Zeros of rank-generating functions of Cohen-Macaulay complexes

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Abstract
Many combinatorial polynomials are related to rank-generating functions of Cohen-Macaulay complexes; notable among these are reliability, chromatic, flow, Birkhoff, and order polynomials. We prove two analytic theorems on the location of zeros of polynomials which have direct applications to the rank-generating functions of Cohen-Macaulay complexes and discuss their consequences for each of the aforementioned classes of polynomials.

0 Introduction

The rank-generating functions of Cohen-Macaulay complexes provide a unified setting for a variety of results and conjectures in the literature which concern the values of coefficients or the location of zeros of some combinatorial polynomials. Among these polynomials are the chromatic and flow polynomials of a graph (or matroid), the reliability polynomial of a (cographic) matroid, the order polynomial of a partially ordered set, and the Birkhoff (or characteristic) polynomial of a geometric lattice. Thus the study of these generating functions can be seen as a natural avenue of attack both on Rota's "critical problem" (cf. Chapter 16 of [6]) *via* the location of zeros of Birkhoff polynomials, and on the Read-Hoggar conjecture concerning logarithmic concavity of the coefficients of chromatic polynomials (cf. [11, 7]), as well as on other

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more recent conjectures. Our purpose here is to present two new theorems, and to see to what extent these theorems relate to the conjectured behaviour of the polynomials in question.

1 Results

Let Δ be a (finite) simplicial complex with vertex-set V of size n; that is, Δ is a subset of 2^V which contains all singletons and is closed by taking subsets. We naturally identify the singletons with the elements of V. Elements of Δ are called *faces*; maximal faces are called *facets*. If all facets of Δ have the same size then Δ is called *pure*.

The primary numerical invariant of a simplicial complex is its f-vector, or equivalently, its rank-generating function. Let Δ have $f_i(\Delta)$ faces of size i, for each natural number i. The *rank-generating function* of Δ is

$$\mathrm{rgf}(\Delta; x) := \sum_{i \in \mathbb{N}} f_i(\Delta) x^i.$$

Since Δ is finite, $\mathrm{rgf}(\Delta; x)$ is a polynomial in x, and has constant term 1 corresponding to $\emptyset$. The sequence of coefficients $\{f_i(\Delta)\}$ is the *f-vector* of Δ.

A less obvious but even more important numerical invariant is the h-vector of a simplicial complex. Let d be the maximum size of a facet of Δ, so that $\mathrm{rgf}(\Delta; x)$ has degree d. The polynomials $x^i(x+1)^{d-i}$ for $i = 0, ..., d$ form a free basis for the additive group of polynomials of degree at most d in $\mathbb{Z}[x]$. Thus we may expand $\mathrm{rgf}(\Delta; x)$ uniquely in terms of this basis:

$$\mathrm{rgf}(\Delta; x) = \sum_{i=0}^{d} h_i(\Delta) x^i (x+1)^{d-i}. \tag{1}$$

This sequence of coefficients $\{h_i(\Delta)\}$ is the *h-vector* of Δ. For us it will be useful to define the *h-generating function* of Δ to be

$$\mathrm{hgf}(\Delta; x) := \sum_{i=0}^{d} h_i(\Delta) x^i.$$

Clearly $\mathrm{rgf}(\Delta; x)$ and $\mathrm{hgf}(\Delta; x)$ are related as follows:

$$\mathrm{rgf}(\Delta; x) = (x+1)^d \mathrm{hgf}(\Delta; x/(x+1))$$

$$\mathrm{hgf}(\Delta;x) = (1-x)^d \mathrm{rgf}(\Delta; x/(1-x)).$$

Many of the simplicial complexes which occur in nature share a structural condition called shellability. For a face F of a simplicial complex Δ, let $\bar{F}$ be the set of all subsets of F. A simplicial complex Δ is *shellable* when it is pure (of facet-size d, say), and the set of its facets may be ordered $F_1, F_2, \ldots, F_m$ in such a way that for each $j = 2, ..., m$, every facet of the subcomplex $\bar{F}_j \cap (\bar{F}_1 \cup ... \cup \bar{F}_{j-1})$ has size $d-1$. Given such a *shelling order* for the facets of Δ, for each $j = 1, ..., m$ let $\nu(j)$ denote the number of faces of size $d-1$ of $\bar{F}_j \cap (\bar{F}_1 \cup ... \cup \bar{F}_{j-1})$; hence $\nu(1) = 0$ and $\nu(j) > 0$ for all $j = 2, ..., m$. Notice that when $\bar{F}_j$ is adjoined to $\bar{F}_1 \cup ... \cup \bar{F}_{j-1}$ it contributes $x^{\nu(j)}(x+1)^{d-\nu(j)}$ to the rank-generating function of Δ. Consequently, we have the following proposition (a very special case of Theorem 6 of [14]).

PROPOSITION 1.1 *Let Δ be a shellable simplicial complex with facet-size d, and let $F_1, ..., F_m$ be a shelling order for the facets of Δ. Then $h_i(\Delta) = \#\nu^{-1}(i)$ for all $i = 0, ..., d$. In particular, $h_i(\Delta) \geq 0$ for all $i = 0, ..., d$.*

In fact, the nonnegativity of the h-vector follows from a more general (ring-theoretic) condition on the simplicial complex, known as Cohen-Macaulayness, but Proposition 1.1 will suffice for the applications we have in mind. It is also worth noting that Stanley has given a complete characterization of the h-vectors of shellable and Cohen-Macaulay complexes (cf. Theorem 6 of [14]).

We are almost ready to state the main results of this paper. For a natural number j, let $x_{\langle j \rangle} = x(x-1)\cdots(x-j+1)$ denote the j-th *falling factorial polynomial*, and define a linear transformation $S : \mathbb{R}[x] \to \mathbb{R}[x]$ by $Sx_{\langle j \rangle} = x^j$ and linear extension. We call S the *Stirling tansformation* since $x_{\langle j \rangle}$ is the generating function for the Stirling numbers of the first kind and Sx^j is the generating function for the Stirling numbers of the second kind.

THEOREM 1.2 *Let $p \in \mathbb{R}[x]$ be any polynomial, say $p(x) = \sum_{i=0}^{d} c_i x^i (x+1)^{d-i}$. If $c_i \geq 0$ for all $i = 0, ..., d$ then $Sp(x)$ has only real nonpositive zeros.*

THEOREM 1.3 *Let $p \in \mathbb{R}[x]$ be a polynomial such that $p(0) = 0$, say $p(x) = x\sum_{i=0}^{d} c_i x^i (x-1)^{d-i}$. If $c_i \geq 0$ for all $i = 0, ..., d$ then $Sp(x)$ has only real nonpositive zeros.*

Proofs of these theorems are deferred until Section 2. In view of Proposition 1.1 and formula (1) the following corollary is immediate.

COROLLARY 1.4 *Let Δ be a finite simplicial complex with nonnegative h-vector. Then $S\mathrm{rgf}(\Delta;x)$ and $Sx\mathrm{rgf}(\Delta;-x)$ have only real nonpositive zeros.*

Before turning to applications of these results, let's consider some of their ramifications in general. Let $\varepsilon : \mathbb{R}[x] \to \mathbb{R}[x]$ be the $\mathbb{R}$-algebra automorphism defined by $\varepsilon x = -x$ and linear and multiplicative extension: for any $p \in \mathbb{R}[x]$ we have $\varepsilon p(x) = p(-x)$. In particular, $\varepsilon x_{(j)} = (-1)^j x^{\langle j\rangle}$, where $x^{\langle j\rangle} = x(x+1)\cdots(x+j-1)$ is the j-th *rising factorial polynomial.* Conjugating S by ε we obtain the linear transformation $T = \varepsilon S \varepsilon$ defined by $Tx^{\langle j\rangle} = x^j$ and linear extension. In Theorems 1.2, 1.3, and Corollary 1.4 we may conjugate by ε to obtain the following statements.

COROLLARY 1.5 (a) *Let $p \in \mathbb{R}[x]$ be any polynomial, say $p(x) = \sum_{i=0}^{d} c_i x^i (x-1)^{d-i}$. If $c_i \geq 0$ for all $i = 0, ..., d$ then $Tp(x)$ has only real nonnegative zeros.*
(b) *Let $p \in \mathbb{R}[x]$ be a polynomial such that $p(0) = 0$, say $p(x) = x\sum_{i=0}^{d} c_i x^i (x+1)^{d-i}$. If $c_i \geq 0$ for all $i = 0, ..., d$ then $Tp(x)$ has only real nonnegative zeros.*
(c) *Let Δ be a finite simplicial complex with nonnegative h-vector. Then $Tx\mathrm{rgf}(\Delta;x)$ and $T\mathrm{rgf}(\Delta;-x)$ have only real nonnegative zeros.*

The condition that a polynomial has only real nonpositive zeros places strong restrictions on the values of its coefficients; see Chapter 8 of [8]. For example, we have the following special case of a theorem of Schoenberg (Theorem 7.1 in Chapter 8 of [8]). Another extension of this result can be found in Theorem 1.3 of [4].

PROPOSITION 1.6 *Let $p(x) = \sum_{i=0}^{d} c_i x^i$ be such that $c_d > 0$ and $c_0 \neq 0$ and if $p(z) = 0$ then $|\pi - \arg(z)| < \pi/3$. Then $c_i > 0$ for all $i = 0, ..., d$, and $c_i^2 > c_{i-1}c_{i+1}$ for all $i = 1, ..., d-1$.*

One difficulty arises in applying Proposition 1.6 in conjunction with Corollaries 1.4 and 1.5: we are primarily interested in the zeros and coefficients of $\mathrm{rgf}(\Delta;x)$, but our conclusions concern $S\mathrm{rgf}(\Delta;x)$ and $Tx\mathrm{rgf}(\Delta;x)$. Accordingly, let's consider more closely the condition that $Sp(x)$ has only real nonpositive zeros. Proposition 1.7(a) is implicit in Section 4 of [15]; parts (b) and (c) comprise Theorem 4.7 of [4].

PROPOSITION 1.7 *Let $p(x), q(x) \in \mathbb{R}[x]$.*
(a) *Suppose that $Sp(x)$ and $Sq(x)$ both have only real nonpositive zeros. Then $S[p(x)q(x)]$ also has only real nonpositive zeros.*

(b) *Let $p(x)$ be a polynomial such that $x_{(m)}$ divides $p(x)$. If every real zero of $p(x)$ is at most m and every complex zero of $p(x)$ is in the parabolic region*

$$\mathcal{R}_S(m) := \{s + it : 4t^2 \leq 1 + 4m - 4s\}$$

then $Sp(x)$ has only real nonpositive zeros.
(c) *A quadratic polynomial $p(x)$ without real zeros is such that $Sp(x)$ has only real nonpositive zeros if and only if the zeros of $p(x)$ lie in $\mathcal{R}_S(0)$.*

Conjugating by ε, we also see that the parabolic regions

$$\mathcal{R}_T(m) := \{s + it : 4t^2 \leq 1 + 4m + 4s\}$$

play the same rôle for the transformation T as the $\mathcal{R}_S(m)$ do for S. In an as yet imprecise way, Proposition 1.7 suggests that if all the zeros of $Sp(x)$ are real and nonpositive then "most" of the zeros of $p(x)$ lie inside the parabolic region $\mathcal{R}_S(m)$, where m is the multiplicity of 0 as a zero of $Sp(x)$. The accompanying figures corroborate this impression. In Figure 1 we plot all zeros other than -1 of the 941 rank-generating functions of Cohen-Macaulay complexes with $d \leq 4$ and $h_1 = 4$, as well as the boundaries of $\mathcal{R}_S(0)$ and $\mathcal{R}_T(1)$. In Figure 2 we plot all zeros other than -1 of the 813 C-M RGFs with $d \leq 5$ and $h_1 = 3$. Figure 3 is a similar plot for the 520 C-M RGFs with $d \leq 8$ and $h_1 = 2$. Of course, a quantitative converse to Proposition 1.7(b) is very much to be desired.

One simple bound on the location of zeros of $\mathrm{rgf}(\Delta; x)$ follows immediately from nonnegativity of the h-vector (cf. Theorem (1,1) of [9]), but in view of our computations it seems that much stronger results are possible.

PROPOSITION 1.8 *Let $p(x) = \sum_{i=0}^{d} c_i x^i$, where $c_i \geq 0$ for all $i = 0, ..., d$ and $c_0 > 0$. It follows that if $p(z) = 0$ then $|\arg(z)| \geq \pi/d$. Consequently, if Δ is a simplicial complex with nonnegative h-vector and maximum facet-size d then the following hold:*
(a) *all zeros z of $\mathrm{hgf}(\Delta; x)$ satisfy $|\arg(z)| \geq \pi/d$, and*
(b) *all zeros z of $\mathrm{rgf}(\Delta; x)$ satisfy $|\arg(z) - \arg(z+1)| \geq \pi/d$.*

One can also apply other classical results on location of zeros (cf. Chapter VII of [9]), but the resulting bounds do not seem to adequately describe the observed location of zeros of C-M RGFs.

Finally, consider the following easy example. For $n \geq 1$ let $[n] := \{1, ..., n\}$ and let $\Gamma_n := 2^{[n]} \setminus \{[n]\}$. Then Γ_n is a shellable simplicial complex (the boundary

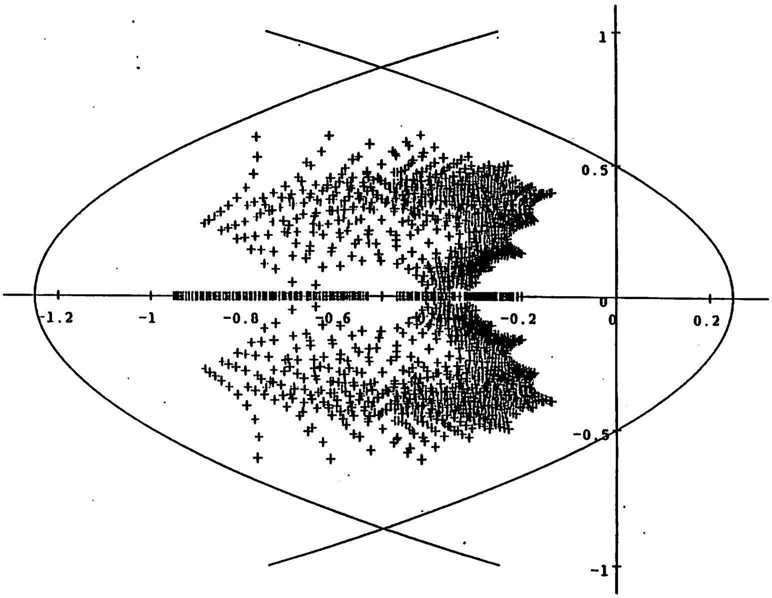

Figure 1: Zeros of C-M RGFs with $d \leq 4$ and $h_1 = 4$.

of a simplex) with $\mathrm{rgf}(\Gamma_n; x) = (x+1)^n - x^n$ and $\mathrm{hgf}(\Gamma_n; x) = 1 + x + \cdots + x^{n-1} = (x^n - 1)/(x-1)$. As $n \to \infty$ the limit distribution of zeros of $\mathrm{hgf}(\Gamma_n; x)$ is uniformly concentrated on the circle $\{z \in \mathbb{C} : |z| = 1\}$. Applying the transformation $z \mapsto z/(1-z)$ takes the zeros of $\mathrm{hgf}(\Gamma_n; x)$ to the zeros of $\mathrm{rgf}(\Gamma_n; x)$; this amounts to inversion in the unit circle centered at 1, followed by reflection in the imaginary axis. It is a simple matter to check that all the zeros of $\mathrm{rgf}(\Gamma_n; x)$ lie on the line $\mathrm{Re}(z) = -1/2$, and that as $n \to \infty$, the proportion of zeros of $\mathrm{rgf}(\Gamma_n; x)$ in the region $\mathcal{R}_S(0) \cap \mathcal{R}_T(1)$ tends to $2/3$.

Chromatic and Birkhoff Polynomials

Let $\mathcal{M}$ be a loopless matroid on the set E, with no parallel elements. Then $\mathcal{M}$ is determined by its geometric lattice $\mathcal{L}$ of flats, or closed subsets of E. Let the minimal and maximal elements of $\mathcal{L}$ be denoted by $\hat{0}$ and $\hat{1}$, respectively, let the rank of $p \in \mathcal{L}$ be denoted $r(p)$, and let $d = r(\hat{1})$. The *Birkhoff (characteristic) polynomial* of $\mathcal{L}$ is

$$B(\mathcal{L}; x) := \sum_{p \in \mathcal{L}} \mu(\hat{0}, p) x^{d-r(p)},$$

where $\mu(\cdot,\cdot)$ is the Möbius function of $\mathcal{L}$. When $\mathcal{M}$ is the graphic matroid of a

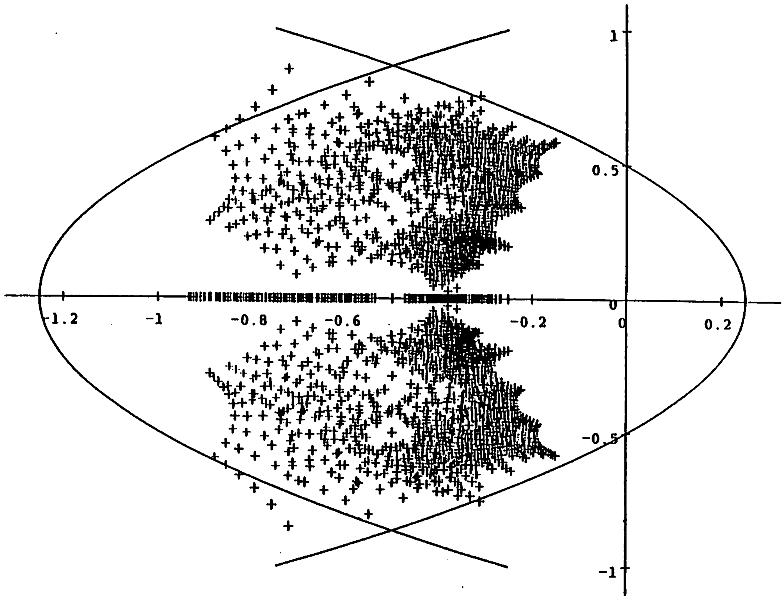

Figure 2: Zeros of C-M RGFs with $d \leq 5$ and $h_1 = 3$.

connected simple graph G, the lattice $\mathcal{L}$ is the lattice of contractions of G, and $P(G;x) := xB(\mathcal{L};x)$ is the *chromatic polynomial* of G (cf. Section 9 of [12]). A theorem of Whitney [18], generalized by Rota (cf. [12], p.359), asserts that for any geometric lattice $\mathcal{L}$ there is a shellable simplicial complex $\mathcal{B}$, called the *broken circuit complex* of $\mathcal{L}$, such that

$$B(\mathcal{L};x) = (-x)^d \mathrm{rgf}(\mathcal{B};-1/x).$$

Thus Corollaries 1.4 and 1.5(c) imply that the zeros of $S(-x)^d B(\mathcal{L};-1/x)$ are all real and nonpositive, and that the zeros of $T(-x)^{1+d}B(\mathcal{L};-1/x)$ are all real and nonnegative. This suggests that "most" of the zeros of $(-x)^d B(\mathcal{L};-1/x)$ lie in $\mathcal{R}_S(0) \cap \mathcal{R}_T(1)$. Replacing x by $-1/x$ amounts to inverting in the unit circle and reflecting in the imaginary axis. Then $\mathcal{R}_S(0)$ is transformed to the exterior of the cardiod $\mathcal{C} := \{re^{i\theta} \in \mathbf{C} : r \geq 2(1-\cos\theta)\}$ and $\mathcal{R}_T(1)$ is transformed to the unbounded region

$$\mathcal{C}' := \{s+it \in \mathbf{C} : 5(s^2+t^2)^2 - 4(s^3+st^2+t^2) \geq 0\}.$$

Hence we expect "most" zeros of Birkhoff polynomials of geometric lattices to lie in $\mathcal{C} \cap \mathcal{C}'$.

As seen in [4], there is a great deal of evidence for the following conjecture.

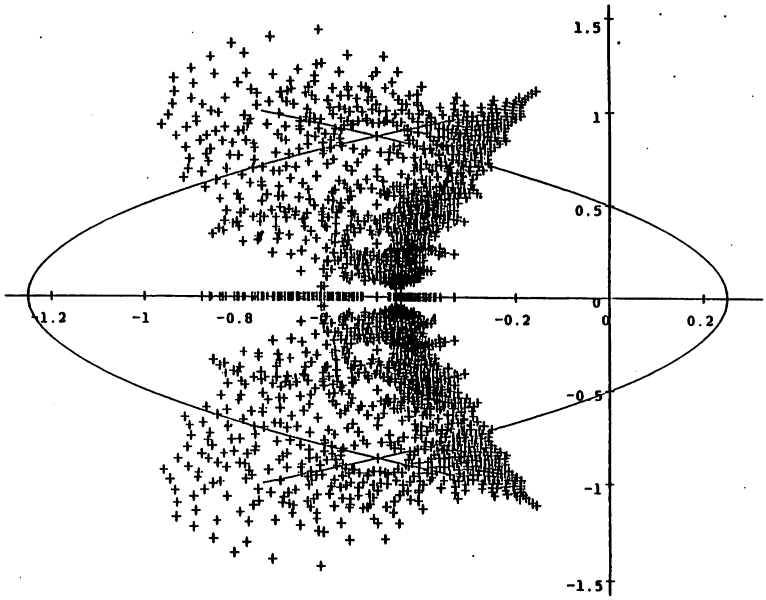

Figure 3: Zeros of C-M RGFs with $d \leq 8$ and $h_1 = 2$.

CONJECTURE 1.9 *Let G be a connected simple graph with chromatic polynomial $P(G;x)$. Then $TP(G;x)$ has only real nonnegative zeros.*

Unfortunately, Corollaries 1.4 and 1.5(c) seem to be working in exactly the wrong direction with regard to this conjecture, as our results show that if G has n vertices then $T(-x)^n P(G;-1/x)$ has only real nonnegative zeros. However, since there are graphs G for which $SP(G;x)$ has nonreal zeros (see [4]), perhaps we should not put much faith in Conjecture 1.9.

Reliability Polynomials

Let $\mathcal{M}$ be a loopless matroid on a set E of size m, and let $\mathcal{I}$ be the collection of independent sets of $\mathcal{M}$; then $\mathcal{I}$ is a shellable simplicial complex (cf. Proposition 4.2 of [2]), and so it is pure, of facet-size d, say. Suppose that each element of E fails independently with probability q; we are interested in the probablility $\mathrm{Rel}(\mathcal{M};q)$ that the set of failing elements is in $\mathcal{I}$. For example, let G be a finite connected loopless multigraph with edge-set E, and let $\mathcal{M}$ be the cographic matroid of G: a set of edges is in $\mathcal{I}$ if and only if its complement induces a connected spanning subgraph of G. Hence in this case $\mathrm{Rel}(\mathcal{M};q)$ is the probablility that G remains connected when the edges fail independently with probability q.

We may partition the event that the set of failing elements X is in $\mathcal{I}$ into its constituent subevents: that $X = F$ for a given face F of $\mathcal{I}$. This leads to the following expansion of the probability $\mathrm{Rel}(\mathcal{M}; q)$:

$$\mathrm{Rel}(\mathcal{M}; q) = \sum_{i \in \mathbf{N}} f_i(\mathcal{I}) q^i (1-q)^{m-i} = (1-q)^m \mathrm{rgf}\left(\mathcal{I}; \frac{q}{1-q}\right) = (1-q)^{m-d} \mathrm{hgf}(\mathcal{I}; q).$$

Thus the probability $\mathrm{Rel}(\mathcal{M}; q)$ is a polynomial function of q, called the *reliability polynomial* of $\mathcal{M}$.

Since $\mathrm{rgf}(\mathcal{I}; x) = (x+1)^m \mathrm{Rel}(\mathcal{M}; x/(x+1))$, Corollaries 1.4 and 1.5(c) imply that all the zeros of $S(x+1)^m \mathrm{Rel}(\mathcal{M}; x/(x+1))$ are real and nonpositive, and that all the zeros of $Tx(x+1)^m \mathrm{Rel}(\mathcal{M}; x/(x+1))$ are real and nonnegative. This suggests that "most" of the zeros of $(x+1)^m \mathrm{Rel}(\mathcal{M}; x/(x+1))$ are in $\mathcal{R}_S(0) \cap \mathcal{R}_T(1)$. Now $\theta \neq -1$ is a zero of $\mathrm{rgf}(\mathcal{I}; x)$ if and only if $\theta/(\theta+1)$ is a zero of $\mathrm{Rel}(\mathcal{M}; x)$. Mapping z to $z/(z+1)$ amounts to inverting in the unit circle centered at -1 and then reflecting in the imaginary axis. Thus $\mathcal{R}_S(0)$ is transformed to the unbounded region $1 - \mathcal{C}'$ and $\mathcal{R}_T(1)$ is transformed to the exterior of the cardiod $1 - \mathcal{C}$. Hence we expect "most" zeros of reliability polynomials other than $q = 1$ to lie in in region $1 - (\mathcal{C} \cap \mathcal{C}')$.

Brown and Colbourne [5] make the following conjecture:

CONJECTURE 1.10 *For any loopless cographic matroid* $\mathcal{M}$ *all the zeros of the polynomial* $\mathrm{Rel}(\mathcal{M}; q)$ *are in the disc* $\{z \in \mathbf{C} : |z| \leq 1\}$.

Note that this is equivalent to saying that all the zeros of $\mathrm{rgf}(\mathcal{I}; x)$ have real part greater than or equal to $-1/2$. Again, Corollaries 1.4 and 1.5(c) seem to have little to do with this conjecture. This is not surprising, as these results rely merely on the nonnegativity of the h-vector while a proof of the Brown-Colbourne conjecture must make use of the matroid structure since the conjecture fails to hold for all shellable complexes.

Order Polynomials

Let P be a nonempty partially ordered set, and for each positive integer m let $\Omega(P; m)$ denote the number of order-preserving functions $\phi : P \to \mathbf{m}$ from P into the m-element chain $\mathbf{m} = \{1 < 2 < ... < m\}$. Then $\Omega(P; m)$ is a polynomial function of m, called the *order polynomial* of P, and has the expansion

$$\Omega(P; x) = \sum_{j \geq 1} e_j(P) \binom{x}{j}$$

in which $e_j(P)$ is the number of order-preserving surjections from P onto a j-element chain (cf. Proposition 13.1 of [13]).

To see the connection with rank-generating functions, define a new partially ordered set $\mathcal{S}$ as follows. The elements of $\mathcal{S}$ are the order-preserving surjections from P onto a nonempty chain, and two surjections $\phi : P \to \mathbf{j}$ and $\psi : P \to \mathbf{k}$ are related by $\phi \leq \psi$ in $\mathcal{S}$ if and only if there is an order-preserving map $\sigma : \mathbf{k} \to \mathbf{j}$ such that $\phi = \sigma\psi$. One can check that this is the same thing as the order complex of the proper part of the finite distributive lattice of order ideals of P. As such, S. Provan showed that $\mathcal{S}$ is a shellable simplicial complex; see Theorem 3.7 and the remarks after Corollary 3.2 of [1]. It is clear that the rank-generating function of $\mathcal{S}$ is $\mathrm{rgf}(\mathcal{S};x) = \sum_{j\geq 1} e_j(P)x^{j-1}$. Now a simple calculation with geometric series yields the identity

$$\sum_{m\geq 0} \Omega(P;m)t^m = \frac{t}{(1-t)^2}\mathrm{rgf}\left(\mathcal{S};\frac{t}{1-t}\right) = \frac{t\cdot \mathrm{hgf}(\mathcal{S};t)}{(1-t)^{2+d}}.$$

Neggers [10] made a conjecture equivalent to the following in 1978. In 1986 Stanley made an analogous conjecture for all labelled posets (private communication). See [3, 17] for recent work on these conjectures.

CONJECTURE 1.11 *For any nonempty poset P, all the zeros of* $\mathrm{rgf}(\mathcal{S};x)$ *are in the interval* $[-1,0]$.

If we assume that this conjecture holds then by Proposition 1.7(b) and its conjugate by ε we may conclude that $\mathcal{S}\mathrm{rgf}(\mathcal{S};x)$ has only real nonpositive zeros, and that $\mathcal{T}x\mathrm{rgf}(\mathcal{S};x)$ has only real nonnegative zeros. But this we know to be true, by Corollaries 1.4 and 1.5(c); hence these results are consistent with the validity of Conjecture 1.11.

2 Proofs

In order to prove Theorems 1.2 and 1.3 we require a few definitions and lemmas. A polynomial $p(x)$ is called *standard* if either $p(x) = 0$ identically or its leading coefficient is positive. For a subset I of the complex plane, $p(x)$ is called *I-rooted* if either $p(x) = 0$ identically or $p(z) = 0$ implies $z \in I$. Let $p(x)$ and $q(x)$ be two $\mathbb{R}$-rooted polynomials; let the zeros of $p(x)$ be $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_r$ and let the zeros of $q(x)$ be $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_s$. We say that $p(x)$ *interlaces* $q(x)$ when $s = 1 + r$ and

$$\theta_1 \leq \xi_1 \leq \theta_2 \leq \xi_2 \leq \cdots \leq \theta_r \leq \xi_r \leq \theta_{r+1}.$$

Also, we say that $p(x)$ *alternates left of* $q(x)$ when $s = r$ and

$$\xi_1 \leq \theta_1 \leq \xi_2 \leq \cdots \leq \theta_{r-1} \leq \xi_r \leq \theta_r.$$

We will use the notations $p \dagger q$ for p interlaces q, and $p \langle\!\langle q$ for p alternates left of q. Lemmas 2.1 to 2.4 follow easily from the Intermediate Value Theorem. The method of proof is explained in detail in Section 3 of [16]. We use D to denote the differentiation operator d/dx.

LEMMA 2.1 *Let p be a* $\mathbf{R}$*-rooted polynomial. Then $Dp \dagger p$.*

LEMMA 2.2 *Let p, q be nonzero standard* $\mathbf{R}$*-rooted polynomials such that $p \dagger q$. Then $p \dagger p + q$ and $p + q \langle\!\langle q$. Also, $p \dagger q - p$ and $q \langle\!\langle q - p$.*

LEMMA 2.3 *Let p, q be nonzero standard* $\mathbf{R}$*-rooted polynomials such that $p \langle\!\langle q$. Then $p \langle\!\langle p + q$ and $p + q \langle\!\langle q$.*

LEMMA 2.4 *Let $p_1, ..., p_m$ be nonzero* $\mathbf{R}$*-rooted polynomials such that $p_i \langle\!\langle p_{i+1}$ for $1 \leq i \leq m - 1$, and $p_1 \langle\!\langle p_m$. Then $p_i \langle\!\langle p_j$ for all $1 \leq i \leq j \leq m$.*

Lemma 2.5 is a simple induction on $\deg p$ (cf. Proposition 5.1(f) of [4]).

LEMMA 2.5 *For any $p \in \mathbf{R}[x]$ and $\alpha \in \mathbf{R}$, $S(x - \alpha)p(x) = (x(1 + D) - \alpha)Sp(x)$.*

To prove Theorem 1.2 we consider the polynomials $\phi_i^d := Sx^i(x + 1)^{d-i}$, for $i = 0, ..., d$, and $d \geq 0$.

PROPOSITION 2.6 *Each ϕ_i^d is nonzero, standard, and $(-\infty, 0]$-rooted. Furthermore, $\phi_i^d \langle\!\langle \phi_j^d$ for all $0 \leq i \leq j \leq d$.*

PROOF: The proposition is true for $d = 0$ since $\phi_0^0 = 1$, and true for $d = 1$ since $\phi_0^1 = x + 1$ and $\phi_1^1 = x$. By induction suppose that the proposition is true for $d - 1$. Note that for any $i = 0, ..., d - 1$ we have by Lemma 2.5 that

$$\phi_i^d = (x(1 + D) + 1)Sx^i(x + 1)^{d-1-i} = (1 + D)x\phi_i^{d-1} \tag{2}$$

and that

$$\phi_{i+1}^d = Sx^{i+1}(x + 1)^d = x(1 + D)\phi_i^{d-1}. \tag{3}$$

It follows that $\phi_0^d = [(1+D)x]^d 1$ and that $\phi_d^d = [x(1+D)]^d 1$ for all $d \geq 0$, and hence that $\phi_0^d = (1+D)\phi_d^d$. By induction ϕ_i^{d-1} is $(-\infty, 0]$-rooted, so it follows from (3) and Lemmas 2.1 and 2.2 that ϕ_{i+1}^d is also $(-\infty, 0]$-rooted, and that $\phi_i^{d-1} \dagger \phi_{i+1}^d$. Now by (2) we have $\phi_i^d = \phi_{i+1}^d + \phi_i^{d-1}$, and it follows from Lemmas 2.1 and 2.2 that ϕ_i^d is $(-\infty, 0]$-rooted and that $\phi_i^d \langle\!\langle \phi_{i+1}^d$. We have shown that $\phi_0^d \langle\!\langle \ldots \langle\!\langle \phi_d^d$ and that all these polynomials are $(-\infty, 0]$-rooted. Now, since $\phi_0^d = (1+D)\phi_d^d$, Lemmas 2.1 and 2.2 imply that $\phi_0^d \langle\!\langle \phi_d^d$. By Lemma 2.4 this suffices to finish the induction step and the proof. □

PROPOSITION 2.7 *Let $f_0, \ldots, f_d$ be any sequence of nonzero standard $\mathbb{R}$-rooted polynomials such that $f_i \langle\!\langle f_j$ for all $0 \leq i \leq j \leq d$. Then for any nonnegative numbers $c_0, \ldots, c_d$ the polynomial $p = c_0 f_0 + \ldots + c_d f_d$ is $\mathbb{R}$-rooted and $f_0 \langle\!\langle p \langle\!\langle f_d$.*

PROOF: By induction on d; the basis $d = 0$ is trivial, and the case $d = 1$ is Lemma 2.3. For the induction step let $f_i' = f_i$ for $i < d-1$ and let $f_{d-1}' = c_{d-1} f_{d-1} + c_d f_d$. By Lemma 2.3 we have $f_{d-1} \langle\!\langle f_{d-1}' \langle\!\langle f_d$, and hence by Lemma 2.4 we find that $f_0', \ldots, f_{d-1}'$ satisfy the inductive hypothesis. Putting $c_i' = c_i$ for $i < d-1$ and $c_{d-1}' = 1$ we find that $p = c_0' f_0' + \ldots c_{d-1}' f_{d-1}'$ is $\mathbb{R}$-rooted and that $f_0' \langle\!\langle p \langle\!\langle f_{d-1}'$. But since $f_0' = f_0$, $f_{d-1}' \langle\!\langle f_d$, and $f_0 \langle\!\langle f_d$, we conclude from Lemma 2.4 that $f_0 \langle\!\langle p \langle\!\langle f_d$, which completes the induction step and the proof. □

PROOF OF THEOREM 1.2: We have $Sp(x) = \sum_{i=0}^d c_i \phi_i^d(x)$ and the result follows directly from Propositions 2.6 and 2.7.

To prove Theorem 1.3 we consider the polynomials $\psi_i^d := Sx^i(x-1)^{d-i}$ for $i = 1, \ldots, d$, and $d \geq 1$.

PROPOSITION 2.8 *Each ψ_i^d is nonzero, standard, and $(-\infty, 0]$-rooted. Furthermore, $\psi_j^d \langle\!\langle \psi_i^d$ for all $1 \leq i \leq j \leq d$.*

PROOF: The proposition is true for $d = 1$ since $\psi_1^1 = x$, and true for $d = 2$ since $\psi_1^2 = x^2$ and $\psi_2^2 = x(x+1)$. By induction suppose that the proposition is true for $d-1$. Note that for any $i = 1, \ldots, d-1$ Lemma 2.5 implies that

$$\psi_i^d = (x(1+D)-1)Sx^i(x-1)^{d-1-i} = (x(1+D)-1)\psi_i^{d-1} \tag{4}$$

and that

$$\psi_{i+1}^d = Sx^{i+1}(x+1)^d = x(1+D)\psi_i^{d-1}. \tag{5}$$

Thus we have the relation $\psi_{i+1}^d = \psi_i^d + \psi_i^{d-1}$ for all $i = 1, ..., d-1$. By the induction hypothesis, ψ_i^{d-1} is $(-\infty, 0]$-rooted. Hence, by (5) and Lemmas 2.1 and 2.2, ψ_{i+1}^d $(-\infty, 0]$-rooted and $\psi_i^{d-1} \dagger \psi_{i+1}^d$. It follows from (4) and Lemma 2.2 again that $\psi_i^d = \psi_{i+1}^d - \psi_i^{d-1}$ is $\mathbb{R}$-rooted and that $\psi_{i+1}^d \langle\!\langle \psi_i^d$. We have deduced that $\psi_d^d \langle\!\langle \ldots \langle\!\langle \psi_1^d$ are all nonzero, standard, and $\mathbb{R}$-rooted.

Now we claim that for any $k \in \mathbb{N}$, $Sx(x-1)^k = xSx^k$. For $k = 0$ this is trivial. By induction we calculate, using Lemma 2.5, that

$$\begin{aligned} Sx(x-1)^k &= (x(1+D)-1)Sx(x-1)^{k-1} \\ &= (x(1+D)-1)xSx^{k-1} \\ &= x^2(1+D)Sx^{k-1} = xSx^k, \end{aligned}$$

as desired. This identity implies that $\psi_1^d = x\psi_{d-1}^{d-1}$ for all $d \geq 2$. Since ψ_{d-1}^{d-1} is $(-\infty, 0]$-rooted by the inductive hypothesis, it follows that all ψ_i^d are $(-\infty, 0]$-rooted. Above, we found that $\psi_{d-1}^{d-1} \dagger \psi_d^d$; since $\psi_1^d = x\psi_{d-1}^{d-1}$ we can now conclude that $\psi_d^d \langle\!\langle \psi_1^d$. By Lemma 2.4 this suffices to finish the induction step and the proof. □

Proof of Theorem 1.3 Theorem 1.3 now follows directly from Propositions 2.7 and 2.8.

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