Decompositions of Matroids and Exponential Structures

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Abstract

We associate to a simple matroid (resp. a geometric lattice) a partially ordered set whose upper intervals are set partition lattices. Indeed for some important cases they are exponential structures in the sense of [Sta78]. Our construction includes the partition lattice, the poset of partitions whose size is divisible by a fixed number d, and the poset of direct sum decompositions of a vector space. If we start with a modularly complemented matroid the resulting poset is CL-shellable. This generalizes results of B. Sagan [Sag86] and M. Wachs and settles the open problem of the shellability of the poset of direct sum decompositions. Finally we give a formula for the Möbius number of the poset of direct sum decompositions of a vector space.

1 Matroids, Flats and Exponential Structures

The notation of exponential structures was introduced by Stanley [Sta78]. It provides a general setting for the treatment of a wide class of interesting posets. The poset of set partitions and the poset of partitions whose block sizes are divisible by a fixed number d [Sta78], [Sag86] are well studied examples. Another example, also mentioned by Stanley, is the poset of direct sum decomposition of vector spaces. This example led us to a more general approach to "decompositions" of matroids by which we retrieve some important classes of exponential structures.

For a finite matroid M of rank $r = rank(M)$ we denote by $\mathcal{F}(M)$ the geometric lattice of its flats. We write $\vee$ and $\wedge$ for the join and meet in $\mathcal{F}(M)$. We call a subset $\{F_1, \ldots, F_k\}$ of flats (of rank ≥ 1) of $\mathcal{F}(M)$ a decomposition of M if

$$\text{(A)} \quad \sum_{i=1}^{k} rank(F_i) = r \text{ and}$$

$$\text{(B)} \quad F_1 \vee \cdots \vee F_k = M.$$

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We order the set $\mathcal{D}(M)$ of decompositions of M by refinement. This means that for two decompositions $F = \{F_1,\dots,F_k\}$ and $E = \{E_1,\dots,E_l\}$ of M the inequality $F \le E$ holds if and only if for all $1 \le j \le k$ there exists an $1 \le i \le l$ such that $F_j \le E_i$. Since in general there does not exist a decomposition which refines all the others, we will always add a least element 0 to $\mathcal{D}(M)$ in order to make $\mathcal{D}(M)$ to a bounded poset. It is easy to see (Proposition 1.2) that $\mathcal{D}(M)$ carries for some important cases the properties of elements of exponential structures. Because we also want to cover the poset of set partitions where each block is divisible by a certain number d we define a type selected subposet of $\mathcal{D}(M)$. More precisely we write $\mathcal{D}(M)_d$ for the poset of all elements E of $\mathcal{D}(M)$ such that either $E = 0$ or each block of E is a flat in $\mathcal{F}(M)$ whose rank is divisible by d. For $d = 1$ we simply retrieve the original poset $\mathcal{D}(M)$. Obviously it makes no sense to look for flats of rank greater then r in a matroid of rank r. Therefore we restrict d to the interval $[r] := \{1,\dots,r\}$. For a $d \in [r]$ which does not divide r the poset $\mathcal{D}(M)$ consists only of the single element 0. Hence we can further restrict our considerations to the case when d divides r.

It turns out that quite a lot of exponential structures investigated so far arise by this construction from a series of matroids. In particular for the projective geometry $\mathcal{P}^r\mathsf{F}_q$ of dimension r over the field with q elements the poset $\mathcal{D}(\mathcal{P}^r\mathsf{F}_q)$ is the poset of direct sum decompositions of the vector space F_p^{r+1}. The shellability of $\mathcal{D}(\mathcal{P}^r\mathsf{F}_q)$ has been left as an open problem in the paper of Sagan [Sag86]. More generally we prove that $\mathcal{D}(M)_d$ is shellable if $\mathcal{F}(M)$ is a modularly complemented geometric lattice. A geometric lattice (resp. a matroid) is called modularly complemented if there is a base of the matroid such that each subset of the base generates a modular element. Modularly complemented geometric lattices have been introduced by Stonesifer [Sto80]. Their classification by Kahn and Kung [KK86] shows that they are actually not too far away from being a modular geometric lattice. But they enlarge the class by some interesting examples such as the Dowling lattices [Dow73].

For a poset P and two elements $a, b \in P$ we denote by $[a,b] = \{c \mid c \in P. a \le c \le b\}$ the interval of all elements between a and b. We call a poset P exponential of rank n if

Exp_1) P is graded of rank n and P has a greatest element 1.

Exp_2) For every atom a of P the interval $[a,1]$ is isomorphic to the lattice Π_n of set partitions.

Exp_3) If a is an atom of P and if $b \in P$ is an element of the interval $[a,1] \cong \Pi_n$ then the type $(\lambda_1,\dots,\lambda_k)$ of b in Π_n does not depend on the choice of a. Furthermore there are graded posets $Q_0,\dots,Q_{r+1}$ such that Q_i has rank i and if $b \in P$ has type $(\lambda_1,\dots,\lambda_k)$ then $\{c \in P \mid c \le b\} \cong Q_{\lambda_1} \times \dots \times Q_{\lambda_k}$.

According to Stanley we call a sequence $(P_1,\dots,P_t,\dots)$ of posets an exponential structure if each P_i is an exponential poset of rank $i+1$ and $Q_i \cong P_i$ for the posets in Exp_3 of the definition of an exponential poset.

Lemma 1.1 *Let M be a finite matroid of rank $r = d \cdot n$ for some divisor d of r. Then $\mathcal{D}(M)_d - \{0\}$ satisfies Exp_1 and Exp_2.*

Proof : From the fact that $r = n \cdot d$ and the fact that $\mathcal{F}(M)$ is a geometric lattice it follows that there exist flats $F_1,\dots,F_n$ of rank d in $\mathcal{F}(M)$ such that $rank(F_1 \vee \dots \vee F_n) = d \cdot n$. Moreover since the intervals of the geometric lattice $\mathcal{F}(M)$ are also geometric we can refine each nontrivial element $\{F_1,\dots,F_k\}$ of $\mathcal{D}(M)_d$ to a decomposition of M into n flats of rank d. Hence all atoms of $\mathcal{D}(M)_d$ are of the form $\{F_1,\dots,F_n\}$ where each F_i is a flat of rank d in $\mathcal{F}(M)$. Now we fix an atom $F = \{F_1,\dots,F_n\}$ of $\mathcal{D}(M)_d$. Then the mapping which assigns to each partition $\pi = A_1 + \dots + A_k$ in Π_n the element $\{\bigvee_{j\in A_1} F_j,\dots,\bigvee_{j\in A_k} F_j\}$ establishes an isomorphism between Π_n and the interval $[F,1]$. This facts follows immediately from $F_1 \vee \dots \vee F_n = M$ and $rank(\bigvee_{j\in A_i} F_j) = d \cdot |A_i|$. ∎

Proposition 1.2 *Let M be a finite matroid of rank $r = d \cdot n$ for some divisor d of r. Then $\mathcal{D}(M)_d - \{0\}$ is an exponential poset of rank n if and only if for any fixed i, $1 \leq i \leq n$ the posets $\mathcal{F}(N)_d$ are isomorphic for all $(d \cdot i)$-flats N of M.*

Proof : Let $b = \{B_1, \ldots, B_k\}$ be an element of $\mathcal{D}_d(M) - \{0\}$. Obviously any refinement $a \leq b$ corresponds bijectively to a k-tuple of elements of the decompositions posets $\mathcal{D}_d(B_i) - \{0\}$ of the matroids determined by the flats B_i. Hence the interval $[0, b]$ in $\mathcal{D}(M)_d$ is isomorphic to the direct product $(\mathcal{D}_d(B_1)_d - \{0\}) \times \cdots \times (\mathcal{D}_d(B_k) - \{0\})$.

If b contains only one flat then condition Exp_3 is trivially satisfied by b. If b contains more than one flat then by induction $\mathcal{D}(B_i)_d$ depends up to isomorphy only on the rank of B_i. Hence condition Exp_3 holds.

On the other hand assume that Exp_3 holds. Then assume that there is are flats B and B' of the same rank in $\mathcal{F}(M)$ for which $\mathcal{F}(B)_d$ and $\mathcal{F}(B')_d$ are not isomorphic. Since B and B' are different flats of the same rank their rank is not 0 and not maximal. Hence there exists a decomposition b (resp. b') in $\mathcal{D}(M)_d$ which consist of the block B (resp. B') and some blocks of rank d. By construction the interval $[0, b]$ (resp. $[0, b']$) is isomorphic to $\mathcal{D}(B)_d$ (resp. $\mathcal{D}(B')_d$). But this contradicts Exp_3 and we are done. ∎

Before we show, that if $\mathcal{F}(M)$ is modularly complemented then $\mathcal{D}(M)_d$ is shellable, we give some examples :

i) Let $M = U_{n,r}$ be the uniform matroid of rank r on an n-element set. For this matroid the poset $\mathcal{D}(M)$ is the disjoint union of $\binom{n}{r}$ partition lattices Π_r where the greatest elements are identified and an extra 0 is added. For an integer d dividing $r = k \cdot d$ the poset $\mathcal{D}(U_{r,r})_d$ is isomorphic to the poset $\Pi_r^{(d)}$ of partitions of r whose block sizes are divisible by d.

ii) If $M = \mathcal{P}^r\mathbb{F}_q$ is the projective geometry of dimension r over the field $\mathbb{F}_q$ then $\mathcal{D}(M)$ is isomorphic to the poset of direct sum decompositions of the vector space $\mathbb{F}_q^{r+1}$ ordered by refinement.

iii) If $M = \Pi_r$ is the partition lattice then every nontrivial element $b = \pi_1 / \cdots / \pi_k \in \mathcal{D}(\Pi_r)$ represents a family of partitions such that only $(1 \cdots r)$ is a partition coarser than all π_i. Additionally for all proper subsets of the set $\{\pi_1, \ldots, \pi_k\}$ there is a partition different from $(1 \cdots r)$ which is coarser than all π_i.

In contrast to examples i) and ii) $\mathcal{D}(\Pi_r)$ is not an exponential poset for $r \geq 4$.

We saw that for $r < n$ the proper part of $\mathcal{D}(U_{r,k})$ is a disconnected poset of rank $r - 2$. Hence $\mathcal{D}(U_{r,k})$ is not shellable for $r > k \geq 2$. But we will prove in the next section that $\mathcal{D}(M)_d$ is shellable for a suitably sized class of matroids.

2 Shellable decomposition posets

For this section we assume that M is a modularly complemented matroid of rank r. We assume further that $\mathrm{F} = \{F_1, \ldots, F_r\}$ is a base of M such that each subset of the base generates a modular element. Such a base exists by the definition of modularly complemented. For a subset $J \subseteq [r]$ we set $F^J := \bigvee_{j \in J} F_j$; for $J = \emptyset$ we simply denote by F^J the least element of $\mathcal{F}(M)$. By $\mathcal{A}(M)$ we denote the atoms of $\mathcal{F}(M)$. For our purposes we can restrict ourselves to simple matroids. Therefore the set $\mathcal{A}(M)$ is just the set on which the matroid M is defined.

Having this notation fixed we define a mapping v from the set $\mathcal{A}(M)$ into the set of vectors $\{0,1\}^r$. We set $v(E) := (v_1,\ldots,v_r)$ where $v_j = 1$ if and only if $E \not\leq F^{[r]-\{j\}}$. We write $v(E)_l$ to denote the l-th entry v_l in $v(E)$. By the usual linear order $0 < 1$ on $\{0,1\}$ we can define the reverse lexicographic $\leq_l$ order on $\{0,1\}^r$. Via the map v the order $\leq_l$ defines a partial order $\mathcal{A}(M)$. For this order we write also $\leq_*$ and we set $E \leq_* G :\Leftrightarrow v(E) \leq_l v(G)$. Now we order the bases of the k-flats of $\mathcal{F}(M)$ by an arbitrary (but fixed) linear extension $\leq_{\mathcal{B}}$ of the lexicographic order on the k-tuples $(E_1 <_* \cdots <_* E_k)$. We omit an additional index k for the order $\leq_{\mathcal{B}}$.

Furthermore we will use an extension of $\leq_{\mathcal{B}}$ to compare decompositions which correspond to integer partitions $d+d+\cdots+d = r$. Here we associate to a decomposition $\{E_1,\ldots,E_k\}$ the integer partition $rank(E_1) + \cdots + rank(E_k) = r$ of r. The necessity of ordering decompositions which correspond to partitions of integers in equal parts of size greater 1 arises since we will have to order the atoms of $\mathcal{D}(M)_d$ also in the case $d \neq 1$. We impose the order in the following way. We order the flats of the same rank simply by comparing the associated least bases by $\leq_{\mathcal{B}}$. In order to keep the notation for this relation consistent with order relation on the atoms (= 1-flats) we will denote the order relation also by $\leq_*$. So far we have transformed a decomposition into parts of rank d into an ordered decomposition. Now we can order decompositions into blocks of rank d by the lexicographic order induced by the order $\leq_{\mathcal{B}}$ on the blocks of the decomposition. For the atoms of $\mathcal{D}(M)$ we simply get the old order. Hence we can use $\leq_{\mathcal{B}}$ unambiguously for the atoms of $\mathcal{D}(\mathcal{F})_d$ for all d dividing the rank of $\mathcal{F}$.

Now the main aim of this section is to show that the order $\leq_{\mathcal{B}}$ induces a recursive atom order for $\mathcal{D}(M)_d$. To verify this we have to prove the following two properties [BW83] :

i) Let E $\leq_{\mathcal{B}}$ G be two atoms of $\mathcal{D}(M)_d$. Let I be another element of $\mathcal{D}(M)_d$ such that E $\leq$ I and G $\leq$ I holds in $\mathcal{D}(M)_d$. Then there is an atom of the interval [G, I] which covers an atom of $\mathcal{D}(M)_d$ preceding the atom G in the order $\leq_{\mathcal{B}}$.

ii) For every atom E of $\mathcal{D}(M)_d$ there is a recursive atom order of [E, 1] such that the atoms of [E, 1] which cover an atom of $\mathcal{D}(M)_d$ preceding E come first.

Our first two lemmas characterize the least base of a k-flat with respect to $\leq_{\mathcal{B}}$.

Lemma 2.1 *Let G be a k-flat in $\mathcal{F}(M)$. Let* E $= \{E_1 <_* \cdots <_* E_k\}$ *be the least base of G with respect to $\leq_{\mathcal{B}}$. For $1 \leq i \leq k$ let j_i be the greatest index for which $v(E_i)_{j_i}$ is nonzero. Then the following two assertions hold :*

i) For $1 \leq l < i \leq k$ the j_l-th entry in $v(E_i)$ is zero.

ii) The inequalities $j_1 < \cdots < j_k$ are strict.

Proof : Let $1 \leq l < i \leq k$ be integers. Now let J be the set

$$[j_i] - \Big(\{j_l\} \cup \{j \mid j < j_l \text{ and } v(E_i)_j = 0\}\Big).$$

The fact that $E_i <_* E_{i+1}$ certainly implies $j_i \leq j_{i+1}$. Hence J does not contain j_i. If the j_l-th entry in $v(E_i)$ is not zero then $E_i \not\leq F^J$. We also know by construction that $E_l \not\leq F^J$. Since $F^J \vee E_i = F^J \vee E_i \vee E_l = F^{J\cup\{j_l\}}$ we deduce from the modularity of F^J the fact that $rank(F^J \wedge (E_i \vee E_l)) = 1$. Hence there is an atom $E_0 \leq F^J$ which is also an atom in $E_{j_l} \vee E_{j_i}$. By construction $v(E_{j_l})_t = 0$ implies $v(E_0)_t = 0$ for $t > j_l$. On the other hand we infer from $E_0 \leq F^J$ that $v(E_0)_{j_l} = 0$. Hence the atom E_0 precedes E_{j_l} in the order $\leq_*$. But from the fact that $E_0 \vee E_i = E_0 \vee E_l$ we conclude that replacing E_{j_l} by E_0 in the base E we can construct another base. Since $E_0 <_* E_{j_l}$ the constructed base of G precedes the base E. But this contradicts the assumptions.

By the fact that $E_i \leq_* E_{i+1}$ implies $j_i \leq j_{i+1}$ (which we have already used in the first part of the proof), the second assertion is an immediate consequence of the first one. ∎

Lemma 2.2 *Let G be a k-flat in $\mathcal{F}(M)$. For a base $\mathrm{E} = \{E_1 <_* \cdots <_* E_k\}$ of F the following four conditions are equivalent :*

i) The base E is the least base of G with respect to $\leq_{\mathcal{B}}$.

ii) For all integers d, $2 \leq d \leq k$, and for all all d-element subsets J of $[k]$ the base $\{E_j \mid j \in J\}$ is the least base of E^J with respect to $\leq_{\mathcal{B}}$.

iii) There is an integer d, $2 \leq d \leq k$, such that for all d-element subsets J of $[k]$ the base $\{E_j \mid j \in J\}$ is the least base of E^J with respect to $\leq_{\mathcal{B}}$.

iv) For $i \in [k]$ let j_i be the greatest index such that $v(E_i)_{j_i}$ is nonzero. Then $E_i \leq F^J$ for $J = [j_i] - \{j_1, \ldots, j_{i-1}\}$. Furthermore E_i is the least atom in the flat $F^J \wedge G$.

Proof : First we prove the implication *i)* $\Rightarrow$ *ii)*. Let J be a d-element subset of $[k]$. Then any base of E^J can be extended to a base of F by the elements E_j for $j \notin J$. Hence any base of E^J preceding the base $\{E_j \mid \in J\}$ would allow the construction of a base of G which precedes $\mathrm{E} = \{E_1, \ldots, E_k\}$. Since E is the least base of F we are done by assumption.

Obviously the implication *ii)* $\Rightarrow$ *iii)* is true.

Now consider the implication *iii)* $\Rightarrow$ *iv)*. Let us take a d-element subset of $[k]$ which contains i. By assumption $\{E_j \mid j \in J\}$ is the least base of E^J. Now we apply Lemma 2.1 to $G = E^J$. Hence $v(E_i)_{j_l} = 0$ for all $l \in J \cap [i-1]$. Since $d \geq 2$ there exists for every pair of elements of $[k]$ a d element subset of $[k]$ which contains it. Now the preceding reasoning shows the first part of the claim. The second part of iv) follows immediately by the arguments used for the implication i) $\Rightarrow$ ii).

So it remains to prove the implication iv) $\Rightarrow$ i). Assume that the least base $\mathrm{I} = \{I_1 <_* \cdots <_* I_k\}$ of G is strictly smaller than the base E with respect to $\leq_{\mathcal{B}}$. Let i be the first index such that $E_i \neq I_i$. Let s be the greatest index for which $v(I_i)_s = 1$. Since $I_i <_* E_i$ the inequality $j_{i-1} < s \leq j_i$ follows from the choice of I_i. Hence $E^{[i]} = E^{[i-1]} \vee E_i$ and $I^{[i]} = I^{[i-1]} \vee I_i$ are contained in $G \wedge F^{[j_i]}$. First we treat the case $s = j_i$. The equality $s = j_i$ implies $I^{[i]} = E^{[i]}$. Since I is the least base of G the truncation $\{I_1, \ldots, I_i\}$ is the least base of $I^{[i]}$. For this conclusion we can argue in the same way as for the implication i) $\Rightarrow$ ii). Hence Lemma 2.1 shows that $v(I_i)_{j_l} = 0$ for $l \in [i-1]$. Since $rank(E_i \vee I_i) = 2$ we deduce from the modularity of $E^{[i-1]} = I^{[i-1]}$ that there is an atom I_0 in the meet $I^{[i-1]} \wedge (E_i \vee I_i)$. From the fact $v(E_i)_{j_l} = v(I_i)_{j_l} = 0$ for $l \in [i-1]$ we infer that I_0 is contained in $F^{[t]-\{j_1,\ldots,j_i\}}$. But then I_0 cannot be contained in $E^{[i-1]} = I^{[i-1]}$. Hence $\{I_1, \ldots, I_i, I_0\}$ is another base of G. By construction $v(I_0)_{j_i} = 0$ and therefore $I_0 <_* I_i$. But this contradicts the fact that $\{I_0, \ldots, I_i\}$ is the least base of $I^{[i]}$. What is still left is the case $j_{i-1} < s < j_i$ where all inequalities are strict. Let l be the least index such that $I^{[i]} \leq E^{[l]}$. Hence by the modularity of E^J for $J = \{i, \ldots, l\}$ we have

$$l = rank(E^{[l]}) = rank(I^{[i]} \vee E^J) =$$

$$= rank(I^{[i]}) + rank(E^J) - rank(I^{[i]} \wedge E^J) = i + (l - i + 1) - rank(I^{[i]} \wedge E^J).$$

This shows that $rank(I^{[i]} \wedge E^J) = 1$. But for every atom E_0 in $I^{[i]}$ the entry $v(E_0)_t$ is 0 for $t > s$ whereas $v(E_0)_t = 1$ for an atom E_0 in E^J for $j_i > t$. So we have deduced a contradiction from the assumption also in the case that $s < j_i$. Therefore E is the least base of G, which shows i). ∎

To deal with the poset $\mathcal{D}(M)_d$ uniformly we introduce the notion of a d-base. Let G be any flat in the lattice $\mathcal{F}(M)$ whose rank $l = d \cdot l_1$ is divisible by d. A set of l_1 flats of rank d is called a d-base of G if their join is G. For $d = 1$ we simply have reintroduced the notion of a base. As indicated at the beginning of the section it proves to be useful to identify each d-flat with its least 1-base. Since 1-bases are ordered by $\leq_{\mathcal{B}}$ their lexicographic order induces a linear order on the d-bases.

Theorem 2.3 *Let M be a finite modularly complemented matroid of rank $r = n \cdot d$. Then the order $<_{\mathcal{B}}$ is a recursive atom order for $\mathcal{D}(M)_d$. In particular the poset $\mathcal{D}(M)_d$ is CL-shellable.*

Proof : By Proposition 1.1 and by the definition of an exponential posets we know that $\mathcal{D}(M)_d$ is a graded poset and that each interval is isomorphic to Π_r.

It is well known that Π_r is a geometric lattice and hence upper semimodular. Since every order on the atoms of an upper semimodular lattice is a recursive atom order [BW83, Theorem 5.1] is suffices to show C_1 for the order $<_{\mathcal{B}}$ [Sag86, Lemma 3].

Let $\mathrm{E} = \{E_1 <_* \cdots <_* E_n\}$ and $\mathrm{G} = \{G_1 <_* \cdots <_* G_n\}$ be two atoms of $\mathcal{D}(M)_d$ for which $\mathrm{E} <_{\mathcal{B}} \mathrm{G}$ holds. Let $\mathrm{I} = \{I_1, \dots, I_k\}$ be an element of $\mathcal{D}(M)_d$ such that $\mathrm{E} < \mathrm{I}$ and $\mathrm{G} < \mathrm{I}$ (here $<$ is the order relation in $\mathcal{D}(M)$). Since $\mathrm{E} <_{\mathcal{B}} \mathrm{G}$ there is a flat in I for which E contains a d-base strictly preceding its d-base contained in G. We may assume that I_1 is this flat. Hence G does not contain the least d-base for I_1. Now by Lemma 2.2 ii) there is a flat H of rank $2 * d$ contained in I_1 such that G does not contain the least d-base for I_1. Let G_1, G_2 be the d-base of I_1 contained in G. By construction the decomposition $\{H, G_3, \dots, G_k\}$ is an atom of $[\mathrm{G}, 1]$ which is contained in I. Let H_1, H_2 be the least d-base of H; then $\{H_1, H_2, G_3, \dots, G_k\}$ is a d-base of M and hence an atom in $\mathcal{D}(M)_d$ which is covered by $\{H, G_3, \dots, G_k\}$. Again by construction the atom $\{H_1, H_2, G_3, \dots, G_k\}$ precedes the atom G and hence we have verified C_2 for the order $\leq_{\mathcal{B}}$. ∎

Actually we do not know whether the decomposition poset of a modularly complemented matroid $\mathcal{D}(M)_d$ is EL-shellable or not. In a recent paper by M. Wachs [Wac] it is proved that this is true for the free matroid.

Finally we give an example of a not modularly complemented line configuration for which it is easy to verify that the decomposition poset $\mathcal{D}(M)$ of the associated matroid is shellable (even EL-shellable). Figure 1 depicts the line configuration of a geometric lattice which is not modularly complemented. Obviously the corresponding matroid has rank 3. Hence any base of M consist of 3 points. But there are only two modular hyperplanes (the 3-point lines) in the lattice. Therefore there must be two of the base elements which do not generate a modular line. This shows that the matroid is not modularly complemented.

On the other hand the associated decomposition poset is depicted on the right side of Figure 1. It is easily seen that the poset is bounded of rank 3 and moreover its proper part is connected. This implies that the decomposition poset is shellable. But we would like to remark that the described matroid is supersolvable. Indeed we know of no supersolvable matroid for which the decomposition poset is not shellable and we know of no non supersolvable matroid with shellable decomposition poset.

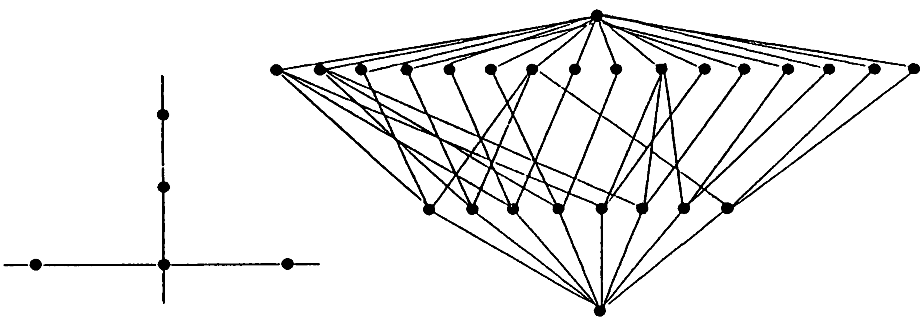

Figure 1

3 The Möbius numbers

For the free matroid M on $r = n \cdot d$ elements the Möbius number $\mu(\mathcal{D}(M))_d = \mu(\Pi_{nd}^{(d)})$ equals the number of permutations with descent set $\{d, 2d, \ldots, nd\}$ [Sta78]. But in the case of projective geometries $\mathcal{P}^r\mathbb{F}_q$ it seems that the Möbius number of $\mathcal{D}(\mathcal{P}^r\mathbb{F}_q)$ has not been investigated so far. We will denote the Möbius number of $\mathcal{D}(\mathcal{P}^r\mathbb{F}_q)$ by $\mu_{r+1}(q)$. The shift in the dimension is for the sake of a more convenient formulation of the formulas. Moreover we will provide some general tools which can be useful for the determination of the Möbius numbers for arbitrary decomposition posets which arise from a modularly complemented matroid.

It is well known that the conditions given Proposition 1.2 are true for $\mathcal{P}^r\mathbb{F}_q$ and therefore $\mathcal{D}(\mathcal{P}^r\mathbb{F}_q)$ is an exponential structure. Hence we can apply the results of Stanley [Sta78] for an analysis of the exponential generating function of $\mu_{r+1}(q)$. It will turn out that a transformation of the generating series leads to a nice formula for the logarithm of the q-hypergeometric series $\sum_{r=0}^{\infty} \frac{q^{r^2}}{\prod_{i=0}^{r}(1-q^i)} z^i$. Additionally we would like to point out that for a prime number p congruent 1 modulo q the Möbius number of $\mathcal{D}(\mathcal{P}^{r-1}\mathbb{F}_q)$ is the same as the Möbius number of the poset $S_p(GL(r,q))$ of nontrival p-subgroups of $GL(r,q)$. Since the order complex of $\mathcal{D}(\mathcal{P}^{r-1}\mathbb{F}_q)$ is even homotopy equivalent to $S_p(GL(r,q))$ [Qui78, Theorem 12.4] we obtain by $|\mu_r(q)|$ the dimension of the $(r-1)$-th homology group of the order complex of $S_p(GL(r,q))$ which is regarded as an analog of the Steinberg module [AS91]. Of course an analysis of the homotopy equivalence eventually shows that also the $GL(r,q)$-module structure is preserved. But the determination of this representation will not be done here. In his proof D. Quillen shows that $S_p(GL(r,q))$ is homotopy equivalent to a subposet which is Cohen-Macaulay. The subposet is given by the elementary abelian p-subgroups. But although by Theorem 2.3 the poset $\mathcal{D}(\mathcal{P}^{r-1}\mathbb{F}_q)$ is CL-shellable it remains open whether the poset of elementary abelian p-subgroups of this $GL(n,q)$ is itself shellable.

Lemma 3.1 *The Möbius number $\mu_r(q)$ is a polynomial of degree $r(r-1)$ in q. The leading coefficient is $\frac{(-1)^r}{r}$.*

Proof : The assertion is trivially fulfilled for $r = 1$. In this case we have $\mu_r(q) = -1$. Now assume $r > 1$. By definition we have

$$\mu_r(q) = - \sum_{\lambda=(\lambda_1\geq\cdots\geq\lambda_k\geq 1)\in P_r-\{(r)\}} \frac{(q^r-1)\cdots(q^r-q^{r-1})}{\prod_{i=1}^k((q^{\lambda_i}-1)\cdots(q^{\lambda_i}-q^{\lambda_i-1}))\cdot j_1^\lambda!\cdots j_r^\lambda!}\cdot(-1)^{k-1}\cdot\prod_{i=1}^k \mu_{\lambda_i}(q).$$

Here we denote by P_r the integer partitions of r and for an integer partition $\lambda \in P_r$ the number j_i^λ is the number of blocks of size i in λ. The fraction (first factor of each summand) counts the number of direct sum decompositions of $\mathbb{F}_q^r$ into blocks of sizes $\lambda_1, \ldots, \lambda_k$. The last two factors stem from the direct decomposition (Exp_3) of the lower intervals in $\mathcal{D}(M) - \{0\}$. The factor $(-1)^{k-1}$ comes up by the fact that the decomposition only exists if the least element is removed. First all factors are easily seen to be polynomials. After the cancellation of the all $(q-1)$'s the first factor has degree $2\binom{r}{2} - 2\sum_{i=1}^k \binom{\lambda_i}{2}$ and the second part has by induction degree $2 \cdot \sum_{i=1}^k \binom{\lambda_i}{2}$. Summing up the degree we obtain as the maximal total degree $r(r-1)$. It remains to show that the coefficient is correct. The first factor has leading coefficient $\frac{1}{j_1!\cdots j_r!}$. The second one has by induction leading coefficient $\frac{(-1)^k}{\prod_{i=1}^k \lambda_i}$. Hence the coefficient for $q^{\binom{r}{2}}$ is

$$\sum_{\lambda=(\lambda_1\geq\cdots\geq\lambda_k\geq 1)\in P_r-\{(r)\}} \frac{(-1)^k}{j_1!\cdots j_r!\cdot\lambda_1\cdots\lambda_k}.$$

Substituting in the power series expansion of $exp(\sum_{i=1}^\infty \frac{x_i}{i})$ the variable x_i by $-z^i$ we obtain

$$\sum_{(\lambda_1\geq\cdots\geq\lambda_k\geq 1)\in P_r} \frac{(-1)^k}{j_1!\cdots j_r!\cdot\lambda_1\cdots\lambda_k}$$

as the coefficient of z^n. Since

$$exp(\sum_{i=1}^\infty \frac{(-z)^i}{i}) = exp(log(1-z)) = 1-z$$

the assertion follows immediately for $r > 1$. ∎

Now we will show that for the analysis of $\mu(\mathcal{D}(M))$ for an modularly complemented matroid it suffices to investigate the $(0-1)$-matrices $(v(E_i)_j)_{(i,j)}$ associated to bases $\mathrm{E} = \{E_1 <_* \cdots <_* E_r\}$ of M. Of course $v(E_i)$ denotes the $(0-1)$-vector introduced in the second section. In the sequel we mean by a descending chain in the decomposition poset $\mathcal{D}(M)$ of a modularly complemented matroid M a maximal chain which has descending labels [BW83] in an arbitrary but fixed CL-shelling induced by $\leq_{\mathcal{B}}$.

Before we can give a more detailed analysis of the Möbius number $\mu_r(q)$ we will derive some general properties of the recursive atom order presented in Theorem 2.3 and properties of CL-shellings induced by that atom order.

Lemma 3.2 *Let* $\mathrm{E} = \{E_1 <_* \cdots <_* E_r\}$ *be a base of a modularly complemented matroid* M. *Then the number of descending chains in* $\mathcal{D}(M)$ *passing through* E *depends only on* $(v(E_i)_j)_{(i,j)}$. *The atoms of the interval* $[\mathrm{E}, 1]$ *which cover an atom of* $\mathcal{D}(M)$ *preceding* E *in* $\leq_{\mathcal{B}}$ *correspond to the lines for which* E *does not contain the least base.*

Proof : By the definition of $\leq_{\mathcal{B}}$ it remains to prove that the number of descending chains passing through E does not depend on the choice of the linear extension of the lexicographic order on the bases induced by $\leq_*$. Assume that E $\leq_{\mathcal{B}}$ F are two bases for which $(v(E_i)_j)_{(i,j)} = (v(F_i)_j)_{(i,j)}$. Assume that there is an atom of [E, 1] which covers F but no other atom of $\mathcal{D}(M)$ preceding E in $\leq_{\mathcal{B}}$. Hence there is a 2-flat for which F contains not the least base with respect to $\leq_{\mathcal{B}}$. This statement actually includes the second assertion of the lemma. Hence one of the equivalent conditions of Lemma 2.2 is violated. But this conditions depend only on the entries of the matrix $(v(F_i)_j)_{(i,j)}$. Therefore by assumption E does also not contain the least base for this 2-flat. Replacing the two base elements of this 2-flat in E by its least base gives a base preceding E in the order $\leq_{\mathcal{B}}$ which is also covered by the crucial atom of [F, 1]. Hence the set of atoms of [E, 1] meeting condition ii) on the recursive atom order is independent on the choice of the linear extension. Therefore the construction of a CL-labeling derived from the recursive atom order shows that the number of descending chains starting in E is also independent of the choice of the linear extension. ∎

In the next step we investigate the configurations in a matrix $(v(E_i)_j)_{(i,j)}$ which are conducive to descending chains for a certain induced CL-shelling. To do this we need some more insight into the structure of descending chains generated by different recursive atom orders of the partition lattice. Here we would like to remind the reader of the trivial fact that an atom in the lattice Π_r of set partitions of the set $[r]$ consists of a 2-block and $r-2$ blocks of size 1. Here we say that a set of atoms E of a geometric lattice generates the chain $\pi_1 < \cdots < \pi_t$ if all π_i are joins of atoms from E. At first we give a lemma which applies even to all geometric lattices.

Lemma 3.3 *Let $\{E_1,\ldots,E_k\}$ be a set of atoms of a geometric lattice L. Let $\{E_1,\ldots,E_k\}$ be an initial segment of a recursive atom order of L. Then the number of descending chains generated by $\{E_1,\ldots,E_k\}$ is independent of the linear order within the set.*

Proof : Let L' be the set of all elements of L which are joins of elements of $\{E_1,\ldots,E_k\}$. Let E be the greatest element of L'. If E is not the greatest element of L then no maximal chain in L is generated by $\{E_1,\ldots,E_k\}$. Obviously this statement is independent of the linear order on $\{E_1,\ldots,E_k\}$.

Now assume that E is the maximal element of L. Therefore L' is another geometric lattice. Since all atom orders of a geometric lattice are recursive atom orders the linear order on $\{E_1,\ldots,E_k\}$ is a recursive atom order for L'. The number of descending chains counts the absolute value of the Möbius number [Sta86]. Hence the number of descending chains generated by $\{E_1,\ldots,E_k\}$ in L' is independent of the actual linear order. Now it remains to show that the descending chains in L' generated by $\{E_1,\ldots,E_k\}$ correspond to the descending chains generated by $\{E_1,\ldots,E_k\}$ in L. But the labeling induced by a recursive atom order on an edge in L which actually lies in L' does only depend on elements of L'. This follows from that fact that the labeling is determined by the position of the top element of an edge relative to the elements prior to the bottom element in the recursive atom order. ∎

The next step is to figure out how we can describe the descending chains generated by a subset $\{E_1,\ldots,E_k\}$ of the set of atoms of the partition lattice Π_r. Here we would like to remind the reader of the fact that the modular elements of the partition lattice Π_r are just the partitions with at most one nontrivial block. An element a of a geometric lattice L is called modular if for all $b \in L$ the equation $rank(a \vee b) + rank(a \wedge b) = rank(a) + rank(b)$ holds.

Lemma 3.4 *Let $\{E_1,E_2,\ldots,E_k\}$ be a set of atoms of Π_r. Let $i \in [r]$ be a number such that there exists a modular coatom of Π_r which is generated by $\{E_1,\ldots,E_k\}$ and for which i is not contained in the nontrivial block of this coatom. Then there exists a CL-shelling of Π_r such that a descending chain generated by $E_1,\ldots,E_k$ satisfies :*

i) Each element of the chain is a modular element of Π_r.

ii) The nontrivial block of every member of the chain contains i.

Proof : If $E_1 \vee \cdots \vee E_k \neq (1 \cdots r)$ then no maximal and in particular no descending chain can be generated by $E_1, \ldots, E_k$. Therefore we may assume that $E_1 \vee \cdots \vee E_k = (1 \cdots r)$. But then there exists at least one maximal chain of modular elements generated by $\{E_1, \ldots, E_k\}$. After a suitable renumbering we may assume that $(1) \cdots (r) < (12)(3) \cdots (r) < \cdots < (1 \cdots r-1)(r) < (1 \cdots r)$ is a chain of modular elements generated by $\{E_1, \ldots, E_k\}$. It is well known that a maximal chain of modular elements in a graded lattice induces an EL-shelling. Indeed in this case the lattice is supersolvable. Since an EL-shelling is also a CL-shelling there is a recursive atom order corresponding to that CL-shelling. Now we order the atoms $\{E_1, \ldots, E_k\}$ according to this recursive atom order. We put the atoms $\{E_1, \ldots, E_k\}$ in the described order as the initial segment of another recursive atom order of Π_r. Now a recursive argument will prove that every descending chain in an induced CL-shelling is a chain of modular elements whose nontrivial block contains r. Hence the descending chains generated by $\{E_1, \ldots, E_k\}$ in the chosen atom order correspond to chains of modular elements generated by $\{E_1, \ldots, E_k\}$ whose nontrivial block contains r. Hence the assertion is fulfilled for $i = r$.
■

Now we can give a preliminary description of the descending chains in a specific CL-shelling induced by $\leq_{\mathcal{B}}$. We will give a more detailed description in Proposition 3.7.

Lemma 3.5 *Let M be a modularly complemented matroid. Then there is a CL-shelling of the poset $\mathcal{D}(M)$ such that a descending chain $0 < G_1 < \cdots < G_r$ satisfies :*

Let $\{F_1, \ldots, F_k\}$ be the atoms of $[G_1, 1]$ which cover an atom of $\mathcal{D}(M)$ preceding $G_1 = \{E_1, \ldots, E_r\}$ in the order $\leq \mathcal{B}$. Let i be the greatest index such that there is a modular coatom of $[G_1, 1]$ generated by a subset of $\{F_1, \ldots, F_k\}$ for which E_i is not contained in its nontrivial block.

i) Then $G_1 < \cdots < G_r$ is a chain of modular elements of the interval $[G_1, 1]$ which is generated by $\{F_1, \ldots, F_k\}$.

ii) The element E_i is contained in the nontrivial block of every decomposition G_i.

Proof : In a CL-shelling induced by $\leq_{\mathcal{B}}$ the chain $0 < G_1 < G_2$ is descending if and only if G_2 covers an atom of $\mathcal{D}(M)$ preceding G_1 in $\leq_{\mathcal{B}}$. Now the assertion follows immediately from the fact that $[G_1, 1] \cong \Pi_r$ and Lemma 3.4. ■

Finally we can classify the bases E of $\mathcal{D}(M)$ such that there are descending chains in the CL-shelling constructed in the preceding lemma which pass through E.

Lemma 3.6 *There is a CL-shelling of $\mathcal{D}(M)$ for a modularly complemented matroid M such that for a base $\mathrm{E} = \{E_1 <_* \cdots <_* E_r\}$ of M the following conditions are equivalent :*

i) There is a descending chain passing through E.

ii) For all $2 \leq i \leq r$ there is an index $t \neq i$ such that $\{E_t, E_i\}$ is not the least base for $E_i \vee E_t$.

iii) Let j_i be the greatest index such that the entry $v(E_i)_{j_i}$ in the vector $v(E_i)$ is nonzero. Then for all $2 \leq i \leq r$ there is an index $t \neq i$ such that the entry $v(E_i)_{j_t}$ is nonzero.

Proof : The equivalence *ii)* $\Leftrightarrow$ *iii)* follows immediately from Lemma 2.1. The equivalence of ii) and iii) can also be deduced along the lines of the proof of Lemma 2.1 ∎

Now we give a final necessary condition on a chains in to be descending. We would like to remark here that it is possible to give a condition which exactly characterizes the descending chains.

Proposition 3.7 *Let M be a modularly complemented matroid. Then there is a CL-shelling of $\mathcal{D}(M)$ induced by $\leq_{\mathcal{B}}$ such that a descending chain $0 < G_1 < \cdots < G_r$ satisfies the following conditions :*

i) The base $G_1 = \{E_1 <_* \cdots <_* E_r\}$ satisfies one of the conditions of Lemma 3.6.

ii) All decompositions G_i are modular elements of the interval $[G_1, 1] \cong \Pi_r$.

iii) Let j be the greatest index such that for each index $1 \leq i \leq r$ and $i \neq j$ there is an index $t \neq i, j$ for which that $\{E_i, E_j\}$ is not the least base of $E_i \vee E_j$. Then E_j is contained in the nontrivial block of every decomposition G_i.

Proof : One easily sees that CL-shelling referred to in Lemma 3.5 is the same as in Lemma 3.6. By the construction of j in condition iii) of the assertion there is a modular coatom in $[G_1, 1]$ such that E_j is not contained in its nontrivial block. Now the result follows immediately from Lemma 3.5 and Lemma 3.6. ∎

Having control over the descending chains allows us to identify the absolute value of the Möbius number of $\mathcal{D}(\mathcal{P}^{r-1}\mathsf{F}_q)$ with combinatorial objects.

Theorem 3.8 *The Möbius number $\mu_r(q)$ factors in a monic polynomial $f_r(q)$ of degree $\binom{r}{2}$ with positive integral coefficients and the polynomial $\frac{(-1)^r}{r}\prod_{i=1}^{r-1}(q^i - 1)$.*

Proof : (Sketch !) Since $\leq_{\mathcal{B}}$ is a recursive atom order the absolute value of the Möbius number is given by the number of descending chains in $\mathcal{D}(\mathcal{P}^{r-1}\mathsf{F}_q)$ in a CL-shelling induced $\leq_{\mathcal{B}}$ [Sta86].

In the sequel we will be concerned with a CL-shelling induced by $\leq_{\mathcal{B}}$ which fulfills the conditions of Proposition 3.7.

Now let $0 < G_1 < \cdots < G_r$ be a descending chain, Then $G_1 = \{E_1 <_* \cdots <_* E_r\}$ is a base of M. Assume j is the index defined in condition iii) of Proposition 3.7. Now the nontrivial blocks B_i of the decompositions G_i determine a sequence $j = t_1, \ldots, t_r$ for which $B_i = E_{j_1} \vee \cdots \vee E_{j_i}$. The crucial point is to show that by some projective transformations which preserve $E_j = E_{t_1}$ one generates $\prod_{i=1}^{r-1}(q^i - 1)$ other descending chains. Counting the multiplicity of the occurrence of each descending chain in this enumeration explains the factor $\frac{1}{r}$. Finally Lemma 3.1 proves the assertions on the degree of $f_r(q)$ and the fact that $f_r(q)$ is monic. ∎

In the following corollary we deduce a surprising relation between a series involving the polynomials $f_r(q)$ and the logarithm of one of the basic q-hypergeometric series. For the formulation of the corollary we abbreviate $\prod_{i=1}^{r}(1 - q^i)$ as usual by $(q, q)_r$.

Corollary 3.9 *The following equation holds*

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \cdot \frac{q^r}{1 - q^r} \cdot q^{\binom{r}{2}} f_r(1/q) \cdot z^r = -log\Big(\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q, q)_r} z^r\Big).$$

Proof : By a general result on the Möbius functions of exponential structures [Sta78, (8)] it follows that

$$\sum_{r=1}^{\infty} \frac{\mu_r(q) \cdot z^r}{r! \cdot M(r)_q} = -log\Big(1 + \sum_{r=1}^{\infty} \frac{z^r}{r! \cdot M(r)_q}\Big).$$

Here $M(r)_q = \frac{\prod_{i=0}^{r-1}(q^r - q^i)}{r! \cdot (q-1)^r}$ is the number of atoms in $\mathcal{D}(\mathcal{P}^{r-1}\mathsf{F}_q)$. Replacing z by $u := (q-1)z$ we easily obtain

$$\sum_{r=1}^{\infty} \frac{q^{\binom{r}{2}} \cdot \mu_r(q) \cdot u^r}{\prod_{i=1}^{r}(q^i - 1)} = -log\Big(\sum_{r=0}^{\infty} \frac{q^{\binom{r}{2}} \cdot u^r}{\prod_{i=1}^{r}(q^i - 1)}\Big).$$

Now an application of Theorem 3.8 shows

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \cdot \frac{q^{\binom{r}{2}} \cdot f_r(q) \cdot u^r}{q^r - 1} = -log\Big(\sum_{r=0}^{\infty} \frac{q^{\binom{r}{2}} \cdot u^r}{\prod_{i=1}^{r}(q^i - 1)}\Big).$$

Replacing q by $1/q$ proves the desired equation after some easy calculations. ∎

We are indebted to D. Stanton for providing an easy transformation which proves that the exponential of $\sum_{r=1}^{\infty} \frac{1}{r} f_r(q)$ equals the Rogers-Ramanujan continued fraction [Sta90] and hence the exponential of the generating series of the q-Catalan numbers $C_n(q)$ [Sta90]. R. Stanley pointed out to us that by a theorem [Sta86, Proposition 4.7.11] about the generating series of very pure monoids this also allows us a more explicit combinatorial interpretation of the coefficients of $f_r(q)$. Vice versa going the other way round the described argumentation would give another proof of Theorem 3.8. But this would certainly not enable us to characterize the descending chains in $\mathcal{D}(\mathcal{P}^r\mathsf{F}_q)$.

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