces, commutations rules, Cartier-Foata trace monoids, trees (binary, ternary, colored, "guingois", ...), basic hypergeometric functions, Ehrhart' theory for convex polytopes, random generation, fractal dimension, braids, computer algebra, critical exponents, phase transition , statistical mechanics.

1. Introduction.

An *elementary cell* is a square $[i,i+1]\times[j,j+1]\subseteq \mathbb{R}\times\mathbb{R}$ with i and j integers. A *polyomino* is a connected union of elementary cells such that the interior is also connected. Polyominoes are defined up to a translation. Two main parameters are defined for polyominoes. The *area* is the number of elementary cells. The *perimeter* is the number of edges (of the lattice $\mathbb{Z}\times\mathbb{Z}$) on the border of the polyomino. A major open problem in combinatorics (and also in statistical physics) is to give a formula for the enumeration of polyominoes according to the area or to the perimeter (or to both parameters). To the knowledge of the author, not a single formula of any kind (recurrence for the number of polyominoes, explicit or implicit equation for the generating function, ...) is known.

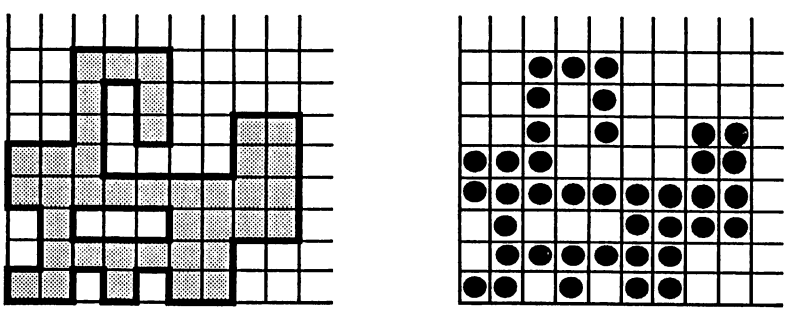

Fig. 1. A polyomino and its associated animal.

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In the past few years, there have been an intense activity for the search of enumerative formulae for subclasses of polyominoes. About a dozen families are known with explicit enumeration formula. Some formulae are surprisingly simple, some are very complicated. Most families are enumerated according to both parameters area and perimeters. Many q-series and algebraic generating functions appear. Slowly, some patterns begin to emerge. In this survey I try to put some order in this jungle of formulae disseminated into about 90 papers listed below. I have classified the methods used into six classes: transition matrix, "Temperley methodology", "DSV methodology", heaps methodology, classical analytic (recurrences, continued fractions,...), pure bijective. Of course this is not a rigid classification, there are many overlaps. Many concepts, tools, models, and bijections of other parts of enumerative and algebraic combinatorics appear here.

The problem takes its roots in physics where polyominoes are incarnate under the equivalent notion of *lattice animals*. An animal is obtained from a polyomino by taking the center of each cell (see Fig. 1). In other words, an animal (on the square lattice) is a set α of points of $Z{\times}Z$ such that any two points can be connected by a path having "elementary steps" North, South, East or West. The physics is the study of models for phase transition and critical phenomena, with computation of the so-called *critical exponents* and *partition function*. For each family of polyominoes (or animals), the asymptotic behavior of the number a_n of polyominoes in the class usually define a critical exponent θ in the form $a_n \sim \mu^n\ n^{-\theta}$. This exponent is analog to the one defined in thermodynamic models (for some models, this analogy can be the identity). Moreover, it appears that some partition functions are exactly the generating function for some classes of animals. Other parameters are also introduced: width, height, ... of the animal; various other lattices are also considered (hexagonal, triangular, "checkerboard", ..), leading to an avalanche of enumeration problems. Thermodynamic models and families of polyominoes are classified in physics by *"universality classes"*, according to their critical exponents. Physical methods can give some approximations to these exponents, or even explicit values (but with no rigorous mathematical proof).

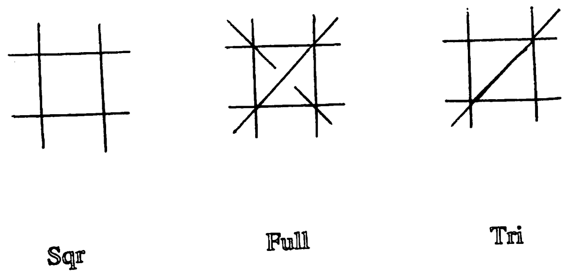
A famous related problem is the enumeration of the so-called *self-avoiding paths* (also completely open). In the case of *polygons* (i.e. closed loops on $Z{\times}Z$), the problem is equivalent to enumerate simply connected (i.e. having no *"holes"*) polyominoes according to the perimeter. A huge literature exists in physics and theoretical chemistry for animals and related paths (walks) problems. Here we will not be concerned with asymptotic considerations and universality classes, although de Gennes says that identification of universality classes for various connected cluster models is crucial in the scaling theory of branched polymer configurations (de Gennes 1979). We limit our interest to the classification and understanding of formulae giving the exact number of polyominoes or the associated generating function. Also, we will not consider here another active area of research around polyominoes: tilings problems with polyominoes where many people bring contributions (J.H. Conway, M.Gardner, S.Golomb, Gordon, D. Klarner, J.C. Lagarias, ... , and the active french school around D.Beauquier and M.Nivat). Note that other connections exist with computer science as for example the appearance of directed animals in binary search networks (Barcucci, Pinzani, Rodella 1990).

2. The polyominoes zoological garden.

Most of the families of the zoo can be defined by combining two main and simple concepts: "convexity" and "directed".

Let denote by $\mathbb{Sqr}$ the square lattice whose vertices are $Z{\times}Z$ and whose edges are the edges of elementary cells. Denote by $\mathbb{Full}$ the (highly non planar) lattice obtained by adding the two diagonal edges of each elementary cells of $\mathbb{Sqr}$. The triangular lattice $\mathbb{Tri}$ and hexagonal lattice $\mathbb{Hex}$ can be identified as sub-lattices of $\mathbb{Full}$. A *path* (or *walk*) on a lattice $\mathbb{La}$ is a sequence of vertices $\omega = (s_0,.., s_i, s_{i+1}, ..., s_n)$ such that each pair (s_i, s_{i+1}) of consecutive vertices (i.e. *elementary steps*) are connected by an edge in the lattice. An *animal* on the lattice $\mathbb{La}$ will be a finite *'connected'* set of vertices, that is a set such that each pair of vertices of the animal can be joined by a path of $\mathbb{La}$ contained in the animal. Usually, $\mathbb{Sqr}$ is referred with 4-connexity and $\mathbb{Full}$ with 8-connexity. Altough some papers deal with animals on other lattices, here we will only consider animals on the square lattice $\mathbb{Sqr}$.

Fig. 2. Lattices



We will denote by V, H, D, Δ respectively the vertical axis, horizontal axis, main diagonal and anti-diagonal (diagonal perpendicular to the main diagonal) of R×R. Let L be any one of these four lines. An animal $\alpha \in Z \times Z$ of the square lattice Sqr is said to be *L-convex* iff the intersection of α with any line parallel to L is a connected set of the lattice Full. Let $\vec{L}$ be one of the 8 oriented possible cardinal directions (East, .., North-East, ..) denoted respectively E, N, W, S, NE, NW, SW, SE. A path of the lattice Sqr is said to be *$\vec{L}$-directed* iff its projection on a line parallel to the direction L never goes backward in the reverse direction to L. An animal α (subset of Z×Z) is said to be *$\vec{L}$-directed* iff any point of the animal can be reached from a single point (called *source point*) by a directed path contained in the animal. For example, a NE-directed animal (here called for short *directed animal*) is an animal such that each point can be reached from the origin (0,0) by a path contained in the animal and having elementary steps only North or East. These qualities will be also defined for polyominoes via their underlying animal.

We can also extend the definition of directed animals with several sources points. We will say that the animal is *$\vec{L}$-cs-directed* (*compact source directed animal*) iff there exist a set of points (called *source points*) on the lattice Sqr which are on a line M perpendicular to the line L associated to $\vec{L}$, which are connected for the connexity in Full (8-connexity), and such that every point of the animal can be reached from one of these source points by a directed path in the square lattice Sqr.

The square lattice zoo.

We can combine the different conditions and define plenty of polyominoes families. All of them can be defined by various combination of H, V, D, Δ -convex, NE, NW, SE, SW -directed, and cs-directed properties. For 'name' = 'convex', 'directed 'or 'cs-directed', we will use the notations "X-Y-..-Z-name" to say that the family of polyominoes has the property 'name' relatively to X, Y, ..., and Z. The *zoo of polyominoes* is the set of all the possible families of polyominoes obtained by various combination of these properties (at least one !) and classified by non-isomorphic classes (i.e. up to the group of symmetries acting on the square and up to the relations existing between these different properties). For example the properties Δ-convex and NE-SE-directed imply V-H-convex (parallelogram polyominoes). Also for L= E, N, W, S, 'L-directed' is equivalent to 'L-cs-directed'. The total theoretical number of possible families is $2^8\, 3^4 - 1$.

In the case of directed animal, the property "convex" is sometimes called in the literature *"compact"* or *"fully compact"*. The E-directed animals are also called *partially directed* animals, and no formula is known. I will called the *friendly zoo* the family of polyominoes of the zoo not using in their definition one of the X-directed (or X-cs-directed) for X=N, E, S, or W. The total theoretical number of families of the friendly zoo becomes $6^4 - 1$. I have not listed the number of non trivial, non isomorphic classes. What I know, is that all the families of polyominoes (square lattice) appearing in the ninety papers listed below giving some explicit enumeration formulae belong to the friendly zoo. Only 14 of these families appear. In some papers appear for technical reasons some secondary sub-families which are some slight modification of the main family. Here are the 14 families.

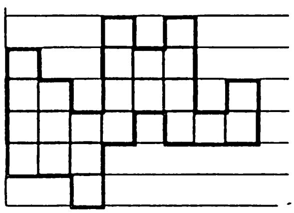


Fig. 3. Column-convex polyomino. (V-convex)

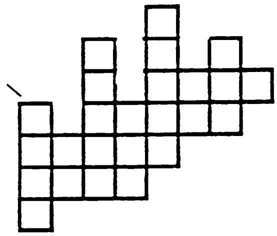


Fig. 4. Directed column-convex animal. (V-convex + NE-directed)

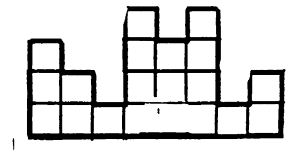


Fig. 5. Wall polyomino (= V-convex + NE-NW directed)

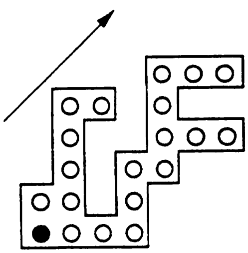


Fig. 6. Directed animal (NE-directed)

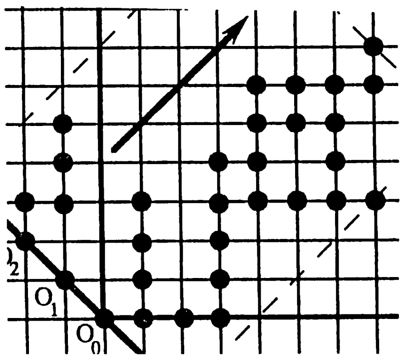


Fig. 7 Compact source directed animal. (NE-cs-directed)

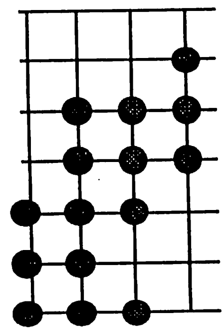


Fig. 8. Directed diagonally convex animal (Δ-convex + NE-directed)

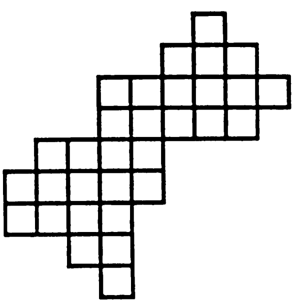


Fig. 9. Convex polyomino. (V-H-convex)

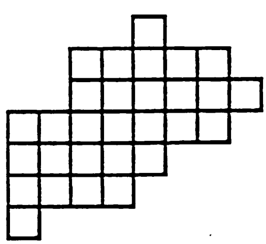


Fig. 10. Directed convex polyomino. (V-H-convex + NE-directed)

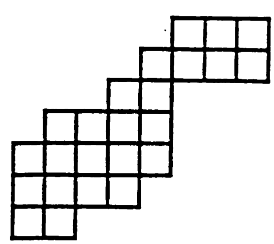


Fig. 11. Parallelogram polyomin (= V-H-Δ-convex + NE-SW-directed)

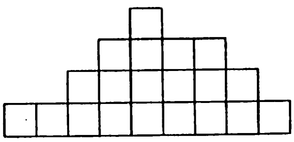


Fig. 12 Stack polyomino.
(= V-H-convex + NE-NW directed)

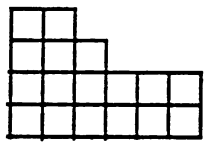


Fig. 13. Ferrers polyomino.
(= V-H-D-convex + NE-directed)

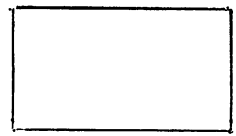


Fig. 14. Rectangle
(=V-H-D-Δ-convex
+ NE-SE-NW-SW-directed (!))

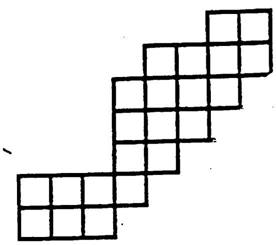


Fig. 15. Circles-wall polyomino
(= V-H-Δ-convex
+ NE-SE-directed + NW-cs-directed)

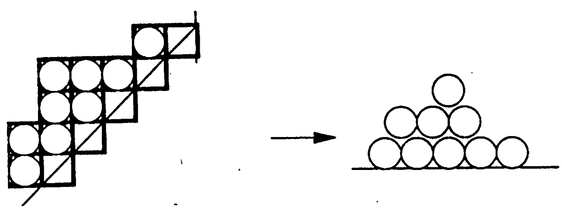


Fig. 16. Circles-stack polyomino
(= V-H-D-Δ-convex
+ NE-SE-directed + NW-cs-directed)

A *convex polyomino* is a polyomino which is both row- and column-convex. A characteristic property is that the perimeter is equal to the perimeter of the minimum rectangle containing the polyomino. There are five non-isomorphic classes of convex polyominoes: Ferrers diagram, stack , parallelogram (also called stair-case, and corresponding to skew Ferrers diagrams) , directed convex and convex. Parallelogram polyominoes are usually defined as polyominoes contained between two paths, each path having only North and East elementary steps, and such that the paths are disjoint, except at their common ending points. For the particular case where the lower path is an "enlarged staircase", we have the polyominoes displayed on Fig. 15. A subclass is in bijection with Andrews's quasi-partitions (Andrews 1981) and with "wall of circles" as considered in Privman, Švrakić 1989.

Each of these 14 families have at least one explicit enumeration formula. In many cases various formulae exist giving the triple generating function according to both area, perimeter and width. For the convex polyominoes, this distribution is equivalent to the distribution according to area, width and height. The only main problem which remains unsolved is the enumeration of the directed animals according to the perimeter. In fact 'perimeter' should be here replaced by 'directed perimeter'. Two perimeters exist for polyominoes: the one defined above is the *bond perimeter,* the *site perimeter* is the number of points outside of the animal which are neighbour (in the square lattice) to a point of the animal. More generally if F is a family of animals, the *F-perimeter* of an animal α of the family F is the number of points x outside the animal α, such that $\alpha \cup \{x\}$ still belongs to the family F. For F = {directed animals}, the F-perimeter is called the *directed perimeter*.

Here are two examples of non trivial enumeration formulae.

$$Y = y\frac{R-\hat{N}}{N},$$

$$N = \sum_{n\geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n}, \quad \hat{N} = \sum_{n\geq 1} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_{n-1} (yq)_n},$$

$$R = y \sum_{n\geq 2} \left[\frac{x^n q^n}{(yq)_n} \left(\sum_{m=0}^{n-2} \frac{(-1)^m q^{\binom{m+2}{2}}}{(q)_m (yq^{m+1})_{n-m-1}} \right) \right].$$

Fig. 17. Generating fucntions for directed convex polyominoes (according to width x , height y and area q)

$$3^n$$

Fig. 18. The number of NE-cs-directed animals.

yes it is really 3^n (!)

The first of the two formulae is not the most complicated. Just have a look at the paper Tzeng, Lin 1991, giving a four variables enumeration of column-convex animals (refinement of the enumeration according to the perimeter) to know what I mean. The second above formula is certainly the most simple non-trivial formula of the garden. Fig. 37 shows a bijection in action (between word of length n on the alphabet {A,B,C} and such cs-directed animals. Try to guess the construction !

If you take a guided tour among the known formulae, methods, technics, bijections related to the friendly zoo, may be you will make the suggestion, as I do , that all polyominoes of the friendly zoo should be enumerated (at least for one parameter) with the same kind of tools.

Curiously, some a priori unrelated regular patterns may appear. For example, look at the appearance of the radical Δ in the following four formulae enumerating, according to width and length respectively, the parallelogram, directed convex, convex polyominoes and a special type of convex polyominoes.

$$X = \frac{1-x-y-\sqrt{\Delta}}{2}, \qquad Y = \frac{xy}{\sqrt{\Delta}}, \qquad \Delta = 1-2x-2y-2xy+x^2+y^2.$$

$$Z = \frac{xy}{\Delta^2}\left(1-3x-3y+3x^2+3y^2+5xy-x^3-y^3-x^2y-xy^2-xy(x-y)^2\right) - \frac{4x^2y^2}{\Delta^{3/2}},$$

$$B = xy\frac{(1-x)(1-x-2y+y^2-xy)}{(1-x-y)\Delta},$$

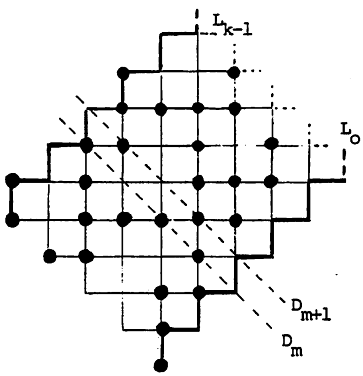
Generating functions for parallelograms, directed convex, convex and convex type B (width and height).

3. Technics and methodologies for polyominoes enumeration.

I have classified the different methods and tools used in the ninety papers listed below into six main classes. Of course, this is not a rigid classification and many papers uses several methods at the same times.

a) Transfer matrix methodology.

This method is very classical in statistical mechanics (see for example Baxter 1982). The combinatorial objects are put in bijection with some paths on a finite graph {1,..., p}, such that the enumerative generating function becomes the generating function of weighted paths on the graph. Denote by A is the p p matrix with term (i,j) equal to the weight of the edge joining i to j. Then apply the inversion matrix formula for (I - At) and get a rational generating function. This method fits for animals on a bounded strip, as for example directed animals on a bounded strip (Fig. 19). Here is a formula for them (conjectured in Nadal, Derrida, Vannimenus, (1982) and proved in Hakim, Nadal, (1983)



$$\overline{a}_n^k(C) = \frac{1}{k}\sum_p (-1)^p \sin\alpha_p \prod_{i=1}^{k-1}\left(\frac{\sin[(i+\frac{1}{2})\alpha_p]}{\sin\frac{\alpha_p}{2}}\right)^{N_i} (1+2\cos\alpha_p)^{n-1} \qquad \alpha_p=(2p+1)\frac{\pi}{2k}$$

Fig 19. Directed animal on a strip with enumerating formula (fixed sources points).

b) Temperley methodology.

This methodology has been introduced in Temperley (1956). It fits well with the different families of convex polyominoes. It has been used intensively used in the papers of Brak, Enting, Guttmann, Lin, Chang, Tzeng, Wu, Privman, Forgacs and Švrakič.

If t,x,y, .. are the different "edges" parameters as perimeter, width, height,.. and q is the parameter referring to the area, then the generating function f is decomposed into a sum of partial generating functions f_k satisfying a recurrence relation of the following type:

$$a_k f_k + (a_{k+1} f_{k+1}) + \dots (a_{k+r} f_{k+r}) = 0,$$

where a_k, ..., a_{k+r} are polynomials in the variables t,x,y, ..., q. Usually, f_k is the generating function for the polyominoes of the family under consideration having the first "machin" of size k, where "machin" means something like first row, first column or first diagonal. In general, for q=1, the polynomials a_{k+i} in the variables t,x,y, .. depends only upon i . The degree in the variable q of the monomials depends upon k. Of course, the resolution of the above recurrence may be not be easy and each case may take a lot of work. Continued fractions expansions may help.

c) DSV-methodology.

This methodology was introduced (and named "DSV") by M.P. Schützenberger (1962, 1963) thirty years ago, in relation with his work with N. Chomsky on the theory of algebraic languages (i.e. context-free). The background comes from linguistic and theoretical computer science (automata and languages theory). It fits very well in the case of an algebraic generating function. The principle is the following.

Denote by A^* the *free monoid* generated by A (or set of words on the alphabet A). A subset of A^* is called a *language*. Let Z<<A>> be the algebra of *non-commutative power series* in variables A and coefficients in Z. To each language $L \subseteq A^*$, we define the non-commutative generating function $\underline{\underline{L}}$ of L as the formal sum of all the words of L. For the reader not familiar with the (classical in computer science) notions of *algebraic* and *rational language*, *algebraic grammar* and *non-ambiguous grammar*, we give a simple (and fundamental) example.

The Dyck language D is the set of words w of $\{x,\bar{x}\}^*$ satisfying the two following conditions:

(i) the number $|w|_x$ of occurrences of the letter x in w is equal to the number $|w|_{\bar{x}}$ of occurrences of the letter $\bar{x}$,

(ii) for any left factor u of w = uv, $|w|_x \geq |w|_{\bar{x}}$.

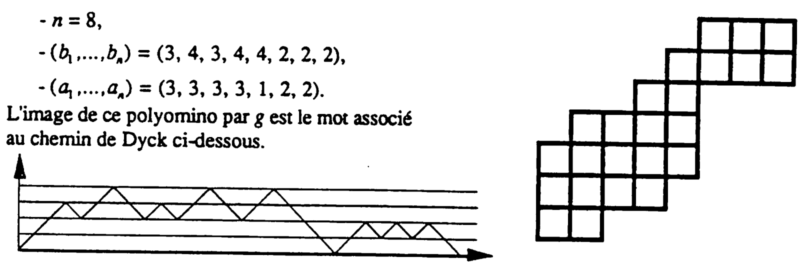


Fig. 20. Bijection between parallelogram polyominoes and Dyck words.

Such words are visualized by the so-called *Dyck paths* (see Fig. 20). Then D can be generated by applying several times the following substitutions rules:

$$D \to x D \bar{x} D \quad \text{or} \quad D \to e,$$

where e denote the *empty word*. This two rules define a "non-ambiguous grammar". The corresponding non-commutative generating function is

$$\underline{\underline{D}} = 1 + x\bar{x} + x\bar{x}x\bar{x} + xx\bar{x}\bar{x} + x\bar{x}x\bar{x}x\bar{x} + x\bar{x}xx\bar{x}\bar{x} + xx\bar{x}\bar{x}x\bar{x} + xx\bar{x}x\bar{x}x + xxx\bar{x}\bar{x}\bar{x} + \dots$$

and satisfies the following algebraic equation in $Z\langle\langle\{x,\bar{x}\}\rangle\rangle$:

$$\underline{\underline{D}} = 1 + x\,\underline{\underline{D}}\,\bar{x}\,\underline{\underline{D}}.$$

This equation appears as a "linearization" of the non-ambiguous grammar. If we send all variables x and $\bar{x}$ onto t, then we go back to the classical commutative case giving the generating function for Catalan numbers: $\underline{\underline{D}}$ becomes the generating function y for the number a_n of words of D length n, the algebraic equation becomes $y = 1 + t^2y^2$.

Another classical algebraic language is the *bilateral Dyck* language, i.e. words of $\{x,\bar{x}\}^*$ satisfying only condition (i) of Dyck words. A visualization is given with path in Fig. 21. They are enumerated by the binomial coefficient $\binom{2n}{n}$.

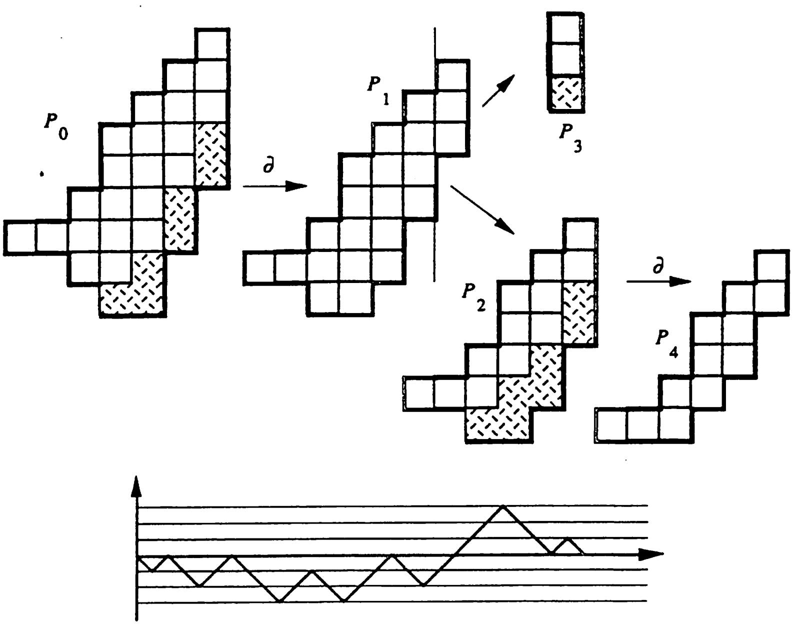


Fig. 21. Bijection between directed convex polyominoes and bilateral Dyck words.

The DSV methodology is in three steps. Let F be a class of combinatorial objects. The most simple form of the problem is to enumerate the objects having *"size"* n (size means any enumerative parameter). Let $f(t) = \Sigma_{n>0}\, a_n\, t^n$ be the corresponding generating function.

- (DSV1) Construct a bijection between the class F and some words of an algebraic language $L \subseteq A^*$ defined by a non-ambiguous grammar such that the the size of the combinatorial objects is the length of the corresponding words (or a linear function of it).
- (DSV2) Write a non-ambiguous algebraic grammar and "translate" it into an algebraic system of equations in the algebra $Z\langle\langle A\rangle\rangle$.
- (DSV3) Applying the morphism sending the letters onto t, we get an algebraic equation for f(t), which can be possibly simplified or computed.

Example: the appearance of the common radical Δ in the four formulae shown at the end of section 2 can be better understood with the DSV methodology. The coding and the corresponding formulae for the perimeter enumeration (obtained by identifying the width and height parameter into a single variable) are the following (respectively parallelogream, directed convex, convex, convex type "B" polyominoes).

Proposition. *(i) Le nombre de polyominos parallélogrammes de périmètre 2n+2 est le $n^{ième}$ nombre de Catalan, soit* $\frac{1}{n+1}\binom{2n}{n}$.
(ii) Le nombre de polyominos convexes dirigés de périmètre 2n+4 est $\binom{2n}{n}$.
(iii) Le nombre de polyominos convexes de périmètre 2n+8 est $(2n+11)4^n - 4(2n+1)\binom{2n}{n}$.
(iv) Le nombre de polyominos de $\mathfrak{B}$ *de périmètre 2n+8 est* $6.4^n + 2^n$.

If the problem contains several enumeration parameters, the method is exactly the same if the coding can be done such that each letter corresponds to one of the parameters, with a linear relation between the length of the word and the "size" of the combinatorial objects. In the last step, we just commute the variables and get the multivariate generating function.

DSV methodology was first illustrated by R. Cori and his students with planar maps (Cori 1975, Cori, Vauquelin 1981, for example), explaining the reason of the algebricity of various Tutte's formulae. After, DSV methodology was used to solve open problems in combinatorics, as for example a problem posed by M. Waterman: enumeration of *secondary structures of single stranded nucleic acids* having a given *complexity* (Vauchaussade de Chaumont, Viennot, 1985). The first open problem in polyominoes enumeration solved by this method is the number of convex polyominoes according to the perimeter (Delest, Viennot, 1984). The reader will find a survey in Viennot, 1985.

In the case of a language accepted by a *finite automaton* (*recognizable* or *rational* langage), then the ordinary generating function will be rational and DSV methodology is similar to transition matrix method.

For all the families of the friendly polyominoes zoo, DSV methodology can be applied (at least for one of the parameter area or perimeter) as shown by Gouyou-Beauchamps and the bordelais group (see papers of Betrema, Bousquet-Mélou, Delest, Dulucq, Fedou, Lalanne, Penaud, Viennot, ..) where DSV is very popular. For the double generating function, a q-analog of DSV can be introduced (see papers of Delest, Fedou and Bousquet-Melou). The area is coded by some powers of the variable q which appears in the algebraic grammar of the language. These *"q-grammars"* have some analogy with the so called *attribute grammars* introduced by Knuth. A recent survey of DSV and q-DSV methodology applied to polyominoes enumeration is given in Delest 1991.

d) Heaps of pieces and Cartier-Foata commutation monoid.

Let P be a set (called set of *basic pieces*). Let C be a binary symmetric and reflexive relation on P called *concurrency relation.* Thus the pair (P,C) defined a graph, called the *concurrency graph.*

A *heap* is finite set E of pairs $\alpha = \{p, j\}$ with $p \in P$, $j \in N$ satisfying the two following relations:
(i) if $\{p, j\}$ and $\{q, k\}$ are two elements of E with $p\,C q$, then $j \neq k$.
(ii) if $\{p, j\}$ is in E and $j = 0$, then there exist $\{q, k\}$ in E such that pCq and $k = j-1$.

The elements $\{p, j\}$ of E are called *pieces*. The basic piece p is called the *projection* of $\{p, j\}$, while i is called the *level* of the piece $\{p, j\}$. The set of all heaps on (P,C) is denoted by Heap(P,C),

This concept was introduced in Viennot (1985), and has been useful in various part of combinatorics including combinatorial proof in classical linear algebra, combinatorial theory of general orthogonal polynomials, algebraic graph theory, and combinatorial problems related to statistical physics, in particular animals enumeration. The intuitive idea behind the heap concept is better understood in the case of heaps of subsets. Here the set P is a certain collection of subsets of a set B. The concurrency relation C is defined by: aCb iff $a \cap b \neq \emptyset$.

Suppose B is R×R and represented by an horizontal board, and that each piece α is represented by a solid piece of wood with small constant width, and projection a on B. Then the concept of heap corresponds to the picture obtained by putting one by one solid pieces on B. See Fig. 23, 24, where the basic pieces are respectively *dominos* (polyominos with two cells) of the chessboard or *hexagons* of the triangular lattice. Another example is displayed on Fig. 25. Here B = Z, the basic pieces are segments [i,j] of Z, the concurrency relation is the same as defined above. A partially order relation, called *"to be below"* can be defined for any heap. Intuitively α is below β iff one has to remove first β in order to remove α. The Hasse diagram of the order relation is displayed on Fig. 28. Conversely, any poset can be represented as a heap of pieces. A *pyramid* is a heap having only one maximal piece.

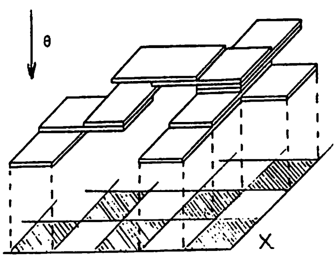


Fig. 23. Heap of "solid dimers" over R R.

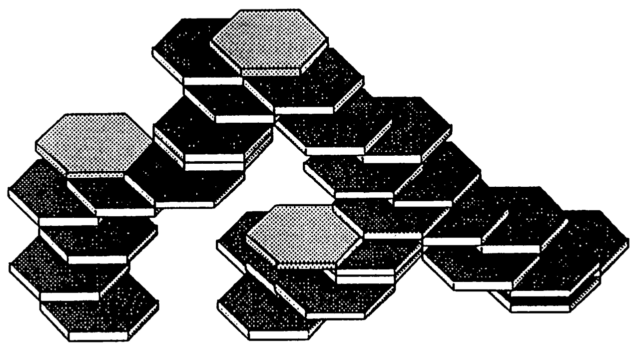
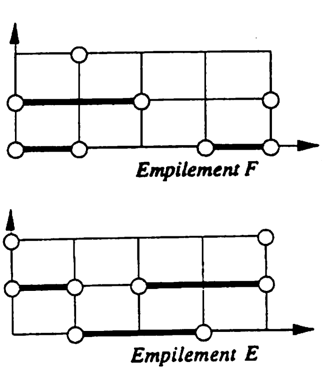


Fig. 24 Heaps of "solid" hexagons.

In fact one can defined a product of two heaps E and F: intuitively put F far above E and let it fall down on E. The set Heap(P,C) becomes a monoid, and this monoid can be defined by some partial commutation rules. Let $\equiv_C$ be the congruence on the free monoid P^* generated by the commutation ab = ba for each pair of basic pieces which are not in concurrence. Then the heap monoid Heap(P,C) is isomorphic to the quotient monoid $P^*/\equiv_C$. Such monoids have been introduced in Cartier, Foata 1969. They have been intensively used in theoretical computer science as a model for concurrency and parallelism problems. They are called *trace monoids* (also *commutation monoid*). Conversely every trace monoid is a heap monoid and the two concepts are equivalent. The advantage of heaps is to provide a geometric interpretation of the equivalence classes of words (called *traces*) with a powerful spatial intuition.



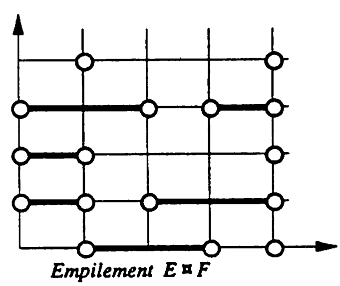


Fig. 25. Product of two heaps.

ad = da
bc = cb
cd = dc

Fig 26. Commutation relation

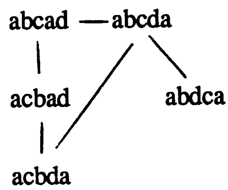


Fig. 27. A trace (equivalence class)

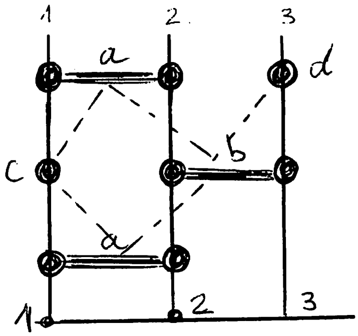


Fig. 28. Realization as a heap and associated Hasse diagram

Heaps have been very useful in polyominoes enumeration. Some families of polyominoes can be put in bijection with some heaps. In particular the parallelogram polyominoes are in bijection with pyramids of segments of N such that the projection of the maximal piece is a segment [0,j]. The directed animals are in bijection with certains pyramids of dimers on Z , and through another bijection with pyramids of momers and dimers on Z (*dimers* are segments [i,i+1], while *monomers* are segments [i]).

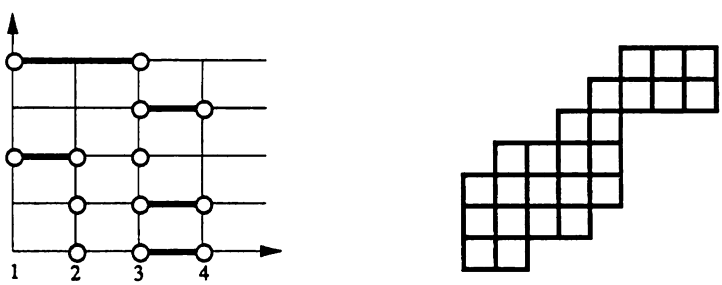


Fig. 29 Bijection between parallelogram polyominoes and pyramid of segments.

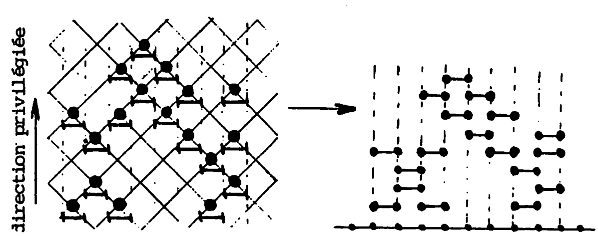


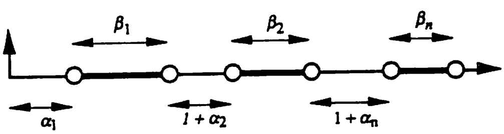
Fig. 30. Bijection between directed animal and heap of dimers.

Three basic facts on heaps are used. These lemma can be summarized schematically by the following:

Paths=Heaps; Pyramids= d/dt(Heaps); Heaps= 1/D and Heaps$^{(*)}$= N/D.

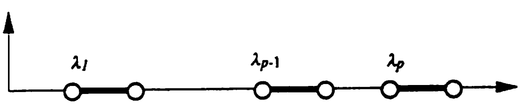
The first lemma described a bijection between paths (on any lattice) and heaps. The three others identities are generating functionsof weighted heaps: each basic piece a is given a weight v(a) and the weight of a heap is the product of the weight of the projection of its pieces. The last equation represents the generating function for weighted heaps such that the projection of the maximal pieces are in a given fixed set $M \subseteq P$. The numerator N and the denominator D are the generating function for trivial heaps (i.e. all the pieces are at level 0). For more details see Viennot, 1985.

In the case of convex polyominoes (parallelograms, directed convex and convex), heaps of segments are considered with weight $v([i,j]) = t\, u^{(j-i)} q^{j}$. Trivial heaps of segments generate the q-Bessel fuunctions (see Fig. 31) appearing in Fig. 18. In the case of trivial heap of dimers, left hand side of Rogers-Ramanujan identies appear (see Fig. 32. in relation with Fig. 15 and 16). (see Bousquet-Mélou 1991 and Bousquet-Mélou, Viennot 1992)



$$N = \sum_{n \geq 0} \frac{(-1)^n x^n q^{\binom{n+1}{2}}}{(q)_n (yq)_n},$$

Fig. 31. Trivial heap of segments over Z
(q-Bessel fonctions).



$$\sum_{n \geq 0} \frac{(-x)^n q^{n+n^2}}{(q)_n} \Big/ \sum_{n \geq 0} \frac{(-x)^n q^{n^2}}{(q)_n},$$

Fig. 32. Trivial heap of dimers over Z
(left hand sides of Rogers-Ramanujan identities).

$$\cfrac{1}{1 - \cfrac{xq}{1 - \cfrac{xq^2}{1 - \cfrac{xq^3}{1 - \dots}}}}.$$

Another example of the power of heap methodology is a very comprehensive proof for the number of directed animals on a bounded circular strip. The formula given in Fig 19 corresponds to a heaps generating function N/D, where N and D are respectively the generating function for trivial heaps of dimers (here q = 1) on a disjoint union of segments and on a circle (i.e. , up to a change of variable, Tchebycheff polynomials of first and second first kind).

e) Classical analytic methods.

In this category I would list the classical analytic tools used in combinatorial problems: recurrence relations, q=Lagrange inversion, continued fractions expansions, ...

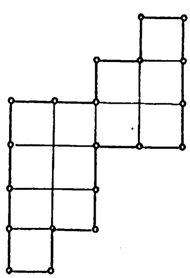
f) Bijective methods

Here I would list enumeration formulae obtained from the construction of explicit bijections between animals and other combinatorial objects which are easier to enumerate. We have already mentioned such bijections with words of algebraic languages and with heaps. Another families of bijections which appear in many papers involve bijections between polyominoes and various families of trees. Fig. 35, 36, 37 show binary trees in the case of parallelogram polyominos, colored binary trees for convex polyominoes, *"guingois"* trees for directed animal. In fact algebraic equation on trees can be defined, analogous to algebraic grammars of words. A surprise is the existence of a bijection between *ternary trees* and directed diagonally convex animals (enumerated according to the directed perimeter).

g) Experimental combinatorics.

A few words about experimental hunting for finding or proving enumeration formulae. Here the tools are a computer with a symbolic algebraic package (as for example your favorite MATHEMACSYMAPLE) and the Sloane'book (thank's to Simon Plouffe, a new version is coming with more than 4000 sequences). Some formulae have been discovered or guessed with these tools. Some technics have been developed in order to guess from the first terms of the sequence (20 to 100) a possible explicit expression for the generating function. In case of a rational power series, excellent algorithms are used based on Pade approximants theory, you cannot miss it. For P-recursive generating functions, the so-called differential Pade method have been introduced in Joyce, Guttmann (1972) and is popular in statistical mechanics for experimental asymptotic results. In the case of algebraic power series, see Brak, Guttmann, (1990a). Contrarily to the case of rational fractions, the method is rather brute force. Nevertheless, some algebraic functions have been guessed this way. The generating function for convex polyominoes was guessed in Guttmann, Enting (1988a), independently of the paper Delest, Viennot (1984). Of course such experimental methods give only mathematical conjectures and have not to be compared with the six methods exposed above. Usually, in the physics literature, the distinction between establishing a formula from computer experiments and giving a mathematical proof is not clearly stated. The experiments can be fundamental for guessing the formula. When the number of known elements of the sequence is much bigger that the number of elements needed to guess the exact formula, the probability that the formula is wrong is infinitesimal, and thus the formula can be considered as an experimental true statement. But mathematically it is just a conjecture waiting for a proof, and may be more: a crystal-clear understanding. Remark that some rigorous proof for polyominoes enumeration, need some huge computation on a computer. The computer is used at an other level, as part of the proof.

4) A nice example: parallelogram polyominoes



All the methodologies mentioned above can be applied for parallelograms polyominoes. The enumeration according to the perimeter gives the classical Catalan numbers. There is plenty of related bijections involving Dyck words, binary trees, heaps of segments, .. Some magic coincidences appears on Fig, 33. Penaud has used similar bijections with parallelogramm polyominoes in order to give a bijective proof of a formula of Riordan-Touchard giving the moments of some q-Hermite polynomials. (see Penaud, 1992 at this collque)

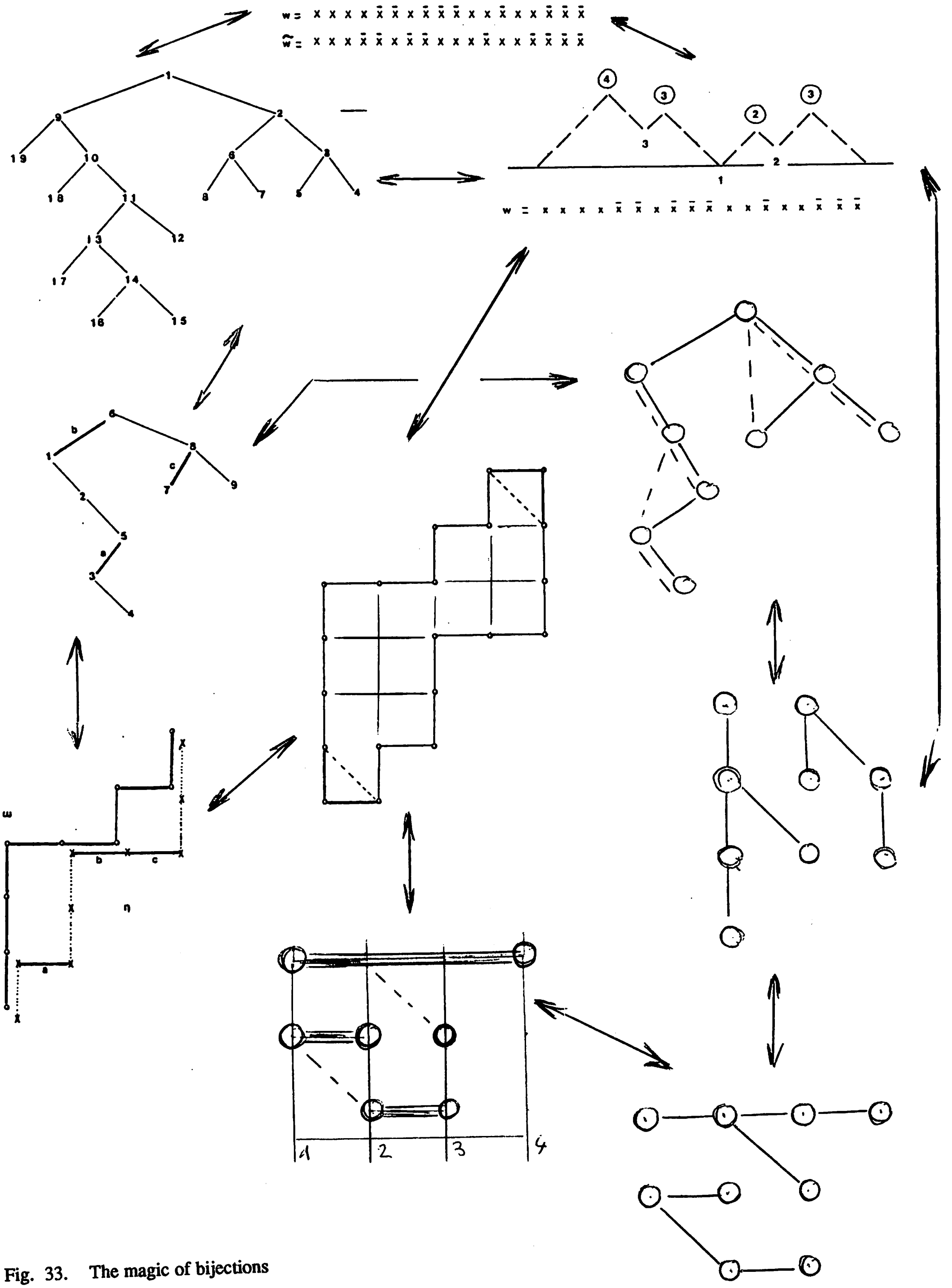


Fig. 33. The magic of bijections

5) The garden of polyominoes bijections.

After the guided tour in the zoo, and the magic network of bijections for parallelograms polyominoes, here are some example of polyominoes bijections. Some are sophisticated.

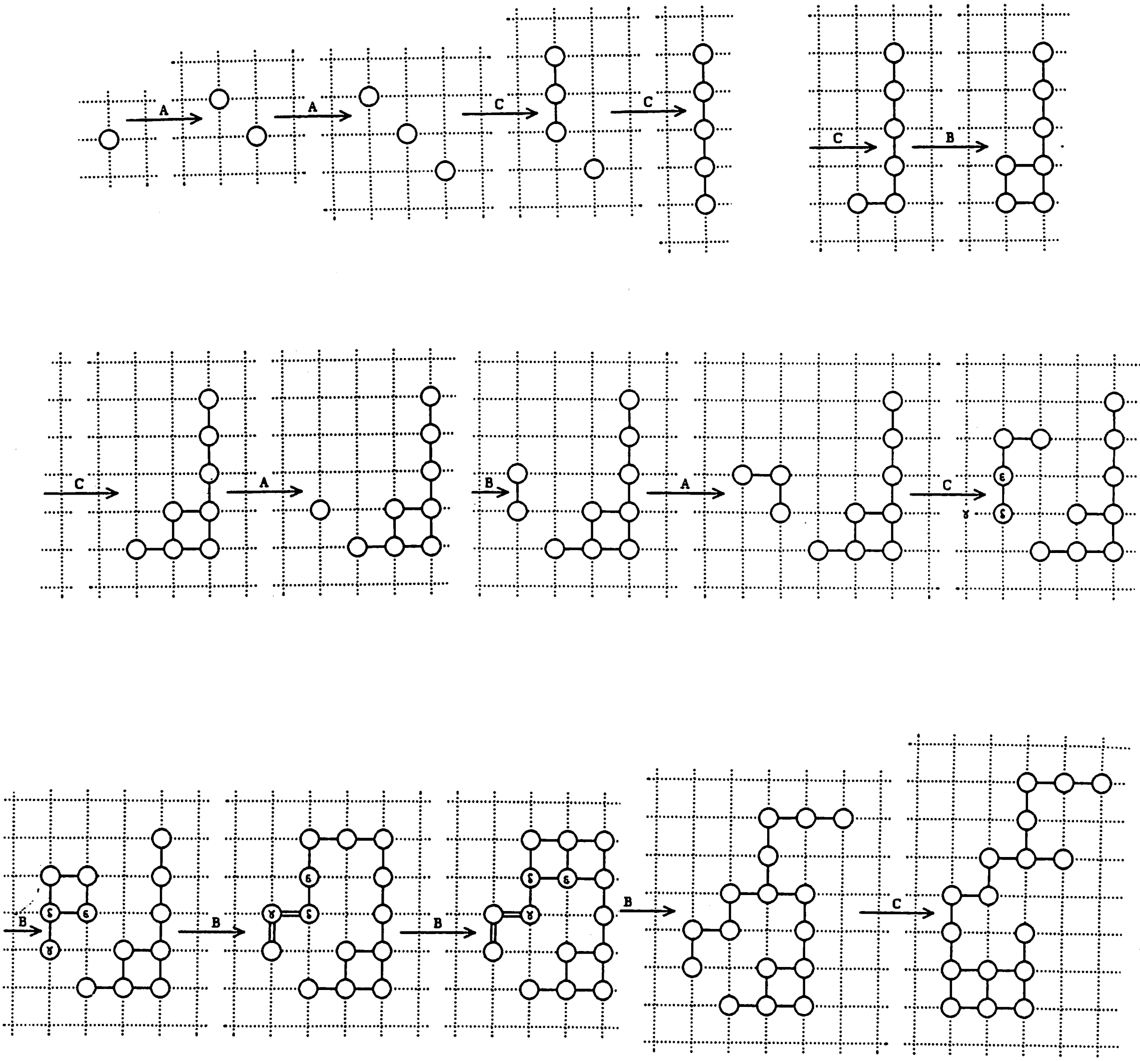


Fig. 34. A bijection for 3^n (compact source directed animals).
try to guess the construction !

(Gouyou-Beauchamps - Viennot, 1988)

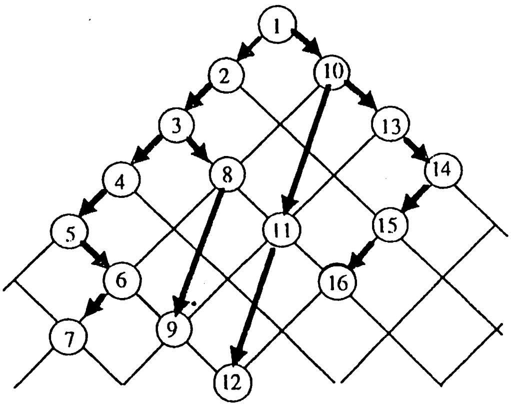


Fig. 35. Bijection between directed animals and "guingois" trees.

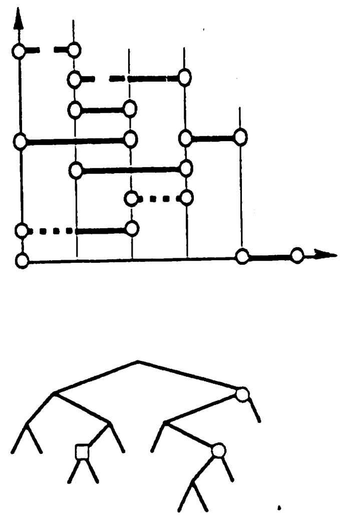


Fig. 36. Bijection between colored heaps of segments and convex polyominoes with projection on acolored binary tree.

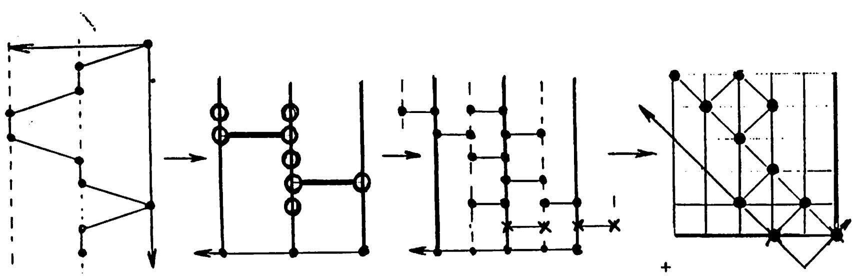


Fig. 37. Path, dimers-monomers heap and directed animal.

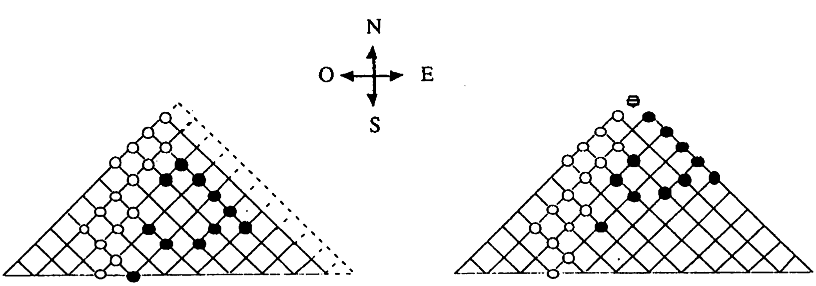


Fig. 38. Penaud's operator for directed animal.

6) Related topics

One of the advantage of using bijective methods in polyominoes enumeration is to show connections with some other apparently completely unrelated topics. For example, we quote here: Ehrhart' theory for counting points with integers coordinate in a convex polytope (see Stanley 1986, Fedou 1989, Delest-Fedou 1991b); basic hypergeometric functions (in particular some q-Bessel functions); the computation of the directed percolation probability (see Fig. 39 and Bousquet-Melou, 1990b); Roger-Ramanujan identidies and continued fractions apppearing in the special class of parallelograms polyominoes displayed on Fig. 15, 16 (in bijection with Andrew's quasi-partitions (Andrews, 1981) and "circles wall" animals (Privman, Švrakič 1989)).

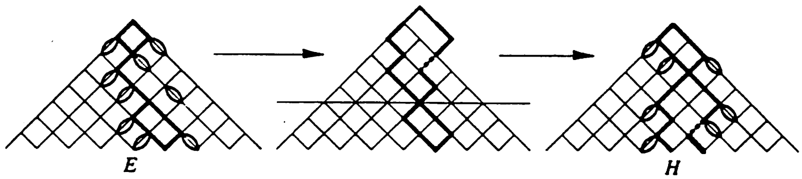


Fig. 39 Modified parallelogram polyominoes appearing in the directed percolation problem.

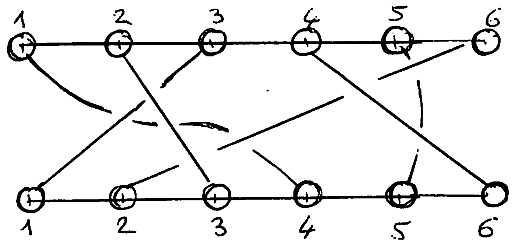
Another advantage of bijective methods is in the random generation of polyominoes. For example, using the bijection with "guingois" trees (Betrema, Penaud, 1991) and an algorithm of Barcucci, Pinzani, 1991, random animals can be generated in linear time. This means that each directed animal having n points appears with same probability (as soon as one can generate equiprobably a number between 1 and k). This operation is supposed to be a primitive with a cost O(1). Fig. 40 shows some random animals. Their fractal dimensions seems to be related to the critical exponents for width and length.

Fig. 40 Random directed animals (size 1000 and 2000).

I will finish with a last surprise. The generating function for parallelogram polyominoes according to the area and having a fixed number of columns is a rationnal fraction of the following form: $(b)^{u}q \;/\; (q;q)_n(q;q)_{n-1}$ where $(q;q)_n = (1-q)(1-q^2)\ldots(1-q^n)$.

The numerator $b_n(q)$ is a polynomial with positive integers coefficients (Fedou 1989). Fedou has just proved (see Fedou 1992 at this colloque) that these coefficients enumerate certain braids of the braid group B_n according to a certain parmeter analog to a Markov trace.

Fig. 41. Heaps and Braids



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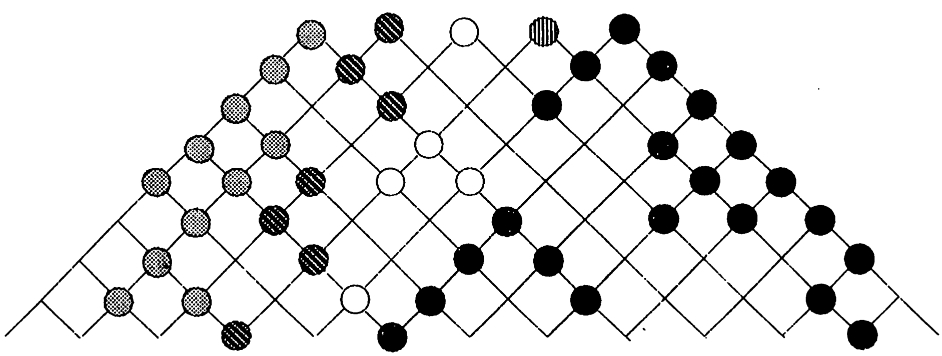

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