Extracting Combinatorics from Discrete Applied Geometry.

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For many discrete applied geometry problems, the analysis separates into layers, described by Felix Klein's classical hierarchy of geometries. This hierarchy focuses on the properties invariant under various groups (categories) of transformations. Combinatorics, and combinatorial topology, appear as several of these layers in the lattice of groups and invariants. Other important layers include projective geometry, affine geometry, and Euclidean geometry. How does the combinatorics appear within this applied geometry? How does the geometry illustate and expand the combinatorics? What common patterns of results/unsolved problems are emerging?

We present a common pattern for extracting those combinatorial layers which are implicit in recent work on

- the rigidity of frameworks (civil engineering);
- correct pictures of spatial polyhedral (computer vision);
- multivariate splines (approximation theory);
- rigidity of polytopial skeletons (h-vectors of polytopes).

In each of these examples, the intial form of the problem is:

(i) an abstract cell complex (e.g a graph, a simplicial polytope, an abstract polyhedral surface, etc.) is given, with a clear combinatorial topology;

(ii) a set of 'geometric realizations' is given for these cell complexes, in $\mathbb{R}^d$ (for example, the vertices of each i-dimensional face span an i-dimensional affine subspace);

(iii) The geometric problem is recorded by a matrix, whose kernal or co-kernal represent the objects of study: the matrix has zero and non-zero entries controlled by the cell complex, (a modification the 'oriented incidence' matrix of the cell complex), and the non-zero entries are polynomials in the coordinates of the realization;

As first clarifying steps, the problem is transformed as follows.

(iv) This basic matrix is re-interpreted as a boundary operator and extended to a chain complex of vector spaces, indexed by the combinatorial faces in each each dimension.

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The top homology, or top two homologies (and the corresponding cohomologies) represent the original geometric properties. (This geometric chain complex modifies the usual chain complex for the reduced homology of the underlying abstract cell complex.)

(vi) The generic rank of the top homology is determined by the combinatorics of the original cell complex: specifically, the rank is determined by the Euler characteristic of the chain complex (which is combinatorial) and the lower homologies (which are also combinatorial for generic realizations). Ideally, we can show that the lower Betti numbers are zero for interesting classes of structures and realizations, so that the top homology is determined by the Euler characteristic of the complex.

(vii) The relevant combinatorics for these generic Betti numbers builds on the matroid for the underlying homology, by a process of matroid union, followed by algebraic specialization (sometimes equivalent to matroid truncation). The known combinatorial algorithms for these problems are built from homology algorithms (disjoint spanning trees, maximal acyclic subcomplexes etc.).

(viii) The residual geometry of the non-generic realizations is captured in the maximal non-zero minors of the generic matrices, representing the 'determinant of the complex', if the sequence of chains and boundary maps is exact.

This underlying pattern is illustrated in explicit form for three of these geometric studies. While the methods apply to singular geometric homology of CW complexes, for simplicity we present only the simplicial theory, beginning with the standard simplicial homology.

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Homology of simplicial complexes

We recall the classical reduced homology of a simplicial cell complex, presented by chains and boundary operators (for example [Munkres, 1984]). An *abstract simplicial complex* is a collection Δ of finite sets (simplices), such that if A is in Δ, then all subsets of A are in Δ. (We shall assume that Δ is a finite collection.) The *d-simplices*, of cardinality $d+1$, are denoted $\Delta^{(d)}$ and the vertices (0-simplices) are $V = \Delta^{(0)}$.

We choose an arbitrary orientation $[v_o, \ldots, v_i \ldots, v_p]$ for each simplex σ. A *p-chain* is a map $\mathbf{c}$ from the oriented p-simplices to the reals, such that $\mathbf{c}(\sigma) = -\mathbf{c}(\sigma')$ if σ and

σ' are opposite orientations of the same simplex. The set of all p-chains is a real vector space $C_p(\Delta)$, and an arbitrary element is written: $\sum_{\sigma \in \Delta^p} c_\sigma \sigma$.

For an oriented p-simplex $\sigma = [v_o, v_1, \ldots, v_p]$ the *boundary* operator is a map from the p-chain σ to the $(p-1)$-chains:

$$\partial_p(\sigma) = \sum (-1)^i [v_o, \ldots, v_{i-1}, \hat{v}_i \ldots, v_p] = \sum_{\{\tau \in \Delta^{(p-1)} | \tau \subset \sigma\}} Sign\,(\tau, \sigma)\tau.$$

This map is extended linearly to all p-chains. We note that $\partial_{p-1}\partial_p(c) = 0$ for any p-chain. (We also include the empty set as $\Delta^{(-1)}$, generating $\mathbb{R}$ as the -1-chains.)

The kernal of ∂_p, $Z_p(\Delta)$, is called the *p-cycles*. The matrix for operator ∂_p, with Z_p as cokernal, is the *oriented incidence matrix*, $M(\Delta^p)$, recording the incidences of the oriented p-simplices and the oriented $p-1$-simplices, with appropriate entries of 0, 1 and -1. The image of ∂_{p+1}, $B_p(\Delta)$ is called the p-boundaries. The row space of the matrix $M(\Delta^p)$ is the $p-1$-boundaries, $B_{p-1}(\Delta)$. With $B_p(\Delta) \subset Z_p(\Delta)$, the vector space $\tilde{H}_p(\Delta) = Z_p(\Delta)/B_p(\Delta)$ is called the *p-th homology* of the complex.

Example 1. For a graph, a complex of dimension 1, this is the usual matrix representation of the graph (leaving the zero entries blank):

$M(\Delta^p)$	λ	a	b	c	d
ab	1	-1	1		
ac	0	-1		1	
ad	-1	-1			1
bc	1		-1	1	
bd	0		-1		1
cd	1			-1	1

The second column gives a row dependence i.e. the coefficients of a 1-cycle, or a polygon. The rank of the matrix is the size of a maximal subgraph with no cycles - a maximal forest.

Example 2. For a set of triangles, we have, for example:

$M(\Delta^p)$	λ	ab	ac	ad	bc	bd	cd
abc	1	1	-1		1		
abd	-1	1		-1		1	
acd	1		1	-1			1
bcd	-1				1	-1	1

.

Again, the row dependence is a 2-cycle, and the rows are the 1-boundaries.

If Δ has dimension d, the *augmented simplicial chain complex* $\tilde{\mathcal{H}}$ is the entire sequence of vector spaces and maps

$$\tilde{\mathcal{H}}:\ 0 \longrightarrow C_d(\Delta) \xrightarrow{\partial_d} C_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} \dots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} \mathbb{R} \longrightarrow 0.$$

There is Euler-Poincaré identity for any such sequence of vector spaces and maps, with $\partial_{p-1}\partial_p = 0$:

$$\chi(\Delta) = \sum_{i=-1}^{i=d} (-1)^i \dim(C_i(\Delta)) = \sum_{i=-1}^{i=d} (-1)^i \dim(\tilde{H}_i(\Delta)).$$

Since $\dim(C_i(\Delta)) = |\Delta^{(i)}|$, the first sum is the a combinatorially defined number called the *Euler characteristic* of the simplicial complex:

$$\chi(\Delta) = \sum_{i=-1}^{i=d} (-1)^i |\Delta^{(i)}|.$$

There are combinatorial criteria under which $\dim(\tilde{H}_i(\Delta)) = 0$ for $i \leq n-2$. (For example, if the complex is a topological wedge of spheres.) When this holds, we have the simple combinatorial identity:

$$\dim(\tilde{H}_n(\Delta)) - \dim(\tilde{H}_{n-1}(\Delta)) = \sum_{i=0}^{i=d} (-1)^{n+i} |\Delta^{(i)}|.$$

Example 3. Consider the standard graph $G = (V, E)$ of graph theory and matroid theory. This gives the chain complex:

$$\tilde{\mathcal{H}}:\ 0 \longrightarrow \mathbb{R}^{|E|} \xrightarrow{\partial_1} \mathbb{R}^{|V|} \xrightarrow{\partial_0} \mathbb{R} \longrightarrow 0.$$

The 1-cycles are generated by the polygons. This complex has Euler characteristic $|E| - |V| + 1$. If the graph is connected, then $\tilde{H}_0 = 0$, and $\tilde{H}_1 = |E| - (|V| - 1)$. The maximal acyclic subcomplex, with $\tilde{H}_1 = 0$, is a polygonal-free subgraph, or a forest, which satisfies: $|E| \leq |V| - 1$.

For graphs, there are simple, polynomial time algorithms for finding the rank of the 'homology matrix', i.e. the size of the maximal forest. This can also be expressed as a polynomial algorithm for the graphic matroid, or the matroid defined by the submodular function on the edges

$$f(E') = |V(E')| - 1$$

where $V(E)$ is the vertices of the subset E'.

We are not aware of similar polynomial time algorithms for the rank of the matrices $M(\Delta)$, $p \geq 2$. (For a general complex, this matroid is not defined by a submodular function.) This issue of algorithms for homology is important in the applications.

There is the corresponding cohomology of the simplicial complex. In fact, the kernal of the matrix $M(\Delta^p)$ is the cocyles, $Z^p(\Delta)$ for the corresponding cohomology:

$$\tilde{\mathcal{H}}: \mathbf{0} \longleftarrow C_d(\Delta) \xleftarrow{\delta_d} C_{d-1}(\Delta) \xleftarrow{\delta_{d-1}} \dots C_0(\Delta) \xleftarrow{\delta_0} \mathbb{R} \longleftarrow \mathbf{0}.$$

Rigidity of frameworks

The theory of static and infinitesimal rigidity of frameworks [Maxwell, 1864] has recently experienced a upsurge of research (Laman, 1970, Asimow & Roth, 1978, Whiteley, 1984 etc.). The underlying abstract structure of a framework is its graph $G = (V, E)$, with an arbitrary orientation of all edges. For frameworks in n-space, the *Euclidean realizations* are maps $\mathbf{p} : V \to \mathbb{R}^n$. For convenience $\mathbf{p}(v_i) = \mathbf{p}(i) = \vec{p}_i$. The realized graph $G(\mathbf{p})$ is called a *bar framework in n-space* provided $\vec{p}_i \neq \vec{p}_j$ for all $\{i, j\} \in E$.

This classical theory studies the kernal and cokernal of the *rigidity matrix* $R_1(\Delta)$. For example:

$R_1(\Delta)$	a	b	c	d
ab	$\vec{b}-\vec{a}$	$\vec{a}-\vec{b}$		
ac	$\vec{c}-\vec{a}$		$\vec{a}-\vec{c}$	
ad	$\vec{d}-\vec{a}$			$\vec{a}-\vec{d}$
bc		$\vec{c}-\vec{b}$	$\vec{b}-\vec{c}$	
bd		$\vec{d}-\vec{b}$		$\vec{b}-\vec{c}$
cd			$\vec{d}-\vec{c}$	$\vec{c}-\vec{d}$

The row dependencies of this matrix are the static *self-stresses* and the column dependencies are the *infinitesimal motions* of the framework. This matrix represents a basic boundary operator for a chain complex on the graph:

The 1-chains are $\mathcal{R}_1 = C_1 = \mathbb{R}^{|E|}$ as defined for homology.

The set of 0-chains is $\mathcal{R}_0 = \oplus_{v_i \in V} \mathbb{R}^n = \mathbb{R}^{n|V|}$.

The boundary map ∂_1 is defined by: $\partial_1(i,j) = -(\vec{p_i} - \vec{p_j})v_i + (\vec{p_i} - \vec{p_j})v_j$.

The set of -1-chains is $\mathbb{R}^{\binom{n+1}{2}}$.

∂_0 is defined by: $(\ldots, (\vec{p_i} - \vec{p_j}), \ldots, [(\vec{p_i})_h(\vec{p_j})_k - (\vec{p_i})_k(\vec{p_j})_h], \ldots)$, where the second part runs over all pairs of distinct coordinates $1 \leq h < k \leq n$.

The complete chain complex is then

$$\mathcal{R}_d : 0 \longrightarrow \oplus_{e\in E}\mathbb{R} \xrightarrow{\partial_1} \oplus_{v\in V}\mathbb{R}^n \xrightarrow{\partial_0} \mathbb{R}^{\binom{n+1}{2}} \longrightarrow 0.$$

The Euler characteristic of this complex is $|E| - n|V| + \binom{n+1}{2}$. If the set of vertices spans at least a hyperplane of $\mathbb{R}^n$, ∂_0 is onto and $\tilde{H}_{-1}(\mathcal{R})) = 0$. If all 0-cycles are 0-boundaries, that is if $\tilde{H}_0 = 0$, the framework is called *statically rigid*.

The dual cohomology is:

$$\mathcal{R}^d : 0 \longleftarrow 2\oplus_{e\in E}\mathbb{R} \xleftarrow{\delta_1} \oplus_{v\in V}\mathbb{R}^n \xleftarrow{\delta_0} \mathbb{R}^{\binom{n+1}{2}} \longleftarrow 0.$$

The 0-cochains (the kernal of $R_1(G)$) are the *infinitesimal motions*, and the 0-coboundaries are called *trivial motions* (they are the derivatives of Euclidean motions). If all infinitesimal motions are trivial motions, that is if $H^0(\mathcal{R}^d(G)) = 0$, the framework is *infinitesimally rigid*. Since the homology and cohomology are isomorphic in these situations, infinitesimal rigidity is equivalent to static rigidity.

Example 4. Rigidity on the line gives the chain complex: $0 \rightarrow \mathbb{R}^{|E|} \xrightarrow{\partial_1} \mathbb{R}^{|V|} \xrightarrow{\partial_0} \mathbb{R} \rightarrow 0$. The boundary operator gives: $\partial_1([i,j]) = (p_j - p_i)v_i - (p_j - p_i)v_j$. This only differs from the homology map by the non-zero constant $(p_j - p_i)$. The cycle spaces are isomorphic, so this is the simplicial homology of the graph.

Example 5. For higher n, the rigidity matrix retains the appearance of the oriented incidence matrix. Each entry has been replaced by an n-vector, with 0 replaced by $\mathbf{0}$ and the two non-zero entries of a row, $+1$ and -1, replaced by the vectors $\vec{p_i} - \vec{p_j}$ and $-(\vec{p_i} - \vec{p_j})$. If we had picked a truly new vector $\vec{b}_{ij}$ for each row, this would be the matroid union of graphic matroids (Whiteley 1988a). The maximum possible rank would be $n(|V| - 1)$ - and the actual rank could be determined by repeated use of the algorithms for maximal forests.

For higher space, the matrix for the boundary map retains the superficial appearance of a matroid union of copies of the homology matroid of the graph. Each entry in the homology matroid has been replaced by a vector, with $\vec{0}$ replacing 0, and the two non-zero entries of a row $+1$ and -1 replaced by the vectors $\vec{p_i} - \vec{p_j}$ and $-(\vec{p_i} - \vec{p_j})$. If we had picked a truly new vector $\vec{b}_{ij}$ for each row, this would be the matroid union. The maximum possible rank would then be rank would then $n(|V|-1)$ - and the actual rank could be determined by repeated use of the algorithms for maximal forests.

However, the actual form of the vector entries imposes a relationship among the entries, and the maximum possible rank is reduced to $n|V| - \binom{n+1}{2}$. What is the impact of this 'specialization' of the entries in the matroid union? For $n = 2$, the maximum rank of the rigidity matrix is $2|V| - 3$, and the matroid is a generic truncation of the matroid union - resulting in precise combinatorial algorithms for the rank (Laman, 1970, Whiteley, 1988a). For $n = 3$, the precise combinatorics are unknown.

However, important partial results support several conjectures on complex combinatorial characterizations (Tay & Whiteley, 1985). In particular, the 1-skeleton of a simplicial convex d-polytope is infinitesimally in d-space (Whiteley, 1984). Because of the underlying projective geometry (see below), crucial tools for these proofs include coning and projections.

Skeletal Rigidity

Recent works of Lee [1991], Filliman [1991] and Tay, White & Whiteley [1992a] have extracted matrices resembling the matrix for rigidity of frameworks, from the face ring of a simplicial complex. Their common goal is to analyse the combinatorics and geometry of the g-theorem of polyhedral combinatorics - a theorem which characterizes the f-vector of a simplicial polytope in n-space:

$$f(\Delta) = (f_o, f_1, \ldots, f_n) = (|\Delta^{(0)}|, |\Delta^{(10}|, \ldots, |\Delta^{(n-1)}|).$$

The g-theorem is a combinatorial theorem with a complex analytic proof, based on the homology of a corresponding toric variety and the hard (or strong) Lefschetz theorem. Major efforts are underway to give a direct combinatorial or geometric proof of the combinatorial result. For g_2, the desired result is

$$0 \leq g_2 = |\Delta^{(1)}| - d|\Delta^{(0)} + \binom{d+1}{2} = |E| - d|V| + \binom{d+1}{2}.$$

This inequality follows from the infiniteismal rigidity of the 1-skeleton of the simplicial complex (Kalai, 1988).

Tay, White & Whiteley [1992b] extends this 'rigidity style' analysis to a complete chain complex. As always, the rank of the highest matrix is expressed by the combinatorics of the complex, and the lower homologies (which should be zero under appropriate assumptions).

We begin with a reexamination of rigidity of bar frameworks. Surprisingly, the ranks of the framework rigidity matrices, or equivalently, the framework matroid, is invariant under projective transformations of the underlying points. The matrices and the chain complex can also be written in a projective form. Let $\tilde{x} = (x_1, \ldots, x_n, 1)$ represent the affine coordinates of the point $\vec{x}$. Then the classical exterior (or Grassman) product of two such points is:

$$\tilde{x} \vee \tilde{y} = (\ldots, x_h y_k - x_k y_h, \ldots) \quad 1 \leq h < k \leq n+1).$$

This extends to the full exterior algebra of a set of points in projective space. All products of d points generate a real vector space $V^{(d)}$. For an oriented simplex $\sigma = [v_0, \ldots, v_d]$, with $\tilde{\sigma} = \tilde{p}_0 \vee \ldots \vee \tilde{p}_n \in V^{(d+1)}$, we have a equivalence relation on elements of $V^{(r)}$

$$P \overset{ker\,\sigma}{=} Q \iff P \vee \tilde{\sigma} = Q \vee \tilde{\sigma}.$$

These equivalence classes, $V^{(r)}/ker\,\sigma$ form the coefficients of σ in our chain complex.

Example 6. Consider again the framework rigidity. Since $\tilde{x} \vee \tilde{y} = \mathbf{0}$, $\tilde{p}_i \vee \tilde{p}_j$ is precisely the desired image for $\partial_0(\tilde{p}_i \cdot v_j)$. It is a simple exercise to show that the chain complex can be rewritten as:

$$\mathcal{R}_d(\Delta^1):\ \mathbf{0} \to \bigoplus_{\{i,j\} \in E} \mathbb{R} \xrightarrow{\partial_0} \bigoplus_{a_i \in \Delta^0} V_d^{(r)}/ker\,a_i \xrightarrow{\partial_0} V_d^{(r+1)} \to \mathbf{0}.$$

This chain complex has the same homologies as the more usual rigidity chain complex (Crapo & Whiteley 1982).

For a simplicial complex Δ realized in projective n-space, the *r-skeletal chain complex* is

$$\mathcal{R}_n(\Delta^r):\ \mathbf{0} \to \bigoplus_{\rho \in \Delta^{(r-1)}} V_d^{(0)} \xrightarrow{\partial_{r-1}} \bigoplus_{\sigma \in \Delta^{(r-2)}} V_d^{(1)}/ker\,\sigma \xrightarrow{\partial_{r-2}} \bigoplus_{\tau \in \Delta^{(r-3)}} V_d^{(2)}/ker\,\tau \xrightarrow{\partial_{r-3}} \ldots$$

$$\ldots \xrightarrow{\partial_1} \bigoplus_{a_i \in \Delta^{(0)}} V_d^{(r-1)}/ker\,a_i \xrightarrow{\partial_0} V_d^{(r)} \to \mathbf{0}.$$

For an elementry d-chain $\tilde{P}\sigma$ the boundary map is defined by:

$$\partial_d(\tilde{P}\sigma) = \partial_d(\tilde{P}[a_0, a_1, \ldots, a_d]) = \sum_{\{j|a_j \in \sigma\}} \tilde{P} \vee \tilde{a}_j \cdot [a_0, a_1, \ldots, \hat{a}_j, \ldots, a_d].$$

This is then extended linearly to all d-chains.

Again, the matrices corresponding to the top operators are formed from the oriented incidence matrices, replacing non-zero entries by appropriate non-zero vectors, and zero entries by corresponding zero vectors. In particular if $r = n + 1$, this is simplicial homology.

For shellable complexes in n-space, and $n \leq 2r + 1$, the lower homologies ($k < r - 2$) are zero, and the study concentrates on showing that $H_{r-2}(\mathcal{R}(\Delta^k)) = 0$ (i.e the r-skeleton is *r-rigid*). We note that, as usual, there are geometric special positions, which depend on the underlying projective geoemtry.

Of course we have the dual *r-skeletal cochain complex* is

$$\mathcal{R}^n(\Delta^r):\ 0 \leftarrow \bigoplus_{\rho \in \Delta^{(r-1)}} V_d^{(0)} \xleftarrow{\delta_{r-1}} \bigoplus_{\sigma \in \Delta^{(r-2)}} V_d^{(1)}/ker\,\sigma \xleftarrow{\delta_{r-2}} \bigoplus_{\tau \in \Delta^{(r-3)}} V_d^{(2)}/ker\,\tau \xleftarrow{\delta_{r-3}} \cdots$$
$$\cdots \xleftarrow{\delta_1} \bigoplus_{a_i \in \Delta^{(0)}} V_d^{(r-1)}/ker\,a_i \xleftarrow{\delta_0} V_d^{(r)} \leftarrow 0.$$

Again, the cohomologies are isomorphic to homologies, but this cohomology gives additional insights into the geometry of the simplicial complex.

Multivariate splines

The theory of mulitivariate splines studies the vector space of all piecwise polynomial (of maximal degree k) globally C^r-functions over a simplicial (or more general polyhedral) decomposition of the domain Ω. Billera [1988] expresses this through a chain complex, as follows. We denote the space of polynomials of degree at most d by P_d. A realized simplex τ generates an ideal I_τ of polynomials which are zero on all vertices of τ, and $(I_\tau)^k$ is the ideal generated by all kth powers.

Consider two polynomial functions f, g of n variables, defined on opposite sides of the hyperplane of a facet τ in $\mathbb{R}^n$ with equation $L(x_1, \ldots, x_n) = 0$. The functions meet with continuity C^r over the simplex τ if and only if:

$$f - g = \beta(x_1, \ldots, x_n)[L(x_1, \ldots, x_n)]^{r+1} \iff f - g \in I_\tau^{r+1}.$$

Assume that the cell complex forms a topological manifold. The functions defined by polynomials of degree at most d over the cells, meeting in pairs over the facets with C^r

continuity, are then the elements of the vector space $\oplus_{\sigma\in\Delta}P_d$ such that, if $\sigma \cap \sigma' = \tau$ for a facet τ, then $f_\sigma - f'_\sigma \in I_\tau^{r+1}$. Thus the splines are cycles for the boundary map: $\partial_{n-1}(f_\sigma\sigma) = f_\sigma\partial_{n-1}(\sigma)$ where $\partial_{n-1}(\sigma)$ is the usual boundary map of homology and the image is taken in $\oplus_{\tau\in\Delta}P_d/(P_d \cap (I_\tau)^{r+1})$.

This boundary operator, expressed in matrix form, looks like a matroid union of the oriented incidence matrix, but, again, the entries in various rows are linked, leaving a specialization of the matroid union.

This basic operation is extended to a chain complex:

$$\mathcal{S}_d^r(\Delta):\ 0 \xrightarrow{\partial_n} \oplus_{\sigma\in\Delta}P_d \xrightarrow{\partial_{n-1}} \oplus_{\tau\in\Delta}P_d/P_d \cap (I_\tau)^{r+1} \ldots \xrightarrow{\partial_0} \oplus_{v\in\Delta}P_d/P_d \cap (I_v)^{r+1} \xrightarrow{\partial_{-1}} 0.$$

Using the Euler characteristic, we find that:

$$\dim(H_n(\mathcal{C}(\Delta)) = \sum_i(|\Delta^{(i)}|) - \sum_{i\leq n-1} H_i(\mathcal{C}(\Delta)).$$

Again the goal is to prove that the lower homologies are trivial, so that this gives the dimension of the space of splines. This has been shown to hold for C_d^1 over generic simplicial manifolds in the plane [Billera, 1988] and in 3-space [Alfeld, Schumaker & Whiteley, 1991].

In practice, this chain complex for splines is also analyzed by means of a short exact sequence of chain complexes - and the resulting long-exact sequence of homologies [Billera, 1988]). Without giving the details, we note that the matrices for all these related chain complexes are formed from the oriented incidence matrix, by inserting appropriate matrices for the non-zero entries across a row (following the other signs) and similarly shaped zero matrices for the zero entries.

In this setting, the corresponding cohomology has not been directly analysed. However it may become a crucial tool for understanding the behaviour of special geometric realizations of the cell complex. We note that, once more, the underlying geometry is projective (Whiteley 1991b). This geometry is important in the use of such tools as coning and projection in the proof of the results.

Polyhedral pictures

The theory of polyhedral pictures studies a plane realization of a set of polygons, together with the vector space of *spatial liftings* which lift the underlying vertices into space, and keeps each of the polygons flat. Without going into detail, we note:

(a) The problem of polyhedral pictures is isomoprphic the the problem of piecewise linear, globally continuous functions, C_1^0 splines, if the polygons do not overlap.

(b) Classical results of James Clerk Maxwell [1864] connect polyhedral pictures with the self-stresses of a planar graph (see also Whiteley, 1982, Crapo & Whiteley, 1986).

(c) These correspondences extend to pictures of 'polytopes' in higher dimensions connecting to splines and skeletal rigidity.

These particular chain complexes are defined only if the underlying cell complex is a manifold (with boundary). However there are also extended theories:

(d) There is an explicit theory of such polyhedral scenes, via a 'geometric' homology theory, elaborated by Crapo & Ryan (1986a,b). This theory is based on a theory of chains and boundary operators different from the standard homology.

(e) Crapo's homology and cohomology is intimately related to the analysis of C^0 splines by sheaves and cohomology (Yuzvinski,1991).

For both this general theory and the manifold theories there are explicit combinatorial algorithms for the lifitings of a generic realization, derived from truncations of a matroid union of appropriate incidence matrices (Sugihara, 1986, Whiteley, 1988b).

Summary.

The oriented incidence matrices of cell complexes, and the corresponding chain complexes, are the combinatorial foundation of an important class of problems in discrete applied geometry. The combinatorial analysis of these extensions draws on everything we know about the homology of the cell complex - and poses additional combinatorial questions for the structures, and their matroids.

The analysis of each of these geometric chain complexes includes the following steps:
(i) A proof that the 'lower homologies' (all but the top two) are zero, under appropriate assumptions (e.g. a strongly connected manifold with boundary). This involves simple arguments, and induction from step (ii), below, for structures in lower dimensions.
(ii) A proof (or conjecture) that the matrix for the top boundary operation has 'maximum rank' (defined from the Euler characteristic), possibly under additional assumptions, so that only the top homology is non-zero. This involves more direct combinatorial analysis of the matrix, and the underlying cell complex.

For example, framework rigidity is analysed by graph theoretic inductions, in ways that apply to all triangulated manifolds ($n \geq 3$) (Tay & Whiteley, 1985, Whiteley, 1984, 1991a). The results actually apply to a larger class of 'minimal homology cycles' - sets of

simplices which are a cycle in the homology of an underlying complex, and are minimal with this property (Fogelsanger, 1989).

(iii) For each problem, the chain complexes for related structures in adjacent dimensions are connected by geometric or combinatorial constructions. For example, cones of structures in lower dimensions connect the homologies of the original structure in dimension $n-1$ and the cone in dimension n (Whiteley, 1983, 1984, Alfeld, Schumaker & Whiteley, 1991). This coning reflects the underlying projective invariance for each of the examples (Crapo & Whiteley, 1982, Whiteley, 1991b).

(iv) If the chain complex is exact for generic realizations (the homologies are all zero) then the geometry of non-generic realizations is captured in the 'determinant' of the complex (Gelfand, Kapranov & Zelivinski, 1991), generalizing the pure conditions of rigidity theory (White & Whiteley 1984).

Some common combinatorial techniques have been developed for these examples. Other basic combinatorial and algorithmic questions remain to be solved. Continuing combinatorial research on each of these problems should expand the results in all these related fields.

Finally, although we have talked about the role of combinatorics in understanding these geometric problems, we should not ignore the role of geometry as a tool to prove combinatorial results. One feature of the the work on the g-theorem is the role of geometric tools (either toric varieties or perhaps skeletal rigidity) in proofs of the combinatorial results. The interaction of combinatorics and geometry is a two-way street.

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