A combinatorial problem in Hamming graphs and an example in Scratchpad

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Abstract

We present a combinatorial problem which arises in the determination of the complete weight coset enumerators of error-correcting codes. This problem is solved by exponential power series with coefficients in a ring of multivariate polynomials. It is worth noting that there is associated to this problem a system of differential equations with coefficients in a field of rational functions and that Scratchpad (or Axiom), thanks to its abstraction capabilities, is able to solve simply and naturally such a differential equation which seems not be the case for the other computer algebra systems now available.

1 A combinatorial problem in Hamming graphs

Let $\mathbb{F}$ be a finite additive abelian group with q elements, let $m = q - 1$ and fix an ordering $\mathbb{F}^* = [a_1, \ldots, a_m]$ of the nonzero elements of $\mathbb{F}$. For x in the cartesian product $\mathbb{F}^n$ the (Hamming) *weight* of x is defined as $w(x) =$ number of nonzero components of x and the *complete weight* of x ([3]) as the list $w^c(x) = [w_{a_1}(x), \ldots, w_{a_m}(x)]$ where $w_a(x) =$ number of components of x which are equal to $a \in \mathbb{F}^*$. The (Hamming) *distance* between x and y is $d(x,y) = w(y - x)$ and the *gap* between x and y is $g(x,y) = w^c(y - x)$.

If Ω is the set of weight one vectors in $\mathbb{F}^n$, then the *Hamming graph* $\Gamma = \Gamma(n,q)$ is the Cayley graph $C(\mathbb{F}^n, \Omega)$ that is the vertex set is $\mathbb{F}^n$ and

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(x, y) is an oriented edge (arrow) iff $y - x \in \Omega$. Set $\Omega_i = \{x \in \Omega \mid$ the only nonzero component of x is $a_i\}$. An arrow (x, y) in Γ will be called of *color* i if $y - x \in \Omega_i$. A *path* of length j joining x to y is a sequence $\gamma = (x_0, x_1, \ldots, x_j)$ where $x_0 = x$, $x_j = y$ and $x_i - x_{i-1} \in \Omega$, $i = 1, \ldots, j$. Set $\mathrm{Path}_j(x, y)$ to be the set of all these paths and

$$\mathrm{Path} = \bigcup_{j \geq 0} \{\mathrm{Path}_j(x, y) \mid x, y \in \mathbb{F}^n\}.$$

We are interested in the various color distributions of the paths in Γ. For this it is convenient to work in the multivariate polynomial ring $\mathbb{Z}[T_{a_1}, \ldots, T_{a_m}]$.

Definition 1. The *weight function* ϕ: Path $\to \mathbb{Z}[T_{a_1}, \ldots, T_{a_m}]$ is defined as follows

1. if (x, y) is an arrow and if $y - x \in \Omega_i$, then $\phi(x, y) = T_{a_i}$;
2. if $\gamma = (x_0, x_1, \ldots, x_j)$ is a path, then $\phi(\gamma) = \prod_{i=1}^{j} \phi(x_{i-1}, x_i)$.

This weight function ϕ is extended to subsets U of Path by the formula

$$\phi(U) = \sum_{\gamma \in U} \phi(\gamma).$$

$\phi(U)$ is called the *inventory* of U.

Problem 1. Determine the inventories $\phi(\mathrm{Path}_j(x, y))$ for all j:

$$\phi(\mathrm{Path}_j(x, y)) = \sum_{j_1 + \cdots + j_m = j} S_{j_1 \ldots j_m}(x, y) T_{a_1}^{j_1} \cdots T_{a_m}^{j_m}$$

where $S_{j_1 \ldots j_m}(x, y)$ is the number of paths of length $j = j_1 + \cdots + j_m$ joining x to y with j_1 arrows of color 1, j_2 arrows of color 2, etc.

Proposition 1. *If* $g(x, y) = g(x', y')$, *then*

$$\phi(\mathrm{Path}_j(x, y)) = \phi(\mathrm{Path}_j(x', y')) = \phi(\mathrm{Path}_j(0, y - x)).$$

In fact $S_{j_1 \ldots j_m}(x, y) = S_{j_1 \ldots j_m}(x', y')$.

PROOF. It is evident that the translation by $-x$ establishes a bijection between $\mathrm{Path}_j(x,y)$ and $\mathrm{Path}_j(0,y-x)$ that preserves coloration. Moreover if $g(x,y) = g(x',y') = w^c(y-x) = [i_1,\ldots,i_m]$, then take a bijection of the set of coordinate places sending the i_1 places where $y-x$ has component a_1 to the corresponding i_1 places in $y'-x'$ etc. This establishes a bijection preserving coloration between $\mathrm{Path}_j(0,y-x)$ and $\mathrm{Path}_j(0,y'-x')$.

By this proposition we may reformulate our problem as follows.

Problem 2. If a complete weight $\vec{\imath} = [i_1,\ldots,i_m]$ of some $x \in \mathbb{F}^n$ is given, determine the inventories

$$S_{\vec{\imath},j} = \phi(\mathrm{Path}_j(0,x)) = \sum_{|\vec{\jmath}|=j} S_{\vec{\imath},\vec{\jmath}} T^{\vec{\jmath}},$$

where $T = [T_{a_1},\ldots,T_{a_m}]$, $\vec{\jmath} = [j_1,\ldots,j_m]$, $|\vec{\jmath}| = j_1 + \cdots + j_m$ and $T^{\vec{\jmath}} = T_{a_1}^{j_1}\cdots T_{a_m}^{j_m}$. The number $S_{\vec{\imath},\vec{\jmath}}$ counts the paths of length $|\vec{\jmath}| = j$ and color distribution $\vec{\jmath}$ joining 0 to a vertex x of complete weight $\vec{\imath}$.

2 Analysis of the problem by exponential generating power series with coefficients in the ring $\mathbb{Z}[T_{a_1},\ldots,T_{a_m}]$

Definition 2. Let $F_s(j_1,\ldots,j_m)$ be the number of sequences in $\mathbb{F}^*$ containing j_1 elements equal to a_1, j_2 elements equal to a_2, … , j_m elements equal to a_m and whose sum is equal to $s \in \mathbb{F}$. We define the power series $f_s(X)$ by

$$f_s(X) = \sum_{j\geq 0}\left[\sum_{j_1+\cdots+j_m=j} F_s(j_1,\ldots,j_m) T_{a_1}^{j_1}\cdots T_{a_m}^{j_m}\right]\frac{X^j}{j!}.$$

The relationship between these exponential generating power series and our problem follows from classical results on shuffle product or "composé partionnel" [2].

Proposition 2. *If $\vec{\imath} = [i_1,\ldots,i_m]$ is the complete weight of some $x \in \mathbb{F}^n$ and j is a natural number, then $S_{\vec{\imath},j}$ is the coefficient of $X^j/j!$ in the expansion of $f_{a_1}^{i_1}(X)\cdots f_{a_m}^{i_m}(X) f_0^{n-|\vec{\imath}|}(X)$*

PROOF. We have to count the paths of length j joining 0 to x paying attention to the various color distributions of these paths.

In any path and for any i, the contribution of pertinent arrows has to sum up to x_i. Let k be the number of coordinates of x that are equal to s. Now by expressing the generating power series $f_s(X)$ in the more convenient form

$$f_s(X) = \sum_{j\geq 0} \left[\sum_{b_1+\cdots+b_j=s} T_{b_1} \dots T_{b_j} \right] \frac{X^j}{j!} \tag{1}$$

we obtain

$$\begin{aligned} f_s^k(X) &= \sum_{j_1,\dots,j_k} \left[\left(\sum T_{b_{11}} \dots T_{b_{1j_1}} \right) \dots \left(\sum T_{b_{k1}} \dots T_{b_{kj_k}} \right) \right] \frac{X^{j_1} \dots X^{j_k}}{j_1! \dots j_k!} \\ &= \sum_{j\geq 0} \left[\sum T_{b_{11}} \dots T_{b_{1j_1}} \dots T_{b_{k1}} \dots T_{b_{kj_k}} \frac{j!}{j_1! \dots j_k!} \right] \frac{X^j}{j!} \end{aligned}$$

where in the inner sum $b_{11} + \cdots + b_{1j_1} = s, \dots, b_{k1} + \cdots + b_{kj_k} = s$. This corresponds in shuffling the j arrows affecting these k different coordinates in such way that the endpoint of the various paths so obtained is s at those k coordinates.

By multiplying all these powers we obtain the result. □

Remark 1. In fact, the coefficients of the series $f_s(X)$ are

$$F_s(j_1, \dots, j_m) = \begin{cases} \binom{j_1+\cdots+j_m}{j_1,\dots,j_m} & \text{if } s = j_1 a_1 + \cdots + j_m a_m \\ 0 & \text{if not,} \end{cases}$$

giving

$$f_s(X) = \sum_{j\geq 0} \left(\sum_{\substack{j_1 a_1+\cdots+j_m a_m=s \\ j_1+\cdots+j_m=j}} \binom{j}{j_1, \dots, j_m} T_{a_1}^{j_1} \dots T_{a_m}^{j_m} \right) \frac{X^j}{j!}.$$

So, at least in the case when the alphabet $\mathbb{F} = \mathbb{Z}/m\mathbb{Z}$ is the ring of integers modulo m and m is not too big, problem 2 is solved by using proposition 2 and any computer algebra system to write down the filtered sums in f_s and to extract coefficients from a product of power series.

Remark 2. From (1) we deduce trivially

$$\sum_{s\in\mathbb{F}} f_s(X) = \exp\{(\sum_{i=1}^{m} T_{a_i})X\}. \tag{2}$$

Example 1. Take $\mathbb{F} = \{0,1,2\}$, $n = 4$, $\vec{\imath} = [2,1]$. We seek the paths joining $0 = [0,0,0,0]$ to $x = [1,1,2,0]$. We have

$$\begin{aligned}
f_0(X) &= 1 + 2T_1T_2\frac{X^2}{2!} + (T_1^3 + T_2^3)\frac{X^3}{3!} + \cdots \\
f_1(X) &= T_1X + T_2^2\frac{X^2}{2!} + 3T_1^2T_2\frac{X^3}{3!} + \cdots \\
f_2(X) &= T_2X + T_1^2\frac{X^2}{2!} + 3T_1T_2^2\frac{X^3}{3!} + \cdots \\
f_1^2(X) &= T_1^2X^2 + 6T_1T_2^2\frac{X^3}{3!} + (24T_1^3T_2 + 6T_2^4)\frac{X^4}{4!} + \cdots \\
f_1^2f_2f_0(X) &= 6T_1^2T_2\frac{X^3}{3!} + (12T_1^4 + 24T_1T_2^3)\frac{X^4}{4!} + (360T_1^3T_2^2 + 30T_2^5)\frac{x^5}{5!} + \cdots
\end{aligned}$$

In Tables 1, 2 and 3, we give a detailed account of what is going on.

3 A differential equation

We have seen that the series $f_s(X)$ are easily determined in some particular cases but it may be worth noting that the series do satisfy a system of linear differential equations with coefficients in a field of multivariate rational functions and that such a system is solved easily and naturally in Scratchpad. This may have interest in other problems where the series f_s are not so easily determined directly.

We first observe the recurrence

$$F_s(j_1,\ldots,j_m) = \sum_{k=1}^{m} F_{s-a_k}(j_1,\ldots,j_k-1,\ldots,j_m)$$

for all $s \in \mathbb{F}$.

This is because we obtain a sequence σ of sum s containing j_1 times a_1, … , j_m times a_m from a sequence of sum $s - a_k$ containing j_1 times a_1, … ,

$(j_k - 1)$ times a_k, ... just by adding an a_k, and all sequences σ are obtained in this fashion.

In differential terms this gives

$$Df_s(X) = \sum_{k=1}^{m} T_{a_k} f_{s-a_k}(X), \quad s \in \mathbb{F}$$

because the derivative $Df_s(x)$ of the series of Definition 2, defined formally as usual, gives here

$$Df_s(x) = \sum_{j\geq 1} \left[\sum_{j_1+\cdots+j_m=j} f_s(j_1,\ldots,j_m) T_{a_1}^{j_1} \cdots T_{a_m}^{j_m} \right] \frac{X^{j-1}}{(j-1)!}$$

$$= \sum_{j\geq 1} \left[\sum_{j_1+\cdots+j_m=j} \sum_{k=1}^{m} f_{s-a_k}(j_1,\ldots,j_k-1,\ldots,j_m) T_{a_1}^{j_1} \cdots T_{a_m}^{j_m} \right] \frac{X^{j-1}}{(j-1)!}$$

$$= \sum_{j\geq 1} \left[\sum_{k=1}^{m} T_{a_k} \sum_{j_1+\cdots+j_m=j} f_{s-a_k}(j_1,\ldots,j_k-1,\ldots,j_m) T_{a_1}^{j_1} \cdots T_{a_k}^{j_k-1} \cdots T_{a_m}^{j_m} \right] \frac{X^{j-1}}{(j-1)!}$$

$$= \sum_{k=1}^{m} T_{a_k} \sum_{j\geq 0} \left[\sum_{j_1+\cdots+j_m=j} f_{s-a_k}(j_1,\ldots,j_m) T_{a}^{j_1} \cdots T_{a_m}^{j_m} \right] \frac{X^j}{j!}$$

This proves the following result.

Proposition 3. *The vector $[f_0(X), f_{a_1}(X), \ldots, f_{a_m}(X)]$ consisting of the exponential generating power series of Definition 2 is the unique solution of the linear system*

$$Df_s = \sum_{k=1}^{m} T_{a_k} f_{s-a_k}, \quad s \in \mathbb{F} \qquad (**)$$

with initial condition vector $[1, 0, \ldots, 0]$.

Remark 3. We may consider this differential equation as having coefficients in the field $K = \mathbb{Q}(T_{a_1}, \ldots, T_{a_m})$ and the solution we seek has components in the differential ring $K[[X]]$. Thanks to its abstraction capabilities, Scratchpad is able to solve easily and naturally such a problem whereas others computer algebra systems available nowaday seem not.

4 An example in Scratchpad

We give an Axiom interactive session to illustrate the preceding in the particular case where the alphabet $\mathbb{F}$ is the additive group of the ternary field $GF(3)$.

4.1 Solution of the differential equation (**)

Creation of the coefficient field

```
-> k := Fraction MultivariatePolynomial([t1, t2], Integer)
```

Specification of the solution

```
-> s := UnivariateTaylorSeries(k, x, 0$k)
-> sol : List s
```

Specification of the right members of (**)

```
-> (f, g, h) : List s -> s
-> f u == t1*u.4 + t2*u.3
-> g u == t1*u.2 + t2*u.4
-> h u == t1*u.3 + t2*u.2
```

Call to Scratchpad command to solve (**)

```
-> )set expose add constructor UnivariateTaylorSeriesODESolver
-> sol := mpsode([1$k, 0$k, 0$k], [f, g, h])
```

4.2 Determination of the numbers $S_{\vec{\imath},j}$

The user gives the length n (PositiveInteger) and the complete weight $\vec{\imath} = [i_1, i_2]$ (List PositiveInteger).

Calculation of the product power series as in Proposition 2

```
-> series := sol.2**i.1*sol.3**i.2*sol.1**(n-i.1-i.2)
```

Stream of the numbers $S_{\vec{\imath},j}$ for $j \in \mathbb{N}$

```
-> c_i := [factorial(j)*coefficient (series,j) for j in 0..]
```

5 The case where $\mathbb{F}$ is a finite field

When the alphabet $\mathbb{F}$ is the additive group of a finite field, the multiplicative structure of the field may be used to reduce the calculation of the series $f_s(X)$ to only one of them, say $f_1(X)$, which is then determined by an order 2 (scalar) differential equation.
Let α be a primitive element of the finite field $\mathbb{F}$ and let us denote by σ the shift operator defined by

$$\sigma(T) = \sigma(T_{\alpha}^{j_1} \dots T_{\alpha^m}^{j_m}) = T_{\alpha}^{j_m} T_{\alpha^2}^{j_1} \dots T_{\alpha^m}^{j_m-1}.$$

By observing that

$$j_1\alpha + j_2\alpha^2 + \cdots + j_m\alpha^m = 1 \iff j_m\alpha + j_1\alpha^2 + \cdots + j_{m-1}\alpha^m = d$$

we may write the power series $f_{\alpha^i}(X)$ in terms of $f_1(X)$ alone. Indeed, if we denote $f_s(X)$ by

$$f_s(X,T) = \sum_{j\geq 0} \left[\sum_{\substack{j_1\alpha+\cdots+j_m\alpha^m=s \\ j_1+\cdots+j_m=j}} \binom{j}{j_1,\dots,j_m} T_{\alpha}^{j_1} \dots T_{\alpha^m}^{j_m} \right] \frac{X^j}{j!},$$

then we have

$$\begin{aligned} f_\alpha(X,T) &= \sum_{j\geq 0} \left[\sum_{\substack{j_1\alpha+\cdots+j_m\alpha^m=1 \\ j_1+\cdots+j_m=j}} \binom{j}{j_1,\dots,j_m} T_{\alpha}^{j_m} T_{\alpha^2}^{j_1} \dots T_{\alpha^m}^{j_m} \right] \frac{X^j}{j!} \\ &= f_1(X,\sigma(T)) \end{aligned}$$

and, in general,

$$f_{\alpha^i}(X,T) = f_1(X,\sigma^i(T)) \tag{3}$$

for $i = 1,\dots,m$.
The system (**) then gives

$$Df_0(X,T) = \sum_{i=1}^{m} f_{\alpha^i}(X,T)T_{-\alpha^i} = \sum_{i=1}^{m} f_1(X,\sigma^i(T))T_{-\alpha^i}$$

and

$$\begin{aligned} Df_1(X,T) &= f_0(X,T)T_1 + \sum_{i=1}^{m-1} f_{\alpha^i}(X,T)T_{1-\alpha^i} \\ &= f_0(X,T)T_1 + \sum_{i=1}^{m-1} f_1(X,\alpha^i(T))T_{1-\alpha^i}. \end{aligned}$$

Hence

$$D^2 f_1(X,T) = \sum_{i=1}^{m} f_1(X,\sigma^i(T))T_{-\alpha^i}T_1 + \sum_{i=1}^{m-1} D_1 f_1(X,\sigma^i(T))T_{1-\alpha^i} \qquad (***)$$

which, together with the initial conditions $f_1(0,T) = 0$ and $Df_1(0,T) = T_1$, determines the series $f_1(X,T)$.
The above relations (2), (3) and $(***)$ then give the following result.

Proposition 4. *Let the alphabet $\mathbb{F}$ be a finite field, let α be a primitive element of $\mathbb{F}$ and let σ be the shift operator defined by $\sigma(T_\alpha,\ldots,T_{\alpha^m}) = (T_{\alpha^m},T_\alpha,\ldots,T_{\alpha^{m-1}})$. Then*

$$f_1(X) = \sum_{j\geq 0} c_j(T)\frac{X^j}{j!}$$

where the coefficients $c_j(T)$ satisfy the order 2 recurrence

$$c_{j+2}(T) = \sum_{i=1}^{m} c_j(\sigma^i(T))T_1T_{-\alpha^i} + \sum_{i=1}^{m-1} c_{j+1}(\sigma^i(T))T_{1-\alpha^i}$$

*with initial values $c_0(T) = 0$, $c_1(T) = T_{\alpha^m} = T_1$.
Moreover, for $i = 1,\ldots,m-1$*

$$f_{\alpha^i}(X,T) = f_1(X,\sigma^i(T))$$

and

$$f_0(X,T) = \exp\{(\sum_{i=1}^{m} T_{\alpha^i})X\} - \sum_{i=1}^{m} f_1(X,\sigma^i(T)).$$

References

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Aknowledgment. We wish to thank Dr. Richard Jenks from IBM Thomas Watson Research Center at Yorktown Heights who has permitted us to use an earlier version of Scratchpad at Université de Sherbrooke, and Mr. Renaud Rioboo and Mr. Daniel Augot for some interesting conversations about Scratchpad. We are also grateful to a referee for his observations which have permitted us to enhence an earlier version of this paper leading in particular to proposition 4.

type	T_1	T_1	T_2	
0	0	0	0	0
path transitions	1	0	0	0
	0	1	0	0
	0	0	2	0
x	1	1	2	0
number of permutations	6			
coefficients of $X^3/3!$	$6T_1^2T_2$			

Table 1: Paths of length 3

type	T_1	T_1	T_1^2		T_1	T_2^2	T_2		T_2^2	T_1	T_2	
0	0	0	0	0	0	0	0	0	0	0	0	0
path transitions	1	0	0	0	1	0	0	0	2	0	0	0
	0	1	0	0	0	2	0	0	2	0	0	0
	0	0	1	0	0	2	0	0	0	1	0	0
	0	0	1	0	0	0	2	0	0	0	2	0
x	1	1	2	0	1	1	2	0	1	1	2	0
number of permutations	12				12				12			
coefficients of $X^4/4!$	$12T_1^4$				$24T_1T_2^3$							

Table 2: Paths of length 4

type	T_1	T_1	T_2	T_1T_2	T_1	T_1	T_1T_2		T_1	T_2^2	T_1^2	
0	0	0	0	0	0	0	0	0	0	0	0	0
path transitions	1	0	0	0	1	0	0	0	1	0	0	0
	0	1	0	0	0	1	0	0	0	2	0	0
	0	0	2	0	0	0	1	0	0	2	0	0
	0	0	0	1	0	0	2	0	0	0	1	0
	0	0	0	2	0	0	2	0	0	0	1	0
x	1	1	2	0	1	1	2	0	1	1	2	0
number of permutations	120				60				30			
coefficients of $X^5/5!$	$360T_1^3T_2^2$											

	T_2^2	T_1	T_1^2		$T_1^2T_2$	T_1	T_2		T_1	$T_1^2T_2$	T_2	
	0	0	0	0	0	0	0	0	0	0	0	0
	2	0	0	0	1	0	0	0	1	0	0	0
	2	0	0	0	1	0	0	0	0	1	0	0
	0	1	0	0	2	0	0	0	0	1	0	0
	0	0	1	0	0	1	0	0	0	2	0	0
	0	0	1	0	0	0	2	0	0	0	2	0
	1	1	2	0	1	1	2	0	1	1	2	0
	30				60				60			

	T_2^2	T_2^2	T_2	
	0	0	0	0
	2	0	0	0
	2	0	0	0
	0	2	0	0
	0	2	0	0
	0	0	2	0
	1	1	2	0
	30			
	$30T_2^5$			

Table 3: Paths of length 5