The Plethystic Inverse of a Formal Power Series (Extended Abstract)

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Abstract

A bijective definition of Littlewood's plethysm of symmetric functions in terms of unlabelled colored combinatorial structures is introduced. The plethystic inverse is understood by examining associated partially ordered sets and using Möbius inversion techniques.

The combinatorial theory of distribution and occupancy (also called "balls in boxes") provides a classical interpretation of symmetric functions. In this view, one counts the number of ways to place distinct balls in distinct boxes such that certain conditions are met (e.g., there is no more than one ball per box). Alternately, the balls can be viewed as "labelled nodes" of a graph, and the boxes can be seen as additional colors assigned to the nodes. Symmetric functions then count the number of colored labelled structures by keeping track of how many use a particular color scheme.

The first systematic development of this viewpoint was given by Doubilet. [4] More recently, Bonetti, Rota, Senato, and Venezia provided set-theoretic interpretations for infinite sums, infinite products, and exponentiation of symmetric functions; this permitted bijective interpretations for many classical symmetric function identities. [2] In this paper, the equivalence classes of colored, labelled structures under group action are examined. This leads to a bijective definition of Littlewood's notion of plethysm of symmetric functions. This theory of symmetric functions is then extended to include posets, and an interpretation of the plethystic inverse is given.

Intuitively, the equivalence classes under the group action may be visualized as unlabelled graphs with colored nodes. This is an extension of types and analytic functors, or "Joyal labellings".[5][6] The combinatorial operations of addition, multiplication, and exponentiation can be readily extended to these unlabelled, colored structures, which will be called colored types.

In addition, the appropriate composition of colored types corresponds to Littlewood's plethysm of the generating functions. Usually, formal power series in infinitely many variables count the number of structures built on objects like partitions of labelled sets, and the resulting interpretations of plethysm can be rather complicated. However, when attention is turned to colored types, the construction of a plethysm becomes remarkably simple. Given two families of colored types M and N, the plethysm of N into M yields M-structures whose colored nodes have been replaced with the appropriate N-structures; two "nodes"

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have the same "color" when the associated N-structures are the "same". This plethysm is a generalization of the usual combinatorial interpretation for Littlewood's plethysm.

The theory of symmetric functions is then extended to include posets. Furthermore, the plethystic inverse is understood by applying Möbius inversion techniques to suitable lattice structures based on colored combinatorial objects.

In Section 1, background material from the theory of symmetric functions and the theory of polynomial species is reviewed. In Section 2, the plethysm is introduced. Then several examples are provided; for example, the plethysm of a symmetric function into a homogeneous symmetric function is shown to correspond to assemblies (sets) of colored types. In Section 3, the plethystic inverse is addressed by examining appropriate lattice-theoretic structures.

1 Background Material

1.1 Symmetric Functions and Littlewood's Plethysm

Let $X = \{x_i\}_{i \in I}$ be a finite or infinite set of variables with index set, I. A *symmetric function* is a formal power series in X with integer coefficients which is invariant under permutation of the variables and of bounded degree. The symmetric functions form a ring denoted by $\Lambda(X)$.

Let $\bigoplus_I \mathbb{N}$ denote the set of sequences of non-negative integers indexed by I with only finitely many non-zero entries. For $\alpha = (\alpha_i) \in \bigoplus_I \mathbb{N}$, x^α denotes the monomial $\prod_i x_i^{\alpha_i}$.

Let $F(X) = \sum_{k \geq 0} f_k(X)$ and $G(X) = \sum_{k \geq 1} g_k(X)$ be two sums of symmetric functions in the variable set X where $f_k(X)$ and $g_k(X)$ are symmetric functions of degree k (these sums are called generating functions of symmetric functions). Suppose that the index set is countably infinite. A special kind of multiplication for these generating functions, called the *plethysm* or *composition*, can be defined. The basic idea is that if $G(X)$ is expressed as a sum of terms of the form x^α, then the plethysm $F * G(X)$ is obtained by substituting the terms of $G(X)$ into the variables x_i in $F(X)$. Formally, write $G(X)$ as the sum of monomials: $G(X) = \sum_\alpha w_\alpha x^\alpha$ where the sum ranges over all $\alpha \in \bigoplus_I \mathbb{N}$.

Define a set of dummy variables, y_i, such that

$$\prod_i (1 + y_i z) = \prod_\alpha (1 + x^\alpha z)^{w_\alpha} .$$

To simplify notation, suppose that the index set, I, is linearly ordered. Littlewood's plethysm is defined as

$$F * G(x_1, x_2, \ldots) = F(y_1, y_2, \ldots).$$

When $F(X)$ is homogeneous of degree n and $G(X)$ is homogeneous of degree k, then $F*G(X)$ is homogeneous of degree nk.

See [9] for more about symmetric functions and plethysm.

1.2 Polynomial Species

The theory of polynomial species provides set-theoretic interpretations of operations on symmetric functions. In this subsection, the definitions for polynomial species and the combinatorial operations of addition, multiplication, and exponentiation will be reviewed.

Let **Ens** be the category of sets and functions, and let **B** be the category of finite sets and bijections. Define a covariant functor $Hom : \mathbf{B} \to \mathbf{Ens}$ as $Hom[E] = \{f : E \to X\}$ where E is a finite set, and f is a function.

Let $M : \mathbf{B} \to \mathbf{Ens}$ be a functor (frequently, M is a (Joyal) species $M : \mathbf{B} \to \mathbf{B}$, and the image of a finite set E under M is called the set of all M-structures with label set E). Let R be a subfunctor of $M \times Hom$. Given a R-structure (m, f), the function f may be visualized as "coloring" the labels (or nodes) of the M-structure with elements from the variable set, X. A bijection $g : E \to E'$ of finite sets induces the morphism

$$\begin{array}{cccc} R[g]E: & R[E] & \to & R[E'] \\ & (m, f : E \to X) & \mapsto & (M[g]m, f \circ g^{-1}). \end{array}$$

The subfunctor R is called a *polynomial species* when for any finite subset $F \subseteq X$, the set $\{(m, f) \in R[E] : \operatorname{im} f \subseteq F\}$ is finite; $\operatorname{im} f$ denotes the set of variables in the image of f. This technical condition ensures that the corresponding generating functions have well-behaved coefficients.

A polynomial species is said to be *symmetric* when for any bijection $u : X \to X$ and for any $(m, f) \in R[E]$, $(m, u \circ f) \in R[E]$.

Given any polynomial species R and any non-negative integer n, let $R_{[n]}$ be the polynomial species where $R_{[n]}[E] = R[E]$ if $|E| = n$, and $R_{[n]}[E] = \emptyset$ otherwise. Let R_0 denote the polynomial species where $R_0[E] = R[E]$ if E is nonempty, and $R_0[E] = \emptyset$ otherwise.

The generating functions associated with a polynomial species will now be covered.

For a finite set E and an arbitrary function $f : E \to X$, let $\operatorname{gen}(f) = \prod_{e \in E} f(e) = x^{\alpha}$ where $\alpha = (\alpha_i) \in \bigoplus_I \mathbb{N}$, and α_i is the cardinality of the set $\{f^{-1}(x_i)\}$. Given any R-structure (m, f), call f the *coloring*, and call α or x^{α} the *color scheme*. Set $\operatorname{gen}(m, f) = \operatorname{gen}(f)$. Define

$$\operatorname{gen}(R[E]) = \sum_{(m,f) \in R[E]} \operatorname{gen}(m, f).$$

Note that every monomial in the sum has a finite coefficient by the definition of a polynomial species. Also, if R is symmetric, then $\operatorname{gen}(R[n])$ is a symmetric function of degree n where $R[n] = R[\{1, 2, \cdots, n\}]$.

The *generating function* of a polynomial species, R, is defined as the formal power series

$$\operatorname{Gen}(R) = \sum_{n \geq 0} \operatorname{gen}(R[n])/n!.$$

Polynomial species form a category whose morphisms are natural maps $\tau : R \to S$ such that for any finite set, E, $\tau_E : R[E] \to S[E]$ is a bijection where $\tau_E(m, f) = (m', f)$ for $(m, f) \in R[E]$. Write $R = S$ when τ is a natural isomorphism. R and S are said to be *equipotent* ($R \equiv S$) if $\operatorname{gen}(R[n]) = \operatorname{gen}(S[n])$ for all non-negative integers n. This is a weaker condition.

The combinatorial operations of addition, multiplication, and exponentiation will now be covered. Let $R \subseteq M_1 \times Hom$ and $S \subseteq M_2 \times Hom$ be two polynomial species. Let E be a finite set.

The *sum* and *product* of R and S are given by

$$R + S[E] = R[E] \bigcup S[E]$$
$$R \cdot S[E] = \textstyle\sum_{E_1 + E_2 = E} \{((m, m'), f) : (m, f \mid E_1) \in R[E_1], (m', f \mid E_2) \in S[E_2]\}.$$

Infinite sums and infinite products may also be defined.

Suppose that a polynomial species, S, is without constant term ($S[0] = \emptyset$). An ***assembly*** of S-structures with associated partition π is an ordered pair (a, f) where $(m_B, f_B) \in S[B]$ for every $B \in \pi$, $a = \{m_B\}$, and $f|B = f_B$ for all $B \in \pi$. The ***k-th divided power*** $\gamma_k(S)$ is the polynomial species of assemblies of S-structures where the associated partition contains k blocks. The *exponential* $\exp(S)$ is the polynomial species of all assemblies of S-structures.

Note that when $|X| = 1$, and M_1 and M_2 are Joyal species, these operations reduce to the corresponding operations on species.

The generating functions for these operations behave as desired.

Theorem 1.1 *Let R and S be polynomial species. Then*

$$\begin{aligned} Gen(R+S) &= Gen(R) + Gen(S) \\ Gen(R \cdot S) &= Gen(R) \cdot Gen(S). \end{aligned}$$

When S is a polynomial species without constant term,

$$\begin{aligned} Gen(\gamma_k(S)) &= Gen(S)^k/k! \\ Gen(\exp(S)) &= \exp(Gen(S)). \end{aligned}$$

2 Colored Types and Plethysm

In the standard combinatorial approach to Littlewood's plethysm (which will be phrased in the language of species), given two Joyal species M and N where $N[0] = \emptyset$, the plethysm corresponds to unlabelled $M \circ N$-structures whose nodes are colored by the variable set X in all ways.[1]

In this section, a more refined plethysm is introduced where we may restrict which color schemes appear. More specifically, let R be a symmetric polynomial species. The group $E!$ of permutations on E acts on $R[E]$ by the morphism of structures. The orbits of $R[E]$ under this group action may be identified with *unlabelled* R-structures. This is a generalization of *types* which correspond to unlabelled structures, and of analytic functors or *Joyal labellings* which correspond to unlabelled structures colored in all ways. Call the unlabelled R-structures *colored types.*

The plethysm of colored types has a remarkably simple combinatorial interpretation. Let R and S be two symmetric polynomial species. Associate a color in X to each unlabelled S-structure. Then the operation of plethysm takes an unlabelled R-structure and replaces each colored node with an unlabelled S-structure of the same associated color.

This operation is related to the plethysm of symmetric functions. While polynomial species are associated with exponential generating functions, orbits of polynomial species (and hence, colored types) correspond to ordinary generating functions. Let $orb(S[E])$ denote the set of orbits of $S[E]$ under the group action. Set $\text{gen}(\underline{q}) = \text{gen}(q)$ where q is a representative of the orbit $\underline{q}$. The generating function for the orbits of S-structures is defined as

$$\text{Gen}(orb(S)) = \sum_{n \geq 0} \sum_{\underline{q} \in orb(S[n])} \text{gen}(\underline{q}).$$

[1]Recall that $M \circ N$ consists of all pairs (a, m) where a is an assembly of N-structures and m is an M-structure on the associated partition of a.

Note that this generating function equals the (ordinary) generating function for the colored types of S where each unlabelled S-structure contributes its color scheme to the sum. It will be shown that the plethysm of the generating functions for the colored types of R and S equals the generating function for the plethysm of the colored types as defined above.

These notions will now be covered rigorously.

2.1 Colored Types

In practice, it is inconvenient to work directly with orbits in these kinds of arguments. For example, let $M : \mathbf{B} \to \mathbf{B}$ be a Joyal species. In the standard approach to Littlewood's plethysm, one begins with the cycle index series $Z_M \in \mathbb{Q}[x_1, x_2, \ldots]$ for M, and the usual plethysm or *wreath product* for formal power series in infinitely many variables (not Littlewood's plethysm) commutes with the combinatorial operation of substitution. [5] Then the results for symmetric functions are obtained by replacing x_n with the power sum symmetric function $p_n(X)$. Combinatorially, this corresponds to taking a labelled structure with automorphism σ and then coloring its nodes by *independently* coloring the cycles of σ; this is equivalent to examining unlabelled structures colored in all ways. In [1], F. Bergeron extends this method by examining structures based on permutations.

In this section, a different approach is taken where we work with the symmetric functions directly rather than going through the intermediate step of power sum substitution. Combinatorially, this corresponds to beginning with a colored labelled structure, and then examining an automorphism of the structure which preserves the coloring. The advantage of this method is that the colorings do not depend upon an underlying permutation of the labelled structure; this results in more control over the coloring schemes. For example, the elementary symmetric functions can be interpreted in this context.

From this point forward, assume that all polynomial species are symmetric unless otherwise specified, and let $X = \{x_i\}_{i \in I}$ be a countably infinite variable set.

Definition 2.1 Let S be a polynomial species. The (polynomial) species of colored types $\tilde{S}$ is defined as follows:

$$\tilde{S}[E] = \{(\sigma, m, f) : (m, f) \in S[E],\ \sigma \in E!, \text{ and } \sigma(m, f) = (m, f)\}$$

for E a finite set. In other words, a $\tilde{S}$-structure is a pair (σ, s) where σ is an automorphism of the S-structure, s. ■

By Burnside's Lemma, the exponential generating function of the polynomial species $\tilde{S}$ is equal to the ordinary generating function for the orbits of S.

Before defining the combinatorial operation of plethysm, colored types will be assigned colors in the variable set X. For now, assume that S is nontrivial.

Theorem 2.1 *Any symmetric polynomial species S may be decomposed into its orbits*

$$S = \sum_{i \in I} S_i$$

where the S_i are polynomial species (not necessarily symmetric), and the sum ranges over the index set, I, of the variable set, X. More formally, for $(m, f) \in S_i[E]$, and $(m', f') \in S_j[E]$, (m, f) and (m', f') are in the same orbit iff $i = j$.

Let x_i be the *associated color* of an S_i-structure (m, f), and let y_i be the color scheme used by this structure ($y_i = \mathrm{gen}(f)$).

The decomposition of S into its orbits induces a decomposition of colored types:

$$\widetilde{S} = \sum_i \widetilde{S}_i.$$

As before, let x_i be the associated color of a $\widetilde{S}_i$-structure. The monomial y_i is simply the color scheme of any $\widetilde{S}_i$-structure; the variables, y_i, will be called the *dummy variables* for $\widetilde{S}$. By Burnside's Lemma, $\mathrm{Gen}(\widetilde{S}_i) = y_i$.

Example 2.1 Let A be the *elementary symmetric species* of uniquely colored sets. $A[0] = \{(\emptyset, f_\emptyset)\}$, and for a nonempty finite set E, $A[E] = \{(E, f) : f \text{ is injective}\}$. $\widetilde{A}$ may be identified with distinctly colored unlabelled sets. Let $\bigoplus_I\{0,1\}$ denote the set of $0-1$ sequences indexed by I with only finitely many non-zero entries. Since two A-structures with the same subjacent set and same coloring scheme are in the same orbit, $\widetilde{A} = \sum A_\alpha$ where the sum ranges over all $\alpha \in \bigoplus_I\{0,1\}$, and $y_\alpha = x^\alpha$. In addition, $\mathrm{Gen}(\widetilde{A})$ equals the generating function for the elementary symmetric functions. ■

Example 2.2 Let U be the *uniform symmetric species* of colored sets. For a finite set, E, $U[E] = \{(E, f) : f : E \to X\}$. $\widetilde{U}$ corresponds to unlabelled, colored sets. Since two U-structures with the same subjacent set and the same coloring scheme are in the same orbit, $\widetilde{U} = \sum U_\alpha$ where the sum ranges over all $\alpha \in \bigoplus_I \mathbb{N}$, and $y_\alpha = x^\alpha$. $\mathrm{Gen}(\widetilde{U})$ is equal to the generating function for the homogeneous symmetric functions. ■

Example 2.3 Let L be the *linear symmetric species* of monochromatic linear orders. $L[0] = \emptyset$, and for a nonempty finite set, E, $L[E] = \{(l, f) : l \text{ linear order on } E, |\mathrm{im}\, f| = 1\}$. $\mathrm{Gen}(\widetilde{L})$ is equal to the generating function for the power sum symmetric functions. ■

2.2 Plethysm of Polynomial Species

Let R and S be nontrivial polynomial species where S is without constant term. Recall that the species of colored type, $\widetilde{S}$, can be decomposed ($\widetilde{S} = \sum \widetilde{S}_i$). The monomials y_i (where y_i is the color scheme of an $\widetilde{S}_i$-structure) are the dummy variables for $\mathrm{Gen}(\widetilde{S})$.

A rigorous definition for the plethysm of polynomial species of colored type may now be given.

Definition 2.2 Let $\widetilde{R}$ and $\widetilde{S}$ be two polynomial species of colored type. The plethysm $\widetilde{R} * \widetilde{S}[E]$ consists of structures of the form $(\sigma, a, f, \hat{\sigma}, m, \hat{f})$ where

1. (a, f) is an assembly of S-structures with automorphism σ i.e., $(\sigma, a, f) \in \widetilde{\exp(S)}[E]$. Let π be the associated partition.

2. The decomposition of S induces a coloring $\hat{f} : \pi \to X$; namely, if $(s, h) \in S_i[B]$ is a member of the assembly built on the block $B \in \pi$, then $\hat{f}(B) = x_i$. Furthermore, since the automorphism σ transports an S-structure in the assembly to itself or to a neighboring S-structure of the same type, σ induces a permutation $\hat{\sigma} : \pi \to \pi$ of the associated partition such that $\hat{f} \cdot \hat{\sigma} = \hat{f}$. Specifically, $\hat{\sigma}(B) = B'$ when there exists representatives $b \in B$, $b' \in B'$ such that $\sigma(b) = b'$.

3. The $\tilde{R}$-structure $(\hat{\sigma}, m, \hat{f})$ is an element of $R[\pi]$ where the coloring $\hat{f}$ and the automorphism $\hat{\sigma}$ are as specified above.

■

Theorem 2.2 *Let R and S be polynomial species. Then*

$$Gen(\tilde{R} * \tilde{S}) \;=\; Gen(\tilde{R}) * Gen(\tilde{S}).$$

2.3 Examples

Theorem 2.3 *$\tilde{A}_{[1]}$ acts as a two-sided identity for any polynomial species $\tilde{S}$ (without constant term).*

$$\tilde{S} * \tilde{A}_{[1]} = \tilde{A}_{[1]} * \tilde{S} = \tilde{S}.$$

Theorem 2.4 *Let R, S, and T be polynomial species where T is without constant term, and let n and k be positive integers. Then*

1. $\tilde{L}_{[n]} * \tilde{L}_{[k]} = \tilde{L}_{[k]} * \tilde{L}_{[n]} = \tilde{L}_{[n \cdot k]}$.
2. $\tilde{T} * \tilde{L}_{[n]} \equiv \tilde{L}_{[n]} * \tilde{T}$.
3. $(\tilde{R} + \tilde{S}) * \tilde{T} = \tilde{R} * \tilde{T} + \tilde{S} * \tilde{T}$.
4. $(\tilde{R} \cdot \tilde{S}) * \tilde{T} = (\tilde{R} * \tilde{T}) \cdot (\tilde{S} * \tilde{T})$.

Let S be any non-trivial polynomial species such that $S[0] = \emptyset$. Intuitively, $\tilde{A} * \tilde{S}$ corresponds to unlabelled assemblies of S-structures where no two members are the "same".

Example 2.4 For any positive integer k, $\tilde{A} * \tilde{A}_{[k]}$ may be associated with unlabelled assemblies of distinctly colored k-sets where no two sets use the same color scheme. Furthermore, $a_n * a_k(X)$, which equals the symmetric function of degree nk in the corresponding generating function, counts the number of unlabelled assemblies (as specified above) with n members.
■

Example 2.5 $\tilde{A} * \tilde{U}_{[k]}$ corresponds to unlabelled assemblies of colored k-sets where no two sets use the same color scheme. Also, $a_n * h_k(X)$, which equals the symmetric function of degree nk in the associated generating function, counts the number of unlabelled assemblies of colored k-sets where there are n sets in the assembly, and no two sets use the same color scheme. ■

Let S be any polynomial species without constant term. The polynomial species $\widetilde{\exp(S)}$ is isomorphic to $\tilde{U} * \tilde{S}$. Intuitively, $\widetilde{\exp(S)}$ corresponds to unlabelled assemblies of S-structures, and therefore the plethysm of a symmetric function into a homogeneous symmetric function counts unlabelled assemblies.

Example 2.6 The polynomial species $\tilde{U} * \tilde{A}_{[k]}$ corresponds to unlabelled assemblies of distinctly colored k-sets. Also, $h_n * a_k(X)$, which equals the symmetric function of degree nk in the associated generating function, counts the number of unlabelled assemblies of distinctly colored k-sets where the assembly contains n members. ■

Example 2.7 For any positive integer k, $\widetilde{U} * \widetilde{U}_{[k]}$ corresponds to unlabelled assemblies of colored k-sets. Furthermore, $h_n * h_k(X)$, which equals the symmetric function of degree nk in the associated generating function, counts the number of unlabelled assemblies of colored k-sets with n members. ∎

3 Plethystic Inverse

Let $F(X) = \sum f_k(X)$ be a generating function of symmetric functions in the variable set X where each $f_k(X)$ contains only non-negative integer coefficients. The function $a_1(X)$ acts as a two-sided identity for $F(X)$:

$$F(X) * a_1(X) = a_1(X) * F(X) = F(X).$$

When $f_0(X) = 0$ and $f_1(X) = a_1(X)$, $F(X)$ has an associated generating function $G(X)$ which acts as its plethystic inverse:

$$G(X) * F(X) = a_1(X).$$

Chen and Read have examined inverses of formal power series under the wreath product. [3][11] Joyal and G. Labelle have studied inversion of species in the context of virtual species, and inversion of indicator series for species under the wreath product. [6][7][8] In this section, the plethystic inverse of a generating function of symmetric functions is given a combinatorial interpretation by examining appropriate lattice-theoretic structures and using Möbius inversion techniques. [12] The general method used here for interpreting inverses was introduced in [10]. First of all, the notion of a polynomial species is extended by taking as the base category not families of colored structures, but families of colored finite partially ordered sets with unique minima and maxima. The generating function for such a *Möbius polynomial species* is computed by replacing the notion of cardinality with the evaluation of the Möbius function. This permits interpretation of symmetric functions with negative terms.

Then given a (symmetric) polynomial species $\widetilde{R}$, a Möbius polynomial species $\widetilde{R}^{[-1]}$ is associated to it which is, in a natural way, its inverse. A partial order is defined on assemblies of R-structures provided with automorphisms by first defining something like a monoid on these structures. This partial order leads to a Möbius polynomial species whose generating function is the plethystic inverse of the generating function for $\widetilde{R}$.

A summary of this procedure is given below. For a fuller exposition, see [14].

3.1 Möbius Polynomial Species

In this subsection, polynomial species are extended to colored families of partially ordered sets by combining polynomial species with Möbius species. [2][10]

A Möbius species is a functor $M : \mathbf{B} \rightarrow \mathbf{Int}$ from the category of finite sets and bijections to the category of finite families of finite posets with $\hat{0}$ and $\hat{1}$ where the morphisms are bijective functions $f : A \rightarrow B$ such that if $f(I) = I'$, then I and I' are isomorphic as posets. A *Möbius polynomial species* is defined as a subfunctor $R \subseteq M \times Hom$ such that for any finite subset F of the variable set, X, the set $\{(m, f) \in R[E] : \text{im } f \subseteq F\}$ is finite. R is said to be symmetric when for any bijection $u : X \to X$, and for any structure $(m, f) \in R[E]$, $(m, u \circ f) \in R[E]$. From now on, assume that a Möbius polynomial species is symmetric.

Set $\text{gen}(m, f) = \mu(m)\text{gen}(f)$ where (m, f) is an R-structure, and $\mu(m)$ is the Möbius function of the interval m evaluated at the two extremes. Define

$$\text{gen}(R[E]) = \sum_{(m,f)\in R[E]} \text{gen}(m, f).$$

Since R is symmetric, $\text{gen}(R[n])$ is a symmetric function of degree n with (possibly negative) integer coefficients.

The *Möbius generating function* for R is defined as

$$\text{Gen}(R) = \sum_{n\geq 0} \text{gen}(R[n])/n!.$$

Addition and multiplication from the theory of polynomial species can be extended to Möbius polynomial species; the corresponding generating function identities behave as desired since the Möbius valuation acts with respect to the sum and product of partially ordered sets in the same way that set cardinality does for ordinary sets. Furthermore, the additive and multiplicative inverses of polynomial species can be interpreted. [13]

3.2 $\mathcal{A}_*$ and the Partial Order

First of all, a little terminology. Recall that any nontrivial (symmetric) polynomial species S may be decomposed into its orbits: $S = \sum S_i$. In subsequent discussions, any polynomial species will come with an implicit decomposition. When S is without constant term, an assembly of S-structures (a, f) with associated partition π has an *associated coloring* $\hat{f} : \pi \to X$ where $\hat{f}(B) = x_i$ for $B \in \pi$ when the corresponding member of the assembly built on B is an S_i-structure (refer to the definition of plethysm for the relevance of $\hat{f}$). Let σ be an automorphism of the assembly (a, f). The *associated partition* of (σ, a, f) is defined as the associated partition of (a, f). The structure (σ, a, f) has an *induced permutation* $\hat{\sigma} : \pi \to \pi$ of the associated partition where $\hat{\sigma}(B) = B'$ when there exists $b \in B$ and $b' \in B'$ with $\sigma(b) = b'$.

For any polynomial species R, let $el(R)$ denote the category whose objects are R-structures, and whose morphisms are isomorphisms of R-structures. Let $\mathcal{A}_*$ be the family whose members are of the form $el(\widetilde{U}_0 * \widetilde{R})$ where R is a (symmetric) polynomial species such that $\text{gen}(R[0]) = 0$ and $\text{gen}(R[1]) = a_1(X)$. For $el(\widetilde{U}_0 * \widetilde{R})$ and $el(\widetilde{U}_0 * \widetilde{S})$ in $\mathcal{A}_*$, define a product $\otimes$ as

$$el(\widetilde{U}_0 * \widetilde{R}) \otimes el(\widetilde{U}_0 * \widetilde{S}) = el(\widetilde{U}_0 * (\widetilde{S} * \widetilde{R})).$$

(Actually, $\mathcal{A}_*$ is a special kind of category, and $\otimes$ is a special kind of bifunctor, but details have been ommitted for brevity.)

Definition 3.1 A *c-monoid* in $\mathcal{A}_*$ is a triple (A, c, I) consisting of a nonvacuous element $A \in A$, a functor $c : A \otimes A \to A$, and an $I \subseteq A$ such that

1. (a) (associativity) The following diagram commutes.

$$\begin{array}{ccccc}
(A \otimes A) \otimes A & \xrightarrow{=} & A \otimes (A \otimes A) & \xrightarrow{id\otimes c} & A \otimes A \\
\downarrow{\scriptstyle c\otimes id} & & & & \downarrow{\scriptstyle c} \\
A \otimes A & & \xrightarrow{\quad c \quad} & & A
\end{array}$$

Note that $A \otimes A$ is a subcategory of the product category $A \times A$. Write $a_1 \cdot a_2$ for $c(a_1, a_2)$.

(b) (left cancellation) For any $a_1, a_2, a_2' \in A$ such that $(a_1, a_2), (a_1, a_2') \in A \otimes A$, if $a_1 \cdot a_2 = a_1 \cdot a_2'$ then $a_2 = a_2'$.

2. (a) (right identity) The product c induces a natural isomorphism $\gamma : A \otimes I \rightarrow A$. In particular, for any $a \in A$, $\gamma(a, i_R(a)) = a$ iff $a \cdot i_R(a) = a$.

   (b) (left identity) The product c induces a natural isomorphism $\beta : I \otimes A \rightarrow A$. In particular, for any $a \in \mathcal{A}$, $\beta(i_L(a), a') = a$ iff a' is isomorphic to a in A and $i_L(a) \cdot a' = a$.

   (c) (no proper divisors of unity) If there is an $i \in I$ and $(a_1, a_2) \in A \otimes A$ such that $a_1 \cdot a_2 = i$ then $a_1 = i$.

   (d) (graded) Let $|a|$ equal the number of blocks of the associated partition of any $a \in A$. Then for any $a_1, a_2, a_2' \in A$ where $a_1 \cdot a_2 = a_3$, $|a_2| = |a_3|$.

■

Example 3.1 Recall that L is the linear symmetric species of monochromatic linear orders. $(el(\widetilde{U}_0 * \widetilde{L}), c, I)$ is a c-monoid where given $(\sigma, a, f, \hat{\sigma}, l, \hat{f}) \in \widetilde{L} * \widetilde{L}[E]$, the product concatenates the linear orders in the assembly a in the order specified by l. Note that all members of (a, f) are linear orders of the same length and color since $\hat{f}$ is a constant function. Also, the automorphisms σ and $\hat{\sigma}$ must be identity permutations. It is simple enough to generalize the product to $\widetilde{U}_0 * \widetilde{L} * \widetilde{L}$-structures. ■

The identity elements for c-monoids are easily described.

Lemma 3.1 *Let $(el(\widetilde{U}_0 * \widetilde{R}), c, I)$ be a c-monoid in $\mathcal{A}_*$. Then for any $(\sigma, a, f) \in \widetilde{U}_0 * \widetilde{R}[E]$,*

1. *$i_L(\sigma, a, f)$ is the unique element of $\widetilde{\gamma_{|E|}(R)}[E]$ with coloring f and automorphism σ.*

2. *$i_R(\sigma, a, f)$ is the unique element of $\widetilde{\gamma_{|\pi|}(R)}[\pi]$ with coloring $\hat{f}$ and automorphism $\hat{\sigma}$ where π is the associated partition of (σ, a, f), $\hat{f} : \pi \rightarrow X$ is the induced coloring of (σ, a, f), and $\hat{\sigma} : \pi \rightarrow \pi$ is the induced permutation of σ.*

A c-monoid $(el(\widetilde{U}_0 * \widetilde{R}), c, I)$ in $\mathcal{A}_*$ induces a partial order on assemblies of R-structures provided with automorphisms by setting

$$(\sigma, a_1, f) \leq_c (\sigma, a_3, f) \text{ iff there exists an } (\hat{\sigma}, a_2, \hat{f}) \text{ with } c((\sigma, a_1, f), (\hat{\sigma}, a_2, \hat{f})) = (\sigma, a_3, f).$$

Note that comparable elements use the same coloring and the same automorphism.

For any finite set E, coloring $f : E \rightarrow X$, and permutation $\sigma : E \rightarrow E$ with $f \circ \sigma = f$, let $P_R[\sigma, E, f]$ be the poset

$$P_R[\sigma, E, f] = \langle \{(\sigma, a, f) \in \widetilde{U}_0 * \widetilde{R}[E]\}, \leq_c \rangle$$

where $\leq_c$ is the restriction of $\langle el(\widetilde{U}_0 * \widetilde{R}), \leq_c \rangle$ to the elements of $P_R[\sigma, E, f]$.

The following properties will be crucial in defining a Möbius polynomial species $\widetilde{R}^{[-1]}$ with the desired properties.

Theorem 3.1 *The family of posets $\{P[\sigma, E, f]\}$ indexed by the triples (σ, E, f) where E is a finite set, $f : E \to X$, and $\sigma : E \to E$ is a permutation with $f \circ \sigma = f$ satisfy the following properties:*

1. *For any bijection $g : E \to F$ between finite sets, the map*

$$\begin{array}{rccc} P_R[g]: & P[\sigma, E, f] & \to & P[g \circ \sigma \circ g^{-1}, F, f \circ g^{-1}] \\ & r & \mapsto & \tilde{U}_0 * \tilde{R}[g]r \end{array}$$

   is an order isomorphism.

2. *$P[\sigma, E, f]$ has a $\hat{0}$ equal to the unique element of $\widetilde{\gamma_{|E|}(R)}[E]$ with coloring f and automorphism σ.*

3. *For any $p \in P[\sigma, E, f]$, the upper order ideal $I_p = \{p' \in P[\sigma, E, f] : p \leq_c p'\}$ is isomorphic to $P[\hat{\sigma}, \pi, \hat{f}]$ where π is the associated partition of p, $\hat{f}$ is the associated coloring, and $\hat{\sigma}$ is the induced permutation.*

3.3 Plethystic Inverse of a Polynomial Species

Given an R-structure (m, f), let $\{(m, f)\}$ denote the assembly containing only the R-structure (m, f). For an $\tilde{R}$-structure (σ, m, f), let $\{(\sigma, m, f)\}$ denote the assembly $\{(m, f)\}$ provided with the automorphism σ.

Definition 3.2 Let $(el(\tilde{U}_0 * \tilde{R}), c, I)$ be a c-monoid in $\mathcal{A}_*$. The inverse Möbius polynomial species $\tilde{R}^{[-1]}$ is defined as

$$\begin{array}{lcl} \tilde{R}^{[-1]}[E] & = & \{([\hat{0}, \{(\sigma, m, f)\}], f) : (\sigma, m, f) \in \tilde{R}[E], [\hat{0}, \{(\sigma, m, f)\}] \subseteq P[\sigma, E, f]\} \\ \tilde{R}^{[-1]}[g]([\hat{0}, \{r\}], f) & = & ([\hat{0}, \{\tilde{R}[g]r\}], f \circ g^{-1}) \end{array}$$

where $g : E \to F$ is a bijection of finite sets. ■

Theorem 3.2 *Let $(el(\tilde{U}_0 * \tilde{R}), c, I)$ be a c-monoid in $\mathcal{A}_*$. Then*

$$Gen(\tilde{R}^{[-1]}) * Gen(\tilde{R}) = a_1(X).$$

Proof: Use Möbius inversion arguments and the previous theorem. ■

3.4 Examples

Example 3.2 (power sum) Recall that $\tilde{U}_0 * \tilde{L}$ has a c-monoid structure where the product is concatenation. Let $(\sigma, l) \in \tilde{L}[n]$ where σ must be the trivial permutation and l is a monochromatic linear order. Any element of the interval $[\hat{0}, \{(\sigma, l)\}]$ is of the form (σ, a) where the assembly a is "obtained" by "cutting" the linear order l into continuous segments of the same length. Therefore, the interval is isomorphic to the lattice of divisors of n, and the Möbius valuation $\mu(\hat{0}, \{(\sigma, l)\}) = \mu(n)$ where $\mu(n)$ is the classical Möbius function of number theory. Let $p_n(X) = \sum_i x_i^n$. Then

$$\text{Gen}(\tilde{L}) = \sum_{n \geq 1} p_n(X) \quad \text{and} \quad \text{Gen}(\tilde{L}^{[-1]}) = \sum_{n \geq 1} \mu(n) p_n(X).$$

■

Example 3.3 (homogeneous) Let π be a partition of a set E, and let τ be a partition of π. The coinduced partition $ind_\pi(\tau)$ is a partition of E whose blocks are the subsets C of E of the form $C = \bigcup\{B : B \in D\}$ as D ranges over the blocks of τ. For example, if $\pi = \{B_1, B_2, B_3\}$ and $\tau = \{\{B_1, B_2\}, \{B_3\}\}$ where $B_1 = \{1,2\}$, $B_2 = \{3,4\}$, and $B_3 = \{5\}$, then $ind_\pi(\tau) = \{\{1,2,3,4\},\{5\}\}$.

$(el(\tilde{U}_0 * \tilde{U}_0), c, I)$ is a c-monoid in $\mathcal{A}_*$ where given $(\sigma, \pi, f, \hat{\sigma}, \tau, \hat{f}) \in \tilde{U}_0 * \tilde{U}_0 * \tilde{U}_0[E]$,

$$c(\sigma, \pi, f, \hat{\sigma}, \tau, \hat{f}) = (\sigma, ind_\pi(\tau), f).$$

When σ is the identity permutation, the interval $[\hat{0}, \{(\sigma, E, f)\}]$ obeys the usual ordering for partitions of E. Note that different automorphisms will result in different intervals. For example, $\sigma_1 = (1)(2)(3)$ and $\sigma_2 = (12)(3)$ are automorphisms of the U-structure $r = (\{1,2,3\}, f)$ where $f(1) = f(2) = x_1$, and $f(3) = x_2$. The interval $[\hat{0}, \{(\sigma_1, r)\}]$ is isomorphic to the lattice of partitions of $\{1,2,3\}$ while $[\hat{0}, \{(\sigma_2, r)\}]$ is isomorphic to a chain of length 2. ■

Example 3.4 (multicolored linear orders) Let R be the symmetric polynomial species of (nontrivial) linear orders colored in all ways. $el(\tilde{U}_0 * \tilde{R})$ has a c-monoid structure where the product is concatenation. Let $(\sigma, l) \in \tilde{R}[n]$ where σ can only be the trivial permutation and l is a colored linear order. Any element of the interval $[\hat{0}, \{(\sigma, l)\}]$ is of the form (σ, a) where the assembly a is obtained by "cutting" l into continuous segments. Since specifying when l is cut completely determines (σ, a), and since more cuts result in a structure lower in the partial order, $[\hat{0}, \{(\sigma, l)\}]$ is isomorphic to the Boolean poset B_{n-1} Therefore,

$$\mathrm{Gen}(\tilde{R}) = \sum_\lambda (-1)^{|\lambda|-1} \frac{|\lambda|!}{\prod \lambda_i!} m_\lambda(X)$$

where the sum ranges over all number partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $|\lambda| > 0$, and $m_\lambda(X)$ denotes the monomial symmetric function. ■

Example 3.5 (monochromatic sets) Let R be the symmetric polynomial species of (nontrivial) monochromatic sets. The induced copartition may be used to define a c-monoid structure.

Consider the interval $[\hat{0}, \{(\sigma, E, f)\}]$ where $(\sigma, E, f) \in \tilde{R}[E]$. Any element of this interval is a triple (σ, π, f) where π is a partition of E with blocks of the same size, f is a constant function, and σ is an automorphism of π. Since the generating function for $\tilde{R}$ equals the generating function for the power sum symmetric functions, this is another (more complicated) way to treat the plethystic inverse for the power sum. ■

Example 3.6 (rooted trees) Let A denote the Joyal species of rooted trees. Given $(a, t_1) \in A_0(A_0)[E]$, define a product $t = p(a, t_1)$ where $t \in A_0[E]$ contains all of the edges in a plus a few more defined as follows: for every pair of trees m_1 and m_2 in a such that the associated blocks form an edge of t_1, insert an edge between the roots of m_1 and m_2. The root of t will be the root of the tree in a whose associated block is the root of t_1.

The polynomial species $\widetilde{A_0 \times Hom}$ corresponds to unlabelled rooted trees colored in all ways; it has a c-monoid where given $(\sigma, a, f, \hat{\sigma}, t, \hat{f}) \in (\widetilde{A_0 \times Hom}) * (\widetilde{A_0 \times Hom})[E]$,

$$c(\sigma, a, f, \hat{\sigma}, t, \hat{f}) = (\sigma, p(a, t), f).$$

A similar c-monoid can be defined for monochromatic rooted trees by restricting the product to the appropriate monochromatic structures.

A *fiber* of a tree vertex is the set of the vertex's children. For a Joyal species $M : \mathbf{B} \to \mathbf{B}$, a tree is said to be M-enriched if each of its fibers is provided with an M-structure (not forgetting the empty fibers). A c-monoid structure in $\mathcal{A}_*$ can be constructed for many kinds of enriched (colored) trees such as trees enriched with linear orders (i.e., ordered trees), trees where each vertex has a multiple of k children (k a positive integer), partition-enriched trees, permutation-enriched trees, and even trees enriched by ordered trees.

For example, let A_L denote the species of ordered trees. An ordered tree is of the form $(T, \{l_x\}_{x\in E})$ where $T \in A[E]$ is a rooted tree and l_x is a linear order on the fiber of the vertex x. Consider an element

$$(a, T_L) = (\{(T_B, \{l_x\}_{x\in B}) : B \in \pi\}, (T_1, \{l_B\}_{B\in\pi})) \in A_L(A_L)[E]$$

where π is the associated partition of the forest of ordered trees, a. Define a product $p(a, T_B) = (T_2, \{l'_x\}_{x\in B})$ where T_2 is obtained by taking the products of the rooted forests $\{T_B : B \in \pi\}$ and $\{T_1\}$ as defined above. Suppose that x is the root of some tree T_B, the linear order $l_B = B_1 B_2 \cdots B_k$, and y_i is the root of tree T_{B_i}. Then l'_x equals the concatenation of l_x with $y_1 \cdots y_k$. When x is not the root of some tree T_B, then $l'_x = l_x$.

Then as done earlier in this example, the product p can be used to define a c-monoid structure for ordered trees colored in all ways. ■

Example 3.7 (graphs) Let G be the Joyal species of simple connected graphs ($G[0] = \emptyset$). Define a product p of $(\{g_B : B \in \pi\}, g_1) \in G(G)[E]$ as the graph $g \in G[E]$ where $\{x, y\}$ is an edge of g if $\{x, y\}$ is an edge of some graph g_B, or the associated blocks of π containing x and y form an edge in g_1. Use the product p to define a c-monoid structure for $G \times \widetilde{Hom}$. ■

Note that in the last few examples, a product p on labelled structures was extended to a c-monoid for the appropriate colored structures. This is a powerful technique for generating examples; that is, products for labelled structures (which correspond to c-monoids for the compositional inverses of exponential generating functions as described in [10]) can be extended to c-monoids in $\mathcal{A}_*$.

Many more examples such as vertebrates (a tree with an ordered pair of not necessarily distinct vertices), bushes (rooted trees with no grandchildren), parenthesizations, and pointed simple connected graphs (a simple connected graph with a special vertex–similar to the root of a tree) fall into this framework.

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