ON THE ENUMERATION OF PERFORATION PATTERNS FOR PUNCTURED CONVOLUTIONAL CODES

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Abstract

Using techniques from combinatorial theory, methods have been elaborated for the enumeration of all the distinct perforation patterns that must be considered in exhaustive searchs for good punctured convolutional codes. An enumeration formula for computing the number of perforation patterns remaining after taking account of equivalences between patterns is derived. Numerical results are provided for the number of distinct patterns for deriving (v, b) punctured codes from $(2,1)$, $(3,1)$ and $(4,1)$ original codes, for values of b varying from 2 to 12.

1 INTRODUCTION

A punctured convolutional code is obtained by periodically eliminating (i.e. puncturing) code symbols from the output of a convolutional encoder. With a low-rate original encoder, the choice of an appropriate puncturing rule allows the specification of a resulting punctured convolutional code with coding rate $R > 1/2$. Since the underlying structure of such high-rate punctured codes is that of low-rate codes, they are more easily decoded than usual high-rate convolutional codes [1]–[3]. This is one of the principal motivations for using punctured convolutional codes.

Using a puncturing rule that specifies the elimination of all but v coded symbols out of every b branches of the original code, a (v, b) punctured convolutional code with coding rate $R = b/v$ is produced. Assuming a low-rate $(v_o, 1)$ original code, the puncturing rule, called the *perforation pattern*, has *period* b bits and is conveniently specified by a binary $v_o \times b$ matrix $\mathbb{P}$

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with entries

$$p_{ij} = \begin{cases} 0 & \text{if symbol } i \text{ of every } j\text{-th branch is punctured} \\ 1 & \text{if symbol } i \text{ of every } j\text{-th branch is not punctured} \end{cases} \quad (1)$$

Given the original code, the resulting punctured code depends only on the perforation pattern.

As is the case with convolutional codes in general, punctured codes are seldom constructed by systematic methods but most often obtained by search techniques [1]–[5]. An original encoder is usually selected at the outset, and punctured codes corresponding to valid perforation patterns are evaluated in order to select the best ones according to certain criteria of error and decoding performances. Since much work is involved in the evaluation process — it usually relies on computing distance properties: weight spectra, column distance function, etc., or on performing lengthy computer simulations — the set of valid perforation patterns should be kept as small as possible in order to reduce the search effort.

In this paper, we consider the task of enumerating all the valid perforation patterns for a given set of parameters b, v, and v_o. A general technique from combinatorial theory is adapted to the specific task of evaluating the number of distinct perforation patterns remaining after equivalences between patterns have been taken into account.

The remainder of the paper is organized as follows. A formalism compatible with the relevant elements of combinatorial theory is established in section 2. Our main result, the enumeration formula for distinct perforation patterns, is presented in section 3. Numerical results derived from the formula are presented in section 4 for several values of parameters b, v and v_o.

2 FORMALISM

It has been observed in [6] that a cyclic shift of the columns of the perforation pattern leaves the ensemble of Hamming distances between sequences of the resulting punctured code unchanged. Since the performances of a convolutional code depend exclusively on this ensemble of distances, cyclic shifts of the perforation patterns yield punctured codes having the same performances.

Let the set of valid perforation patterns for a certain group of parameters b, v, and v_o, be partitioned into classes according to the following equivalence relation: two perforation patterns are equivalent if one may be obtained from the other by cyclic shifts of the columns. Then, with respect to error and decoding performances, equivalent perforation patterns give rise to equivalent punctured codes. For example, whatever the original $(2,1)$ code used, the following four perforation patterns all yield $(5,4)$ punctured codes with equivalent performances

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}. \quad (2)$$

When searching for good punctured codes, only one representative from each equivalence class has to be considered, thus reducing the total number of distinct perforation patterns to consider.

Enumerating distinct perforation patterns therefore amounts to counting the number of equivalence classes induced by this equivalence relation. While this may sound simple, complications arise from the fact that the number of patterns is not the same in every equivalence class. The maximum number of patterns in one class is obviously b, the period of the perforation patterns, as is the case for the class of patterns of (2), but the number of patterns may be smaller. For instance, the two following perforation patterns form a size two equivalence class

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{3}$$

The number of patterns in one class will be called the *effective period* of the class. This may also be viewed as a property of the patterns belonging to the class since it is the smallest number of cyclic shifts that leaves the patterns invariant. In order to evaluate the number of distinct perforation patterns, we utilize a powerful theorem from combinatorial theory: Polyà's enumeration formula (see for example [7, pp. 225–237]). But before we may proceed with this formula, we need to establish a formalism compatible with the framework of the theorem.

Let $X = \{1, 2, ..., n\}$ be a set of n objects. A mapping of X

$$g = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix} \tag{4}$$

into itself is a permutation of degree n. In what is called *cycle notation*, a permutation g may be written as a product of cycles, each cycle being obtained by writing the image under g of each element after the element. Let $\lambda_i(g)$ be the number of cycles of length i in permutation g. The *cycle index* of g is the polynomial in the indeterminates $x_1, x_2, \ldots, x_b$

$$x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_b^{\lambda_b(g)}. \tag{5}$$

With composition of permutations as product, a set of permutations that conforms to the axioms of group theory forms a group G. For our purposes, $X = \{1, 2, \ldots, b\}$ is the set of the b columns of a perforation pattern and the group of interest defined on this set is the cyclic group C_b of order b generated by the circular permutation. It is clear that permutations from this group transform a perforation pattern into equivalent patterns.

The cycle index of a permutation group G is the average of the cycle indices of the permutations of G:

$$P(G; x_1, x_2, \ldots, x_b) = \frac{1}{b} \sum_{g \in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_b^{\lambda_b(g)}. \tag{6}$$

A mapping ψ of X into a set $A = \{a_1, a_2, \ldots, a_m\}$ of colors is called a coloring of the objects. Here, the colors are the $m = 2^{v_o}$ possible binary column vectors. A coloring thus corresponds to a given perforation pattern. For example, the coloring ψ that assigns the following colors to the four columns

$$\psi(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \psi(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \psi(3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \psi(4) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7}$$

corresponds to the perforation pattern

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{8}$$

Let Ψ be a set of colorings. A group G is said to *act* on Ψ if each $g \in G$ induces a permutation g^* on Ψ, called the *action* of g, and if $(g_i \circ g_j)^* = g_i^* \circ g_j^*$. The second condition implies that the set of g_i^* also forms a group. Suppose the group G acts on the set of colorings Ψ. Two colorings ψ_1 and ψ_2 of Ψ belong to the same *scheme* — written $\psi_1 \sim \psi_2$ — if there exists a permutation $g \in G$ whose action takes ψ_1 to ψ_2 (i.e. $g^*(\psi_1) = \psi_2$). A scheme will be denoted $\overline{\psi}$. The relation $\psi_1 \sim \psi_2$ is thus an equivalence relation that partitions Ψ into equivalence classes called schemes. Clearly, when the group of permutation is the cyclic group C_b, schemes correspond to classes of equivalent perforation patterns.

Let us assign to each color a_i a weight w_i for $i = 1, 2, \ldots, m$. For ψ a coloring of the objects in X, define the weight of ψ to be the product of the weights of the colors assigned. The weight of a scheme is the common weight of all the colorings in it

$$w(\overline{\psi}) = w(\psi_i), \quad \psi_i \in \overline{\psi}. \tag{9}$$

For our purpose, the weights of the colors should correspond to the Hamming weights of the binary column vectors in such a way that pertinent information about the perforation patterns may be extracted. This is achieved by the following assignment

$$w(a_i) = z^{W_H(a_i)} \tag{10}$$

where z is an indeterminate and $W_H(a_i)$ is the Hamming weight of column a_i. The exponent of the weight of a coloring is thus the total number of '1's in the corresponding perforation pattern, that is, it is equal to v.

3 ENUMERATION FORMULA

We are now in position to state Polyà's theorem:

Theorem 1 (Polyà) *The sum of the weights of all of the schemes, called the* inventory *of the weights of the schemes, is*

$$\sum w(\overline{\psi}) = P(G; \sum_{i=1}^{m} w_i, \sum_{i=1}^{m} w_i^2, \sum_{i=1}^{m} w_i^3, \ldots, \sum_{i=1}^{m} w_i^b). \tag{11}$$

The number of schemes $\overline{\psi}$ that use α_i times colors a_i, for $i = 1, 2, \ldots, m$, is the coefficient of $w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_m^{\alpha_m}$ in this inventory.

The application of Polyà's enumeration formula is easier than the formulation of the theorem suggests, as the following example demonstrates.

Example 1 *Let us evaluate the number of distinct perforation patterns to consider for producing $(v, 8)$ punctured codes from $(2, 1)$ original codes, i.e., $b = 8, v_o = 2$. The eight permutations in the cyclic group C_8 are listed using cycle notation in Table 1, together with the corresponding cycle indices. The set of colors is the set of possible binary column vectors of dimension $v_o = 2$*

$$A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \tag{12}$$

and the corresponding weights are

$$w\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=1,\ w\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=w\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=z,\ w\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=z^2. \tag{13}$$

From Table 1, the cycle index for C_8 is

$$P(C_8;x_1,x_2,\ldots,x_8)=\frac{1}{8}\left(x_1^8+x_2^4+2x_4^2+4x_8^1\right). \tag{14}$$

On the other hand, we have

$$\sum_{i=1}^{m} w_i = 1+2z+z^2, \tag{15}$$

$$\sum_{i=1}^{m} w_i^2 = 1+2z^2+z^4, \tag{16}$$

$$\sum_{i=1}^{m} w_i^4 = 1+2z^4+z^8, \tag{17}$$

$$\sum_{i=1}^{m} w_i^8 = 1+2z^8+z^{16}. \tag{18}$$

The inventory of the weights of the schemes is therefore

$$\begin{aligned}\sum w(\overline{\psi}) &= \frac{1}{8}\Big[(1+2z+z^2)^8+(1+2z^2+z^4)^4 \\ &\quad +2(1+2z^4+z^8)^2+4(1+2z^8+z^{16})\Big] \qquad (19)\\ &= 1+2z+16z^2+70z^3+232z^4+546z^5+1008z^6 \\ &\quad +1430z^7+1620z^8+1430z^9+1008z^{10}+546z^{11} \qquad (20)\\ &\quad +232z^{12}+70z^{13}+16z^{14}+2z^{15}+z^{16}\end{aligned}$$

This inventory informs us that there are:

1430	*distinct perforation patterns for*	(9, 8)	*punctured codes,*
1008	*distinct perforation patterns for*	(10, 8)	*punctured codes,*
546	*distinct perforation patterns for*	(11, 8)	*punctured codes,*
232	*distinct perforation patterns for*	(12, 8)	*punctured codes,*
70	*distinct perforation patterns for*	(13, 8)	*punctured codes,*
16	*distinct perforation patterns for*	(14, 8)	*punctured codes,*
2	*distinct perforation patterns for*	(15, 8)	*punctured codes.*

Much of the work involved in evaluating the inventory of weights is devoted to finding the cycle index of the group C_b. Fortunately, this operation is highly regular for a cyclic group. One can show that for the cyclic group C_b of order b, the cycle index is [8, p. 171]

$$P(C_b;\ldots)=\frac{1}{b}\sum_{k|b}\phi(k)(x_k)^{b/k} \tag{21}$$

Table 1: Permutations in the cyclic group C_8 and corresponding cycle indices.

permutation	cycle notation	cycle index
g_1	(1)(2)(3)(4)(5)(6)(7)(8)	x_1^8
g_2	(1,2,3,4,5,6,7,8)	x_8^1
g_3	(1,3,5,7)(2,4,6,3)	x_4^2
g_4	(1,4,7,2,5,8,3,6)	x_8^1
g_5	(1,5)(2,6)(3,7)(4,8)	x_2^4
g_6	(1,6,3,8,5,2,7,4)	x_8^1
g_7	(1,7,5,3)(2,8,6,4)	x_4^2
g_8	(1,8,7,6,5,4,3,2)	x_8^1

where the sum is over all integers k, $1 \leq k \leq b$, such that k divides b. $\phi(k)$ is Euler's function and given the factorisation of k

$$k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_q^{\alpha_q}, \tag{22}$$

where the p_i are prime numbers different from 1, it is readily evaluated as

$$\phi(k) = k\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_q}\right). \tag{23}$$

Another simplification comes from the observation that with the weight assignments (10) the sums of weights used in (11) may be evaluated simply as

$$\sum_{i=1}^{m} w_i^j = (1 + z^j)^{v_o}, \; j = 1, 2, \ldots, b. \tag{24}$$

For large values of b and v_o, the computations in (11) become rather involved. Fortunately, using (24), the computations are reduced to sums of binomial powers which may be evaluated using the binomial theorem. With this formulation, the inventory for the former example becomes

$$\sum w(\bar{\phi}) = \frac{1}{8}\left[(1+z)^{16} + (1+z^2)^8 + 2(1+z^4)^4 + 4(1+z^8)^2\right] \tag{25}$$

$$= \frac{1}{8}\left[\sum_{k=0}^{16}\binom{16}{k} z^k + \sum_{k=0}^{8}\binom{8}{k} z^{2k} \right. \tag{26}$$

$$\left. +2\sum_{k=0}^{4}\binom{4}{k} z^{4k} + 4\sum_{k=0}^{2}\binom{2}{k} z^{8k}\right].$$

For instance, the number of distinct perforation patterns for $(12, 8)$ codes may be evaluated as

$$\frac{1}{8}\left[\binom{16}{12} + \binom{8}{6} + 2\binom{4}{3}\right]. \tag{27}$$

Finally, observing that the vast majority of equivalence classes have maximum effective period, a good lower bound to the number of distinct perforation patterns is readily obtained by assuming that all equivalence classes have period b. Then the number of distinct patterns for (v, b) punctured codes derived from $(v_0, 1)$ original codes is lower bounded by

$$\frac{1}{b}\binom{bv_0}{v}. \tag{28}$$

For b prime, the bound (28) is exact: the cycle index for the cyclic group as evaluated by (21) comprises only one term corresponding to equivalence classes with maximum size.

4 NUMERICAL RESULTS

Polyà's enumeration formula has been used to determine the number of distinct perforation patterns to consider in an exhaustive search for high-rate (v, b) punctured codes derived from $(v_o, 1)$ original codes, for periods b ranging from 2 to 12. The results are presented in Tables 2 to 4 for $v_o = 2, 3$, and 4 respectively, for values of v such that $b < v < bv_o$.

From these results, we may infer a number of general observations about the amount of work involved in searchs for good punctured codes. For (v, b) punctured codes derived from $(v_o, 1)$ original codes, the perforation patterns are $b \times v_o$ matrices with v '1' entries and $bv_o - v$ '0' entries. For enumeration purposes, the '1's and '0's play entirely symetric roles in the patterns. Therefore, for given values of b and v_o, the number of distinct patterns as a function of v should be symetric about the value $v = bv_o/2$. It can be seen, particularly from Tables 3 and 4, that such is the case, with $v = bv_o/2$ corresponding to a maximum. On either side of this maximum, the numbers of distinct patterns decrease, down to values of v_o at $v = 1$ and $v = bv_o - 1$. The actual value of the maximum depends on v_o: to larger values of v_o correspond larger maxima, and the rate of increase is rather steep. This means that for a given period b, choosing a lower rate original code implies that considerably more perforation patterns will have to be considered.

When searching for punctured codes, many combinations of parameters b and v yield the same resulting coding rate $R = b/v$. Tables 5 and 6 present the number of distinct perforation patterns to consider for coding rates 2/3 and 3/4 respectively, for $v_o = 2, 3, 4$ and $2 \leq b \leq 12$. For a given coding rate, it can be seen that the number of patterns increases with b, and the rate of change increases rapidly with v_o. It comes as no surprise then that most searchs for good punctured codes have concentrated on perforation patterns with short periods b [1]–[5], even tough it has been observed that longer periods may yield more powerful codes [9].

5 CONCLUSION

Techniques from combinatorial theory have been used for enumerating all the valid perforation patterns remaining after equivalences between patterns are taken into account. An enumeration formula for computing the number of distinct perforation patterns for sets of parameters b, v, and v_o has been derived. Using the formula, numerical results were obtained for the number of distinct patterns for deriving (v, b) punctured codes from $(2, 1)$, $(3, 1)$ and $(4, 1)$ original codes, for values of b varying from 2 to 12. These results provide useful insights on the amount of work involved in exhaustive searchs for good punctured codes.

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Table 2: Number of distinct perforation patterns for deriving (v, b) punctured codes from $(2, 1)$ original codes, $2 \leq b \leq 12$, $b + 1 \leq v \leq 2b - 1$.

v	b										
	2	3	4	5	6	7	8	9	10	11	12
$b+1$	2	5	14	42	132	429	1430	4862	16796	58786	208012
$b+2$		2	8	24	85	286	1008	3536	12618	45220	163504
$b+3$			2	9	38	143	546	2066	7752	29070	108968
$b+4$				2	12	52	232	952	3888	15504	61333
$b+5$					2	13	70	340	1552	6783	28842
$b+6$						2	16	92	489	2394	11240
$b+7$							2	17	114	665	3542
$b+8$								2	20	140	892
$b+9$									2	21	170
$b+10$										2	24
$b+11$											2

Table 3: Number of distinct perforation patterns for deriving (v, b) punctured codes from $(3, 1)$ original codes, $2 \leq b \leq 12$, $b + 1 \leq v \leq 3b - 1$.

v	b										
	2	3	4	5	6	7	8	9	10	11	12
$b+1$	10	42	198	1001	5304	29070	163438	937365	5462730	32256120	192565800
$b+2$	9	42	236	1287	7314	41990	245256	1448655	8649823	52106040	316360752
$b+3$	3	30	198	1287	8110	50388	312018	1931568	11975985	74437200	463992012
$b+4$		12	129	1001	7314	50388	338140	2228700	14542911	94287120	608993010
$b+5$		3	55	603	5304	41990	312018	2228700	15511760	106073010	716458050
$b+6$			18	273	3114	29070	245256	1931568	14542911	106073010	756265484
$b+7$			3	91	1428	16614	163438	1448655	11975985	94287120	716458050
$b+8$				21	516	7752	92001	937365	8649823	74437200	608993010
$b+9$				3	138	2907	43263	520779	5462730	52106040	463992012
$b+10$					27	855	16852	246675	3004809	32256120	316360752
$b+11$					3	190	5313	98670	1430715	17594250	192565800
$b+12$						30	1338	32898	585429	8414640	104308122
$b+13$						3	253	8970	203580	3506100	50067108
$b+14$							36	1950	59423	1262196	21182952
$b+15$							3	327	14253	388368	7845310
$b+16$								39	2751	100688	2521956
$b+17$								3	406	21576	695640
$b+18$									45	3720	162396
$b+19$									3	496	31416
$b+20$										48	4923
$b+21$										3	597
$b+22$											54
$b+23$											3

Table 4: Number of distinct perforation patterns for deriving (v, b) punctured codes from $(4, 1)$ original codes, $2 \leq b \leq 12$, $b + 1 \leq v \leq 4b - 1$.

v	b = 2	3	4	5	6	7	8	9	10	11	12
$b+1$	28	165	1092	7752	57684	444015	3506100	28242984	231180144	1917334783	16077354108
$b+2$	38	264	2016	15504	122661	986700	8064576	66756144	558689224	4719593312	40193414112
$b+3$	28	312	2860	25194	217936	1874730	16128060	139075410	1203322288	10450528048	91105007340
$b+4$	16	264	3238	33592	327008	3067740	28225120	256754400	2320700736	20901056096	187904537509
$b+5$	4	165	2860	36956	416024	4345965	43421700	421810800	4022534528	37883164174	353701790376
$b+6$		76	2016	33592	450872	5348880	58930880	618656016	6285222762	62395799816	609153193728
$b+7$		62	1092	25194	416024	5730948	70715340	811985790	8873237880	93593699724	961820658040
$b+8$		4	464	15504	327008	5348880	75136678	955277400	11338042976	128075589096	1394640117728
$b+9$			140	7752	217936	4345965	70715340	1008348576	13128240840	160094486370	1859519940784
$b+10$			32	3104	122661	3067740	58930880	955277400	13784671388	182965127280	2282138314816
$b+11$			4	969	57684	1874730	43421700	811985790	13128240840	191281723980	2579808294648
$b+12$				228	22480	986700	28225120	618656016	11338042976	182965127280	2687300534584
$b+13$				38	7084	444015	16128060	421810800	8873237880	160094486370	2579808294648
$b+14$				4	1782	169152	8064576	256754400	6285222762	128075589096	2282138314816
$b+15$					340	53820	3506100	139075410	4022534528	93593699724	1859519940784
$b+16$					48	14040	1315024	66756144	2320700736	62395799816	1394640117728
$b+17$					4	2925	420732	28242984	1203322288	37883164174	961820658040
$b+18$						468	113344	10460416	558689224	20901056096	609153193728
$b+19$						54	25172	3362260	231180144	10450528048	353701790376
$b+20$						4	4512	927520	84767616	4719593312	187904137509
$b+21$							620	216436	27343888	1917334783	91105007340
$b+22$							64	41888	7690953	697212652	40193414112
$b+23$							4	6545	1864356	225568798	16077354108
$b+24$								796	383952	64448228	5805722768
$b+25$								70	65804	16112057	1882933364
$b+26$								4	9158	3483688	545063200
$b+27$									988	641732	139758980
$b+28$									80	98728	31446646
$b+29$									4	12341	6135756
$b+30$										1204	1022816
$b+31$										86	142692
$b+32$										4	16240
$b+33$											1444
$b+34$											96
$b+35$											4

Table 5: Number of distinct perforation patterns for deriving (v, b) punctured codes such that $b/v = 2/3$ from $(v_o, 1)$ original codes, $2 \leq b \leq 12.$, $v_o = 2, 3, 4$

b	$v_o = 2$	$v_o = 3$	$v_o = 4$
2	2	10	28
4	8	236	2016
6	38	8110	217936
8	232	338140	28225120
10	1552	15511760	4022534528
12	11240	756265484	609153193728

Table 6: Number of distinct perforation patterns for deriving (v, b) punctured codes such that $b/v = 3/4$ from $(v_o, 1)$ original codes, $2 \leq b \leq 12$, $v_o = 2, 3, 4$.

b	$v_o = 2$	$v_o = 3$	$v_o = 4$
3	5	42	165
6	85	7314	122661
9	2066	1931568	139075410
12	61333	608993010	187904537509