Induced Cycle Structures of the Hyperoctahedral Group

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Abstract

The hyperoctahedral group B_n is treated as the automorphism group of the n-dimensional hypercube, denoted Q_n, which is nowadays understood to be a graph on 2^n vertices. It is well-known that B_n can be represented by the group of signed permutations. In other words, any signed permutation induces a permutation on the vertices of Q_n which preserves adjacencies. Moreover, signed permutations also induce a permutation group on the edges of Q_n, denoted H_n. We study the cycle structures of both B_n and H_n. The technique proposed here is to determine the induced cycle structure of a signed permutation by the number of fixed vertices or fixed edges of a signed permutation in the cyclic group generated by a signed permutation of given type. Here we directly define the type of a signed permutation by a double partition based on its signed cycle decomposition. In this way, we obtain explicit formulas for the number of induced cycles on vertices as well as on edges of Q_n of a signed permutation in terms of its type. By further exploring the connection between cycle indices and the structure of fixed points, we obtain the cycle indices of both B_n and H_n. Our formula for the cycle index of B_n is much more natural and considerably simpler than that of Harrison and High. Meanwhile, the cycle structure of H_n seems to have been untouched before, although it is well motivated by nonisomorphic edge colorings of Q_n as well as by the recent interest in symmetries of computer networks.

1 Introduction

The hyperoctahedral group considered in this paper will be understood as the symmetry group of the n-dimensional hypercube, or simply the n-cube. As in [1] we shall choose to treat the n-cube as a graph, usually denoted Q_n. To be more specific, the vertex set of Q_n consists of all the sequences of 0's and 1's of length n and two such sequences are adjacent whenever they differ at exactly one position. Nevertheless, this standpoint is by no means substantially different from that of treating the hypercube as a regular solid in the n-dimensional Euclidean space. The recent surge of interest in symmetry properties of computer networks has led to the investigation of automorphism groups as well as the induced edge automorphism groups of the currently studied network models, including the hypercube. Throughout, we shall use B_n to denote the group of symmetries of the n-cube (graph automorphisms of Q_n in the present context), and H_n to denote the induced permutation group of B_n on the edges of Q_n. Sometimes, the term *line-group* of a graph G is used for the permutation group on the edges of G induced by the automorphism group of G. In this sense, H_n is the line-group of Q_n.

In view of Pólya theory on enumeration under group action, an important feature of an automorphism group is its cycle structure. In particular, the study of the cycle structure of B_n has an interesting history. From the signed permutation representation of B_n, namely, the fact that B_n can be represented by the wreath product of S_n and S_2, Pólya [13] noticed that the number of types of Boolean functions in n variables equals the number of nonisomorphic vertex colorings of the n-cube using two colors. This led to the question of computing the cycle structure of B_n. Although B_n is isomorphic to the wreath product $S_n[S_2]$, which is a permutation group on $2n$ elements whose cycle index can be obtained by those of S_n and S_2 in terms the operation called *plethysm* or Pólya's composition, B_n itself is a much more sophisticated permutation group on 2^n elements which does not seem to possess relatively simple cycle structure. In fact, Pólya [13] computed the cycle indices of B_n up to $n = 4$. This problem got more attention with the advent of the switching circuit theory. The complete solution was first obtained by Slepian [16] based on Young's results on irreducible representations of B_n. Later on, Harrison and High [9] succeeded in obtaining the cycle index of B_n which also leads to a solution to the problem of counting types of Boolean functions. However, the formula of Harrison and High is rather involved. Our method turns out to be more natural and considerably simpler than that of Harrison and High's, moreover, our approach is more effective regarding its applicability to more general situations such as the cycle structure of the line-group H_n of Q_n, a permutation group on $n2^{n-1}$ edges. It seems that the cycle structure of H_n has been untouched in previous research, although it is well motivated by the enumeration of nonisomorphic edge coloring of Q_n as well as by the recent interest in edge symmetries of computer networks.

Our first objective is to obtain the cycle polynomials of both B_n and H_n. As we know in many circumstances, such as counting types of Boolean functions and vertex coloring of the n-cube, we do not really need all the information contained in the cycle index of B_n. Instead, for a permutation group G, sometimes it suffices to have the following polynomial:

$$K(G;x) = \frac{1}{|G|} \sum_k w_k\, x^k ,$$

where w_k is the number of permutations in G with k cycles. Clearly, $K(G;x)$ can be obtained from the cycle index $Z(G;x_1,x_2,\ldots)$ of G by substituting every x_i with x. We shall call $K(G;x)$ the *cycle polynomial* of G. As expected, cycle polynomials would be much easier to compute than cycle indices. Keeping in mind that the signed permutation representation of B_n is considerably easier than B_n itself, one naturally expects that the cycle structure of B_n should follow in some way from that of signed permutations. First, we observe a simple connection between the cycle structure of a permutation and the Burnside Lemma so that counting cycles reduces to counting fixed points. Secondly, by using a recent result of [1], we unexpectedly found that the number of fixed vertices of a symmetry of Q_n can be easily determined by its signed cycle decomposition. The notion of balanced signed cycles defined in [1] turned out to be crucial in our approach. We remark that our method is not only effective for B_n and H_n, but also for other permutations groups induced from wreath product of two permutation groups. It turns out to be satisfying that the notion of double partitions used in the representation theory of B_n naturally arises in the present context, and we can explicitly give the induced cycle structure of any signed permutation in terms of its type (in the form of a double partition).

By further exploring the connection between induced cycle structure and fixed points, we find that for any induced permutation group its cycle structure is determined by the structure of fixed points (the Cycle Structure Lemma). In this way, we achieve our goal of computing the cycle indices of both B_n and H_n, the second objective of this paper.

2 A Cycle Counting Lemma

Let G be a group and S be a finite set. Let Π be a permutation group on S, namely a subgroup of the symmetric group on S. Given isomorphism ρ from G to Π:

$$\rho\colon g \to \pi_g, \quad g \in G, \quad \pi_g \in \Pi ,$$

we usually say that G is a group acting on S in the sense that a element of G acts on S through its image of under the isomorphism ρ. With the isomorphism ρ being understood, we shall simply call Π an induced group from G. Specifically, as far as we are concerned in this paper, G will be the wreath product $S_n[S_2]$, or the group of signed permutations on n elements. The hyperoctahedral group is an induced group of $S_n[S_2]$, which is a permutation group on the vertices of Q_n. Given a signed permutation π the acting role (i.e., the isomorphism ρ as above) of π on Q_n is explained as permuting the sequence of 0's and 1's and then taking complements in certain positions, the detailed definition will be given in the next section. Another induced group is the edge automorphism group of $S_n[S_n]$. By definition, an automorphism on a graph induces a permutation on the edges of the graph. Thus, $S_n[S_2]$ also induces a permutation group on the edges of Q_n. Our objective is to consider the cycle structures of the above mentioned induced permutation groups on the vertices and edges of Q_n.

Given an element g in a group G, suppose it induces a permutation on S. By the induced cycle structure of g we mean the cycle structure of the induced permutation of g on S. We are going to use the Burnside Lemma to compute the number of cycles of an induced permutation of g. To this end, let's recall some basic terminology related to the Burnside Lemma. Given two elements s_1 and s_2 in S, we say s_1 is *equivalent* to s_2, denoted $s_1 \sim s_2$, if there exists an element $g \in G$ such that

$$\pi_g s_1 = s_2 .$$

Then it is easy to verify that $\sim$ is an equivalence relation on S. For any $g \in G$, we denote by $\psi(g)$ the number of elements $s \in S$ such that $\pi_g s = s$, namely the number of elements fixed by g. Then the Burnside Lemma states that the number of equivalence classes under $\sim$ equals

$$\frac{1}{|G|} \sum_{g \in G} \psi(g) .$$

Using the Burnside Lemma, we may compute the number of cycles of an induced permutation in terms of the number of fixed points of the induced permutation.

Lemma 2.1 (Cycle Counting Lemma) *Let G be a group acting on S, and $g \in G$. Then the number of cycles of the induced permutation π_g of g equals*

$$\frac{1}{o(g)} \sum_{\sigma \in (g)} \psi(\sigma)\,,$$

where $o(g)$ is the order of g in G and $\psi(\sigma)$ is the number of elements $s \in S$ fixed by σ, namely $\pi_\sigma d = d$.

Proof. We simply write π for π_g. It is easy to see that two elements s_1, $s_2 \in S$ are in the same cycle in the decomposition of π if and only if there exists a permutation $\sigma = \pi^i$ for some i such that $\sigma(s_1) = s_2$. Therefore, the number of cycles of π is the same as the number of equivalence classes of D under the permutation group $(\pi) = \{e,\ \pi,\ \pi^2,\ \ldots\}$. Clearly, (π) is finite. By the Burnside Lemma, we have

$$\frac{1}{o(\pi)} \sum_{\varphi \in (\pi)} \psi(\varphi)\,,$$

where $o(\pi)$ is the order of π and $\psi(\varphi)$ is the number of elements $d \in D$ fixed by φ, namely $\varphi d = d$. Since (g) is isomorphic to (π), we have $o(g) = o(\pi)$. This completes the proof. ∎

3 The Cycle Polynomial of B_n

We first recall some definitions from [1]. For any positive integer n, we shall use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. We may represent an element $w \in B_n$ by a *signed permutation* of $[n]$, i.e., a permutation of $[n]$ with a $+$ or $-$ sign attached to each element $1, 2, \cdots, n$. For simplicity of notation we omit the $+$ sign in examples. Thus $(\overset{+}{2}\ \overset{+}{4}\ \bar{5})\ (\bar{3})\ (\overset{+}{1}\ \bar{6})$ or $(2\ 4\ \bar{5})\ (\bar{3})\ (1\ \bar{6})$ represents an element of B_6 with underlying permutation (2 4 5) (3) (1 6) (written in cycle notation). We call each a representation of an element of B_n a *signed cycle decomposition.* A signed permutation w acts on a vertex $u_1 u_2 \cdots u_n$ of Q_n by the rule

$$w(u_1 u_2 \cdots u_n) = \hat{u}_{\pi(1)} \hat{u}_{\pi(2)} \cdots \hat{u}_{\pi(n)}\,,$$

where π is the underlying permutation of w and

$$\hat{u}_{\pi(j)} = \begin{cases} u_{\pi(j)}\,, & \text{if } j \text{ has the sign } +\,, \\ 1 - u_{\pi(j)}\,, & \text{if } j \text{ has the sign } - \quad. \end{cases} \tag{3.1}$$

Thus the action of π on $u = u_1 u_2 \cdots u_n$ can be understood as the action of permuting u into $u_{\pi(1)} u_{\pi(2)} \cdots u_{\pi(n)}$, and then taking complements at positions where π has minus signs. If we define the *sign vector* $(s_1, s_2, \ldots, s_n)$ of a signed permutation w as

$$s_j = \begin{cases} 0\,, & \text{if } j \text{ has the sign } +\,, \\ 1\,, & \text{if } j \text{ has the sign } -, \end{cases}$$

Then (3.1) can be rewritten as

$$\widehat{u}_{\pi(j)} \equiv u_{\pi(j)} + s_j \pmod 2 . \tag{3.2}$$

To make the above definition a little clearer, we may let $v_1 v_2 \cdots v_n = w(u_1 u_2 \cdots u_n)$, then (3.2) becomes

$$v_{\pi(j)} \equiv u_j + s_{\pi(j)} \pmod 2 . \tag{3.3}$$

For two symmetries π and σ of Q_n, we define their product by

$$(\pi\, \sigma)(u_1 u_2 \cdots u_n) = \sigma(\pi(u_1 u_2 \cdots u_n)) ,$$

where $u_1 u_2 \cdots u_n$ is any vertex of Q_n. Note that the above convention is consistent with the usual definition of product of ordinary permutations, i.e., for two permutations π and σ on $[n]$, $\pi\sigma$ is defined by $(\pi\sigma)(i) = \sigma(\pi(i))$ for any i. If no confusion arises, we shall identify a signed permutation π with its underlying permutation when applied to an element in $[n]$ instead a vertex of Q_n.

Proposition 3.1 *Let π_1 and σ_1 be two signed permutations on $[n]$ with underlying permutations π and σ and sign vectors $(s_1, s_2, \cdots, s_n)$ and $(t_1, t_2, \ldots, t_n)$. Then the signed permutation $\pi_1\sigma_1$ has underlying permutation $\pi\sigma$ and sign vector*

$$\left(t_1 + s_{\sigma^{-1}(1)},\ t_2 + s_{\sigma^{-1}(2)},\ \ldots,\ t_n + s_{\sigma^{-1}(n)}\right) \pmod 2 .$$

Proof. Let $u_1 u_2 \cdots u_n$ be any vertex of Q_n and let $v_1 v_2 \cdots v_n = \pi_1(u_1 u_2 \cdots u_n)$, By (3.2), we have

$$v_{\pi(i)} \equiv u_i + s_{\pi(i)} \pmod 2 . \tag{3.4}$$

Let $w_1 w_2 \cdots w_n = \sigma_1(v_1 v_2 \cdots v_n)$. Hence for any j,

$$w_{\sigma(j)} \equiv v_j + t_{\sigma(j)} \pmod 2 . \tag{3.5}$$

Substituting j with $\pi(i)$ in (3.5), we get

$$\begin{aligned} w_{\sigma(\pi(i))} &= w_{(\pi\sigma)(i)} \\ &\equiv v_{\pi(i)} + t_{\sigma(\pi(i))} \pmod 2 \\ &\equiv v_{\pi(i)} + t_{(\pi\sigma)(i)} \pmod 2 . \end{aligned}$$

From (3.4) it follows that

$$w_{(\pi\sigma)(i)} \equiv u_i + s_{\pi(i)} + t_{(\pi\sigma)(i)} \pmod 2 . \tag{3.6}$$

Let $r_i = s_{\sigma^{-1}(i)}$. Then we have,

$$r_{(\pi\sigma)(i)} = s_{\sigma^{-1}((\pi\sigma)(i))} = s_{(\pi\sigma\sigma^{-1})(i)} = s_{\pi(i)} .$$

Therefore (3.6) can be written as

$$w_{(\pi\sigma)(i)} \equiv u_i + r_{(\pi\sigma)(i)} + t_{(\pi\sigma)(i)} \pmod 2 . \tag{3.7}$$

Since $w_1 w_2 \cdots w_n = (\pi_1\sigma_1)(u_1 u_2 \cdots u_n)$, this implies that $\pi_1\sigma_1$ has underlying permutation $\pi\sigma$ and sign vector $(r_1 + t_1,\ \ldots,\ r_n + s_n)$. This completes the proof. ∎

The following corollary will be used later.

Corollary 3.2 *Let π_1 be a signed permutation with underlying permutation π and sign vector $(s_1, s_2, \ldots, s_n)$. Let $\theta = \pi^{-1}$. Then π_1^k has underlying permutation π^k and sign vector*

$$\left(s_1 + s_{\theta(1)} + \cdots + s_{\theta^{k-1}(1)}, \ \ldots, \ s_n + s_{\theta(n)} + \cdots + s_{\theta^{k-1}(n)}\right) \pmod 2 . \qquad (3.8)$$

Proof. We use induction on k. The assertion is trivial for $k = 1$. Suppose it is true for k. Let $\sigma_1 = \pi_1^k$. Then σ_1 has underlying permutation π^k and sign vector (3.8). Let $(t_1, t_2, \ldots, t_n)$ be the sign vector of $\sigma_1 = \pi_1^k$, and $\sigma = \pi^k$ be the underlying permutation of σ_1. By Proposition 3.1, π_1^{k+1} has underlying permutation π^{k+1} and sign vector

$$\left(t_1 + s_{\sigma^{-1}(1)}, \ t_2 + s_{\sigma^{-1}(2)}, \ \ldots, \ t_n + s_{\sigma^{-1}(n)}\right) .$$

Clearly, we have

$$t_i + s_{\sigma^{-1}(i)} = \left(s_i + s_{\theta(i)} + \cdots + s_{\theta^{k-1}(i)}\right) + s_{\pi^{-k}(i)} = s_i + s_{\theta(i)} + \cdots + s_{\theta^k(i)} .$$

The proof is thus complete by induction. ∎

A *double partition* (λ, μ) of an integer n, denoted $(\lambda, \mu) \vdash n$, is a an ordered pair (λ, μ) of partitions such that $|\lambda| + |\mu| = n$ where $|\lambda|$ denotes the sums of parts of λ. A double partition (λ, μ) can also be denoted by $(\lambda, \mu) \vdash (p, q)$, if $|\lambda| = p$ and $|\mu| = q$. The number of parts of λ will be denoted by $\ell(\lambda)$. Given two partitions λ and μ, we shall define $\lambda \cup \mu$ to be the partition obtained by joining the parts of λ and μ together. For example, $2\,2\,1 \cup 3\,2\,1 = 3\,2\,2\,2\,1$. The notion of a double partition is closely relation to the that of balanced cycles introduced in [1]. A signed cycles is said to be *balanced* if it contains an even number of minus signs; otherwise, it is called *unbalanced*. Moreover, a signed permutation is said to be *balanced* or if every cycle is balanced in its cycle decomposition, and it is said to be *totally unbalanced* if every cycle is unbalanced in its cycle decomposition. Given a signed permutation π, the cycle structure of π is defined by a double partition (λ, μ) such that λ is the cycle structure of balanced cycles in the signed decomposition of π, and μ is the cycle structure of unbalanced cycles in the signed decomposition of π. For example the type of the signed permutation $(3\ \overline{7}\ 4)(1\,5\ \overline{6}\ \overline{2})(8\,10)(\ \overline{9})$ is $(2\,4,\ 1\,3)$. From the representation of B_n it is known that irreducible representations of B_n can be indexed by double partitions. For a partition $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}$ of n, i.e., the number i occurs λ_i times in λ for any i, we shall use $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ to denote the number of permutations on $[n]$ of type λ. It is well-known that

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \frac{n!}{1^{\lambda_1}\,\lambda_1!\,2^{\lambda_2}\,\lambda_2!\,\cdots} .$$

Given a double partition $(\lambda, \mu) \vdash (p, q)$ of n, it is not difficult to show that the number of signed permutations of type (λ, μ) equals

$$\binom{n}{p} \begin{bmatrix} p \\ \lambda \end{bmatrix} \begin{bmatrix} q \\ \mu \end{bmatrix} 2^{n-\ell(\lambda)-\ell(\mu)} . \qquad (3.9)$$

Suppose $S \cup T$ is a disjoint union of $[n]$ such that $|S| = p$ and $|T| = q$. Consider all balanced permutations π on S of type λ. Given an underlying cycle of length m, there are 2^{m-1} ways to form a balanced cycle by attaching signs to each element in the underlying cycle. Thus, given an underlying permutation on S of type λ, we can form $2^{p-\ell(\lambda)}$ balanced permutations the same type. A similar argument shows that given any underlying permutation on T of type μ, we may form $2^{q-\ell(\mu)}$ totally unbalanced permutations of the same type. Combining these two arguments, we obtain (3.9).

The following Lemma gives the parity of the number of minus signs in each cycle of the signed permutation π^k, where the underlying permutation of π is a cycle.

Lemma 3.3 (Cycle Splitting Lemma) *Let π be a signed permutation with underlying permutation is a cycle of length n. Suppose π has Δ minus signs. Then π^k can be decomposed into (k,n) signed cycles with each of length $n/(k,n)$. Moreover, the number of minus signs in each signed cycle of π^k is congruent to $k/(k,n)\,\Delta$ modulo 2.*

Proof. Without loss of generality, we may assume that π has underlying permutation $C = (1\,2\,\cdots\,n)$. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ be the sign vector of π; it is known that C^k can be decomposed into (k,n) cycles with each of length $n/(k,n)$. Thus, the underlying cycle decomposition of π^k also has (k,n) cycles with each having length $n/(k,n)$. Let $d = (k,n)$, in general, a cycle of C^k containing the element i has the following form:

$$\begin{array}{ccc} i & \longrightarrow & i+k \\ i+k & \longrightarrow & i+2k \\ & \vdots & \\ i+(n/d-1)k & \longrightarrow & i\,, \end{array}$$

here the numbers in the above diagram are taken modulo n. Let $(\theta_1, \theta_2, \ldots, \theta_n)$ be the sign vector of π^k. Since $C(j) \equiv j+1 \pmod n$, we have $C^k(j) \equiv j+k \pmod n$. Since $C^{-1}(i) \equiv i-1 \pmod 2$, applying Corollary 3.2, it follows that

$$\theta_j \equiv \delta_i + \delta_{i-1} + \cdots + \delta_{i-k+1} \pmod 2\,.$$

The number of minus signs contained in the above cycle equals $\theta_i + \theta_{k+i} + \cdots + \theta_{(n/d-1)k+i}$. Then we have

$$\begin{aligned} \sum_{j=0}^{n/d-1} \theta_{jk+i} &\equiv \sum_{j=0}^{n/d-1} \sum_{l=0}^{k-1} \delta_{jk+i-l} \pmod 2 \\ &\equiv (\delta_i + \delta_{i-1} + \cdots + \delta_{i-k+1}) \\ &\quad +(\delta_{i+k} + \delta_{i+k-1} + \cdots + \delta_{i+1}) + \cdots \\ &\quad +(\delta_{i+(n/d-1)k} + \delta_{i+(n/d-1)k-1} + \cdots + \delta_{i+(n/d-2)k+1})\,. \end{aligned}$$

Rearranging the summands in the above identity, we obtain

$$(\delta_{i-k+1}+\delta_{i-k+2}+\cdots+\delta_i)+(\delta_{i+1}+\delta_{i+2}+\cdots+\delta_{i+k})+\cdots+(\delta_{i+(n/d-2)k+1}+\cdots+\delta_{i+(n/d-1)k})\,.$$

Note that $(n/d)\,k \equiv 0 \pmod n$. Thus $i+(n/d-1)k$ and $i-k+1$ can be regarded as consecutive numbers (mod n) so that all the above summands can be arranged on a circle of length $(n/d)k$. Since all the indices of δ in the above summation are taken modulo n, the above sum can be further simplified to

$$\begin{aligned} & \delta_1 + \delta_2 + \cdots + \delta_{(n/d)\,k} \\ = \; & \delta_1 + \delta_2 + \cdots + \delta_{(k/d)\,n} \\ = \; & (k/d)\,(\delta_1 + \delta_2 + \cdots + \delta_n) \\ = \; & (k/d)\,\Delta\,. \end{aligned}$$

Thus the number of signs in each cycle of π^k is congruent to $(k/d)\,\Delta$ modulo 2. ∎

By the above Lemma, it can be seen that if π is a balanced cycle, then π^k is balanced for any k; and that if π is totally unbalanced, then π^k is balanced whenever $k/(k,n)$ is even or otherwise π^k is totally unbalanced. Furthermore, the Cycle Splitting Lemma can be used to determine the cycle structure of π^k based on the cycle structure of π.

Lemma 3.4 *Let π be an unbalanced cycle of length n. Let k be a positive integer. Now write n and k in the form $n = 2^i s$ and $k = 2^j t$ where s and t are odd. Then π^k is balanced if and only if $j > i$.*

Proof. Since $n = 2^i s$, $k = 2^j t$, and s and t are odd, we have

$$\frac{k}{(k,n)} = \frac{2^j t}{(2^i s,\ 2^j t)} = \frac{2^j}{2^{\min(i,\,j)}} \cdot \frac{t}{(s,\ t)}\,.$$

Then it is easy to see that $k/(k,n)$ is even if and only if $j > i$. By the Cycle Splitting Lemma, it follows that π^k is balanced if and only if $k/(k,n)$ is even. This completes the proof. ∎

We now recall a result from [1] concerning the number of fixed vertices of a symmetry of Q_n. This result altogether with Lemmas 3.3 and 3.4 will be sufficient to give the cycle polynomial of B_n.

Proposition 3.5 *([1]) Let π be a symmetry of Q_n represented by a signed permutation. If π is balanced, then it has 2^k fixed vertices where k is the number of balanced cycles of π; otherwise π has no fixed vertex.*

To describe the main result of this section, we need the following notation. Let λ be a partition of n and π be a permutation on $[n]$ of type λ. We shall use $C_\lambda(x)$ to denote the cycle polynomial of the cyclic group $\langle\pi\rangle$, and we shall call it the *cyclic polynomial* of λ. Clearly, such a definition does not depend on the choice of the permutation π. For a permutation π of type λ, it is easy to see that the order of the π equals $[\lambda]$, where $[\lambda]$ stands for the least common multiple of the components of λ. Let $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}$, for any k, the cycle structure of π^k, denoted λ^k, is given by

$$\lambda^k = \prod_i \left[i/(i,k)\right]^{(i,k)\,\lambda_i}\,. \tag{3.10}$$

As a result, the number of cycles in π^k equals

$$\ell(\lambda^k) = \sum_i (i,k)\,\lambda_i\,. \tag{3.11}$$

Thus the cyclic polynomial of λ is given by

$$C_\lambda(x) \;=\; \frac{1}{[\lambda]} \sum_{k=0}^{[\lambda]} x^{\sum_{i=1}^{n}(i,k)\,\lambda_i}\,. \tag{3.12}$$

We are now are ready to present the main result of this section.

Theorem 3.6 *Let (λ,μ) be a double partition and let i be the maximum number such that 2^i is a factor of some part of μ. Set $r = 2^{i+1}$ if $\mu \neq \emptyset$ otherwise set $r = 1$. Suppose π is a signed permutation of type (λ,μ). Then the number of cycles of π when acting on Q_n equals*

$$\frac{1}{r}\,C_{\lambda^r\cup\mu^r}(2) = \frac{1}{[\lambda^r,\mu^r]} \sum_{k=1}^{[\lambda^r,\mu^r]} 2^{\ell(\lambda^{rk}\cup\mu^{rk})}\,.$$

Proof. By Lemma 2.1, the number of reduced cycles of π on Q_n is determined by the number of fixed vertices of the signed permutations of π^k. It follows from Proposition 3.5 that π^k does not have any fixed vertex if π^k is not balanced. To make π^k balanced, by Lemma 3.4, k has to contain the factor r; otherwise there exists an unbalanced cycle θ of π such that r does not divide the length of θ, it follows that θ^k totally unbalanced. In other words, π^k has no fixed vertex unless $\pi^k \in (\pi^r)$. Clearly, π^r is a balanced permutation of type $\lambda^r \cup \mu^r$. Suppose π is of order m. Since the identity permutation is balanced, it follows that m must contain the factor r. Since π^r is balanced, the order of (π^r) is just the order of an ordinary permutation of type $\lambda^r \cup \mu^r$, which is $[\lambda^r,\mu^r]$. Therefore, the order of π equals $m = r\,[\lambda^r,\mu^r]$. By Lemma 2.1, it follows that the number of induced cycles of π on the vertices of Q_n equals

$$\frac{1}{r\,[\lambda^r,\mu^r]} \sum_k 2^{\text{the number of cycles of } \pi^{rk}} \;=\; \frac{1}{r}\,C_{\lambda^r\cup\mu^r}(2)\,. \qquad \blacksquare$$

Corollary 3.7 *The cycle polynomial of B_n is given by*

$$\frac{1}{2^n n!} \sum_{p+q=n} \binom{n}{p} \sum_{(\lambda,\mu)\vdash(p,q)} \begin{bmatrix} p \\ \lambda \end{bmatrix} \begin{bmatrix} q \\ \mu \end{bmatrix} 2^{n-\ell(\lambda)-\ell(\mu)}\, x^{C_{\lambda^r\cup\mu^r}(2)/r}\,,$$

where r is given as in Theorem 3.6.

By Pólya's theorem, the number of nonisomorphic vertex colorings of Q_n using m colors equals the cycle polynomials of B_n evaluated at $x = m$. In particular, for $m = 2$ it yields the number of types of Boolean functions in n variables.

4 The Cycle Polynomial of H_n

In this section, we shall restrict ourselves to induced permutations of signed permutations on the edges of Q_n. In a similar vein of the preceding section, one expects that the number of cycles in the induced permutation is dependent only on the type of the original signed permutation. Thus, the aim of this section is to compute the number of induced cycles (i.e., the number of cycles of the induced permutations) of a signed permutation of type (λ, μ). To this end, we first consider the number of fixed edges of a signed permutation of given type, again, a signed permutation is considered to act on edges of Q_n through its induced permutation. Now we need the following result from [1]: let π be a signed permutation acting on the edges on Q_n, then π has a fixed edge if and only if π is balanced and contains a 1-cycle or π contains a unbalanced 1-cycle and all the other cycles are balanced, i.e., π is of type $(\lambda, 1)$, where $\lambda \vdash n-1$. Using this result, we may derive the number of fixed edges of a signed permutation of given type.

Proposition 4.1 *Let π be a signed permutation acting on the edges of Q_n. If π is balanced and of type λ, then it has $\lambda_1 2^{\ell(\lambda)-1}$ fixed edges. If π is of type $(\lambda, 1)$, then it has $2^{\ell(\lambda)}$ fixed edges.*

Proof. We first consider the case when π is balanced. If $\lambda_1 = 0$, i.e., π has no 1-cycle, then it has no fixed either. So we may assume that $\lambda_1 \geq 1$. As in [1], an edge of Q_n is represented by a sequence of $n-1$ 0's or 1's with one occurrence of the symbol $*$. For example, 00101*10 denotes the edge joining the vertices 00101010 and 00101110. Treating π as a symmetry on the vertices of Q_n, it then fixes an edge, $a_1 \cdots a_{i-1} * a_{i+1} \cdots a_n$, if and only if π contains the 1-cycle (i) (by the separation argument in [1]). In such a case, $a_1 \cdots a_{i-1}\, a_{i+1} \cdots a_n$ becomes a fixed vertex for the signed permutation π' obtained from π by removing the cycle (i). Thus by Proposition 3.5, there are $2^{\ell(\pi')} = 2^{\ell(\pi)-1}$ choices for the subsequence $a_1 \cdots a_{i-1}\, a_{i+1} \cdots a_n$. Moreover, for any 1-cycle (i) of π we may place $*$ in the ith position of the above edge representation. Thus, there are λ_1 choices for the position of $*$, so that the total number of fixed edges of π equals $\lambda_1 2^{\ell(\pi)-1}$.

Let us now consider the case when π is of type $(\lambda, 1)$, that is, π contains only one unbalanced 1-cycle, say, $(\,\bar{i}\,)$, and all other cycles of π are balanced. Then the separation argument of [1] shows that the symbol $*$ must appear at the ith position in the above representation of fixed edges of π. Thus, a fixed edge of π is of the form $a_1 \cdots a_{i-1} * a_{i+1} \cdots a_n$, and the number of choices for the subsequence $a_1 \cdots a_{i-1}\, a_{i+1} \cdots a_n$ equals $2^{\ell(\lambda)}$, which makes the number of fixed edges of π. ∎

Analogous to the strategy of computing the cycle polynomial of B_n, here we need to count the number of induced cycles on edges of Q_n of a signed permutation of given type. Because of the appearance of two cases in the above Proposition 4.1, we shall proceed according to these two cases. For a partition α, we shall use $\beta_j(\alpha)$ to denote the number occurrences of j in α. Let $\lambda = 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n}$. From (3.10) it follows that

$$\beta_1(\lambda^k) = \sum_{i \mid k} i\, \lambda_i\,. \tag{4.1}$$

We now give the main result of this section which leads to the cycle polynomial of the induced edge automorphism group H_n of the n-cube.

Theorem 4.2 *Suppose π is a signed permutation of type $(\lambda, 1)$, then the number of induced cycles of π equals*

$$\frac{1}{2\,[\lambda^2]}\left(\sum_{k=1}^{r\,[\lambda]} 2^{\ell(\lambda^k)} + \sum_{2k\le 2\,[\lambda^2]} \beta_1(\lambda^{2k})2^{\ell(\lambda^{2k})}\right). \tag{4.2}$$

If π is a signed permutation of type (λ, μ) where $\mu \neq 1$, then the number of induced cycles of π is given by

$$\frac{1}{r\,[\gamma]} \sum_{k=1}^{[\gamma]} \beta_1(\gamma^k)\,2^{\ell(\gamma^k)-1}, \tag{4.3}$$

where r is defined as in Theorem 3.6 and $\gamma = \lambda^r \cup \mu^r$.

Proof. We first prove (4.2). Suppose π is of type $(\lambda, 1)$. Recall that for π, the number r equals 2; the order of π is thus $2\,[\lambda^2]$. If k is odd, then π^k is of type $(\lambda^k, 1)$. By Proposition 4.1, the number of edges fixed by π^k equals $2^{\ell(\lambda^k)}$. If k is even, then π^k is balanced with type $\lambda^k \cup 1$. Then Proposition 4.1 shows that the number of fixed edges of π^k equals

$$\beta_1(\lambda^k \cup 1)\,2^{\ell(\lambda^k\cup 1)-1} = \left(\beta_1(\lambda^k)+1\right)\,2^{\ell(\lambda^k)} = 2^{\ell(\lambda^k)} + \beta_1(\lambda^k)\,2^{\ell(\lambda^k)}.$$

Hence by Lemma 2.1, the number of induced cycles of π adds up to (4.2).

Next we prove (4.3). Suppose π is of type (λ, μ) where $\mu \neq 1$. We claim that π^k does not have any fixed edges unless π^k is balanced. We may assume that $\mu \neq \emptyset$ otherwise the claim holds trivially. Suppose π^k is not balanced, this implies that there exists an unbalanced cycle θ of π such that θ^k contains an unbalanced cycle. By the Cycle Splitting Lemma, every cycle of θ^k must be unbalanced. Let i be the length of the cycle θ, then θ^k contains (i,k) cycles with each having length $i/(i,k)$. If $i > 1$, then either $(i,k) > 1$ or $i/(i,k) > 1$, that is, θ^k contains either an unbalanced cycle of length at least 2 or at least two unbalanced 1-cycles. By Proposition 4.1, π^k cannot have any fixed edge. We now consider the case when θ^k is balanced for every unbalanced cycle θ of π with length at least two. If such a cycle θ exists, then k must be even. Thus π must be balanced because for any signed 1-cycle σ, σ^k is balanced whenever k is even. Finally, we are left to the case when π does not have any unbalanced cycles of length at least two. Since $\mu \neq 1$, π has at least two unbalanced 1-cycles. If π has at least two unbalanced 1-cycles, then for any odd number k, π^k has the same number of unbalanced 1-cycles as π, which implies that π^k has no fixed edge and for any even number k, π^k becomes balanced. Thus, we arrive at the conclusion that π^k does not have any fixed edge unless π^k is balanced. As we showed in the proof of Theorem 3.6, π^k is balanced if and only if $\pi^k \in (\pi^r)$. By Proposition 4.1, π^{rk} has $\beta_1(\gamma^k)2^{\ell(\gamma^k)-1}$ fixed edges. Since π is known of order $r\,[\gamma]$ and γ has order $[\gamma]$, by Lemma 2.1 we obtain (4.3). ∎

Similar to Corollary 3.7, the preceding Theorem actually gives the cycle polynomial of H_n by summing over all double partitions of n. Let $K(H_n; x)$ be the cycle polynomial of H_n, then by Polya's theorem, $K(H_n; m)$ gives the number of nonisomorphic colorings on the edges of Q_n using m colors.

5 A Cycle Structure Lemma

In this section, we propose a method to compute the cycle index of a permutation group G in terms of the number of fixed points of an element in G. We will first give a general formula and then apply it to the hyperoctahedral group B_n and its induced edge automorphism group H_n. In view of Pólya's theorem, using the cycle index of a permutation group one obtains the generating function of nonisomorphic coloring patterns, which is more detailed than just the number of nonisomorphic colorings. For this purpose, sometimes it is necessary to know the cycle index of a permutation group. We shall achieve this goal for both B_n and H_n. The cycle index of B_n has been computed by Harrison and High [9] in a rather complicated way, but our formula is much more natural and clearer. The cycle structure of H_n seems to have been untouched before, although it is well motivated by the nonisomorphic edge colorings of Q_n as well as by the recent interesting in edge symmetries of computer networks, and our formulas for H_n are believed to be new.

Let G be a permutation group on a finite set S. As in Lemma 2.1, for any $\pi \in S$ we shall use $\psi(\pi)$ to denote the number of elements of S that are fixed by π. Then the following Lemma establishes a connection between the cycle structure of π and the number of fixed points of a permutation in G.

Lemma 5.1 (Cycle Structure Lemma) *Let π be a permutation on S, then the number of k-cycles of π is given by*

$$\sum_{i|k} \mu(k/i)\, \psi(\pi^i)\,, \tag{5.1}$$

where μ is the classical Möbius function.

Proof. Let $f_k(\pi)$ denote $\psi(\pi^k)$ and $g_k(\pi)$ denote the number of elements x of S such x is fixed by π^k but not by any permutation π^r for $r < k$. We are going to establish the following relation:

$$f_k(\pi) = \sum_{i|k} g_i(\pi)\,. \tag{5.2}$$

Let x be a fixed point of π^k and i be the small number such that x is fixed by π^i. We claim that $i|k$; otherwise we assume $k = q\,i + r$ where $0 < r < i$. Since x is fixed by both π^k and π^i, it follows that $\pi^{qi}(x) = x$ and

$$\pi^r(x) = \pi^r(\pi^{qi}(x)) = \pi^k(x) = x\,,$$

which contradicts the definition of i. Thus, we have shown that $i|k$, which yields (5.2). From (5.2) and the Möbius inversion, we obtain

$$g_k(\pi) = \sum_{i|k} \mu(k/i)\, f_i(\pi)\,. \tag{5.3}$$

What remains to be proved is that $g_k(\pi)$ equals the number of k cycles of π. It is not difficult to see that if x is in a k-cycle of π, then it must be fixed by π^k but not by any π^r for $r < k$. Therefore, k must be the smallest number such that π^k fixes x. Conversely, if k is the smallest number such that π^k fixes x, then x must be in a k-cycle of π. The proof is thus complete. ∎

As expected, the purpose of the remainder of this paper is to obtain the induced cycle structure of a signed permutation of given type. In accordance with the above Lemma, this problem reduces to the computation of the number of fixed vertices and fixed edges of the signed permutation π^k, given the type of π. At this point, we have already encountered these numbers in computing the cycle polynomials of B_n and H_n. In the proofs of Theorem 3.6 and Theorem 4.2 we have actually shown the following two propositions. Recall that for a double partition (λ, μ), the number r is determined by μ as in Theorem 3.6.

Proposition 5.2 *Let π be a signed permutation of type (λ, μ), then π^k has $2^{\ell(\lambda^k \cup \mu^k)}$ fixed vertices if $r|k$; otherwise π^k has not fixed vertex.*

Proposition 5.3 *Suppose π is a signed permutation of type $(\lambda, 1)$, then π^k has $2^{\ell(\lambda^k)}$ fixed edges if k is odd; otherwise π^k has $\left(\beta_1(\lambda^k) + 1\right) 2^{\ell(\lambda^k)}$ fixed edges. If π is a signed permutation of type (λ, μ) where $\mu \neq 1$, then the number of fixed edges of π^k is given by $\beta_1(\lambda^k \cup \mu^k) 2^{\ell(\lambda^k \cup \mu^k)-1}$ if $r|k$; otherwise π^k has no fixed edges.*

Finally, we note that the maximum length of an induced cycle of a signed permutation π is bounded by the order of π, which has been shown to be $r\,[\lambda^r \cup \mu^r]$. Since the number of signed permutations of a given type is determined in (3.9), like Corollary 3.7, the cycle indices of B_n and H_n can be obtained by summing the cycle structures of signed permutations π of type (λ, μ) over all double partitions (λ, μ).

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