DECOMPOSITION OF CERTAIN PRODUCTS OF CONJUGACY CLASSES OF S_n

Alain Goupil[1]

Department of Mathematics,
University of California at San Diego
San Diego, California 92093

ABSTRACT. Using the character theory of the symmetric group S_n, we study the decomposition of the product of two conjugacy classes $K_\lambda * K_\mu$ in the basis of conjugacy classes. This product takes place in the group algebra of the symmetric group and the coefficient of the class K_γ in the decomposition, called structure constant, is a positive integer that counts the number of ways of writing a given permutation of type γ as product of two permutations of type λ and μ. In this paper, we present new formulas for the decomposition of the products $K_{1^r n-r} * K_{1^s n-s}$, $K_{(r,n-r)} * K_{(s,n-s)}$ and $K_{1^r n-r} * K_{(s,n-s)}$ over a restricted set of conjugacy classes K_γ. These formulas generalize the formula for the decomposition of the product of the class of full cycles with itself $K_{(n)} * K_{(n)}$.

Introduction

Let $\lambda = (1 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k)$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$, be a partition of n denoted $\lambda \vdash n$. We write also $\lambda = 1^{m_1} 2^{m_2} \ldots n^{m_n}$ when m_i parts of λ are equal to i $(i = 1, \ldots, n)$. Consider the conjugacy classes C_λ, C_μ of permutations in the symmetric group S_n whose cycle type is given by the partitions λ and μ of n. Let $Q[S_n]$ be the group algebra of the symmetric group over the field Q of rationnals and let $C[S_n]$ be the center of this group algebra. Let K_λ be the element of $C[S_n]$ defined by

$$K_\lambda = \sum_{\sigma \in S_n} \chi(\sigma \in C_\lambda)\sigma \qquad I.1$$

where χ is the usual characteristic function. We shall also call K_λ a conjugacy class. A product $K_\lambda * K_\mu$ of conjugacy classes in $C[S_n]$ can always be decomposed in the basis of conjugacy classes with non negative integer coefficients:

$$K_\lambda * K_\mu = \sum_{\gamma \vdash n} c^\gamma_{\lambda\mu} K_\gamma \qquad I.2$$

The structure constants $c^\gamma_{\lambda\mu}$ that we will also write $c^\gamma_{\lambda\mu} = (\gamma, \lambda, \mu)$ count the number of ways of writing a permutation of type γ as a product of a permutation of type λ with a permutation of type μ .

[1] Work supported by NSERC (Canada).

Boccara ([1],corollary 4.8), Stanley ([9],theorem 3.1) and Goupil ([3],corollary 1) have previously given different equivalent formulas for the number of decompositions (γ, n, n) of a given permutation of type γ as a product of two n-cycles. Boccara ([1], theorem 7.2) has also given a recursive formula that permits the computation of $(\gamma, 1^k n-k, 1^r n-r)$. In section two, we show that if $\gamma = (k+1)^{m_{k+1}}, \ldots, n^{m_n}$ is restricted to partitions with parts of size at least $k+1$, then we can derive a closed formula for the decomposition $(\gamma, 1^k n-k, 1^k n-k)$ of a permutation of cycle type γ as product of two (n-k)-cycles. From this we easily obtain a formula for the expansion of $(\gamma, 1^k n-k, 1^s n-s)$. In section three, we similarly provide a formula for the coefficients $(\gamma, (k, n-k), (k, n-k))$ with the same restriction on the partition γ and we expand it to a formula for $(\gamma, (k, n-k), (s, n-s))$. These formulas are generalizations of [3], corollary 1 and show potential for a combinatorial interpretation.

To develop our results, we will use a method given by Murnaghan ([7]) to construct polynomials in several variables that are in bijection with the irreducible representation of S_n. These character's polynomials provide the evaluation of the irreducible characters on each conjugacy class and they are used and described in [3]. For character theory concepts that will be used, we refer the reader to Macdonald's book ([6]), but in order to make our treatment as self contained as possible, we present in section one an overview of the concepts used in this paper.

1. Notations and Terminology

1.1. The *sum* of two partitions of possibly different integers $\mu = 1^{m_1} 2^{m_2} \ldots n^{m_n} \vdash n$ and $\mu' = 1^{m'_1} 2^{m'_2} \ldots k^{m'_k} \vdash k$, $k \leq n$, is the partition of the integer $n+k$ defined by $\mu + \mu' = 1^{m_1+m'_1} 2^{m_2+m'_2} \ldots n^{m_n+m'_n}$. The partial order $\triangleleft$ on the set of partitions is defined by the condition:

$$\mu' \triangleleft \mu \Leftrightarrow m'_i \leq m_i, \qquad \forall\, i = 1, \ldots, n \tag{1.1}$$

1.2. A *Ferrers's diagram* of shape λ is a set of rows positioned on top of each other with the ith row, starting from the top, having λ_i nodes. A *hook* is a partition of shape $1^i n-i$, $i = 0, \ldots, n-1$. The *lenght* of a hook is the number of nodes it contains. A double hook is a partition of shape $1^i 2^j m, t$. Its Ferrers's diagram consists of two hooks, one inside the other. Figure 1.2 shows the Ferrers's diagram of a double hook of shape $1^2 2^3 4, 5$ with inner hook of lenght 6.

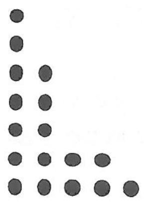

figure 1.2

The inclusion order $\subset$ on the set of partitions is defined by saying that $\lambda' \subset \lambda$ iff the diagram of λ contains the diagram of λ'. The dimension of the irreducible representation of S_n associated with a partition λ is noted f^λ and is given by the *hook formula*:

$$f^\lambda = \frac{n!}{\prod_{i,j} h_{i,j}} \tag{1.2}$$

In particular, when λ is a hook, we have:

$$f^{1^i n-i} = \binom{n-1}{i} \qquad i = 0, \ldots, n-1 \tag{1.3}$$

1.3. The *Murnaghan-Nakayama's rule* computes recursively the values χ^λ_μ of the irreducible characters λ on the conjugacy classes indexed by μ. It relates the values χ^λ_μ and $\chi^{\lambda'}_{\mu-i}$ where $\mu - i = 1^{m_1} \ldots i^{m_i - 1} \ldots n^{m_n} \vdash n-i$ and λ' is obtained from λ by taking a border strip out of its Ferrers's diagram. A *border strip* s is a connected subset of λ that lies on its northeast most part which contains no $2x2$ block of nodes. The *lenght* $|s|$ of a border strip s is the number of nodes it contains and its *height* h(s) is one less than the number of rows it occupies in λ. The Murnaghan-Nakayama rule is given by the formula:

$$\chi^\lambda_\mu = \sum_{\substack{\lambda' \vdash n-i \\ \lambda' \subset \lambda}} (-1)^{h(\lambda-\lambda')} \chi^{\lambda'}_{\mu-i} \qquad \forall\, \lambda, \mu \vdash n \tag{1.4}$$

where the sum is taken over all $\lambda' \subset \lambda$ such that the diagram $\lambda - \lambda'$ is a border strip of lenght i. A description of *Murnaghan's method* and how we use it to construct *characters' polynomials* associated with the irreducible representations of S_n, is presented in [3] section 1, and in the original paper of Murnaghan [7]. A table of characters' polynomials is given in [5]. In particular the following identity can easily be obtained from Murnaghan's method:

$$\chi^{1^i n-i}_{1^k n-k} = \binom{k-1}{i} + (-1)^{n-k-1} \binom{k-1}{i-n+k} \qquad \forall i = 0, \ldots, n-1 \tag{1.5}$$

2. The products $K_{1^k n-k} * K_{1^s n-s}$

Our point of departure is the character formula (see [3],section 3)

$$(\gamma, \lambda, \mu) \;=\; \frac{|C_\lambda||C_\mu|}{n!} \sum_{\alpha \vdash n} \frac{\chi^\alpha_\lambda \chi^\alpha_\mu \chi^\alpha_\gamma}{f^\alpha} \tag{2.1.}$$

When we have the restriction $\lambda = \mu = 1^k n - k$, formula 2.1 becomes:

$$(\gamma, 1^k n-k, 1^k n-k) \;=\; \frac{|C_{1^k n-k}|^2}{n!} \sum_{\alpha \vdash n} \frac{\chi^\alpha_{1^k n-k} \chi^\alpha_{1^k n-k} \chi^\alpha_\gamma}{f^\alpha} \tag{2.2.}$$

Our goal is to evaluate the characters in the right hand side of 2.2 and then evaluate the sum. The next result serves that purpose:

Theorem 2.1 *Let $\lambda \vdash n$ and $\gamma = (k+1)^{m_{k+1}} \ldots n^{m_n}$ be a partition of n with smallest part of size $l \geq$ k+1. The product $\chi^{\lambda}_{1^k n-k} \chi^{\lambda}_{\gamma}$ is non zero only if λ has one of the following two shapes:*

i) $\lambda = 1^i n-i, \quad 0 \leq i \leq k-1$ or $n-k \leq i \leq n-1$
ii) $\lambda = 1^i 2^j$k+1-j,t; *i.e. λ is a double hook with inner hook of lenght k.*

Notice that when $k \leq n-k$, the condition $0 \leq i \leq k-1$ or $n-k \leq i \leq n-1$ gives rise to two conjugate sets of partitions. To establish theorem 2.1, we first derive the following lemmas.

Lemma 2.1 *Let $\lambda = 1^i 2^j m, t \vdash n$ be a double hook, then we have:*

$$f^{\lambda} = \frac{\binom{j+m-2}{j}\binom{n}{j+m-1}\binom{n-m-j}{i}\binom{j+t-1}{j}(n-i-2j-2m+1)}{\binom{i+j+1}{j}(i+j+m)}$$

Proof. This is a straighforward application of the hook formula 1.2. ■

Lemma 2.2 *Let $\lambda = 1^i 2^j k-j+1, t \vdash n$ be a double hook with inner hook of lenght k, then:*

$$\chi^{\lambda}_{1^k n-k} = (-1)^{i+j+1}\binom{k-1}{j}$$

Proof. We make use of Murnaghan's method and see that when $\lambda = 1^i 2^j m, t \vdash n$ with i+j+m < n-k, the character $\chi^{\lambda}_{1^k n-k}$ is obtained by evaluating only the contribution of the fixed points in the character's polynomial of λ. Recall that the fixed points contribution in a character's polynomial is the monomial obtained by replacing n by k in the hook formula for λ. The only other possible non zero term in the character's polynomial of λ is annihilated by the condition i+j+m < n-k. So we have

$$\chi^{1^i 2^j m,t}_{1^k n-k} = \frac{\binom{j+m-2}{j}\binom{k}{j+m-1}\binom{k-m-j}{i}\binom{k-i-m-j-1}{j}(k-i-2j-2m+1)}{\binom{i+j+1}{j}(i+j+m)} \qquad (2.3)$$

and lemma 2.2 follows from replacing m by k-j+1 in 2.3. Observe that when we have k≤n-k, if a double hook does not satisfy the condition i+j+m < n-k in lemma 2.2, then its conjugate diagram does. ■

Lemma 2.3 *Let $\lambda \vdash n$ and $\gamma = (k+1)^{m_{k+1}} \dots n^{m_n} \vdash n$ be partitions with the smallest part of γ of size $l \geq k+1$. Then we have:*

i) *$\chi^{\lambda}_{1^k n-k}$ is non zero only if a hook shape can remain after removing k dots from the Ferrers's diagram of λ.*

ii) *χ^{λ}_{γ} is non zero only if we can remove a border strip of lenght l from λ and obtain a Ferrers's diagram.*

Proof. These two observations are straightforward consequences of the Murnaghan-Nakayama's rule. ∎

Proof of theorem 2.1 Combining the two observations in lemma 2.3, it is easy to realize that the product $\chi^{\lambda}_{1^k n-k}\chi^{\lambda}_{\gamma}$ is non zero only if λ itself is a hook or a double hook. If $\lambda = 1^i n-i$ is a hook, then 1.5 imposes that $0 \leq i \leq k-1$ or $n-k \leq i \leq n-1$. If λ is a double hook $1^i 2^j m,t$, then by lemma 2.3 (i), the inner hook of λ is of lenght at most k, and also $k < n-k-1$ because otherwise $\gamma = (n)$ and λ has to be a hook. Suppose that the inner hook of λ is of size j+m-1<k and $\chi^{\lambda}_{1^k n-k} \neq 0$, then γ must contain at least two parts of size at least $k+1$ and we have $n \geq 2k+2$. The inner hook of λ must be part of only one border strip that is removed via γ, otherwise there would have a border strip of size smaller than k+1 that would remain in the recursion process. Let the border strip that contains the inner hook be of size $l = i' + j + m \geq k+1$ $(i' \leq i)$, then we obtain at the same time, using the assumption and lemma 2.2:

$$i' > k - j - m$$
$$0 \leq i' \leq i \leq k - j - m$$

and this is a contradiction. Thus the inner hook of λ must be of lenght k. ∎

Proposition 2.1 *Let $\gamma = (k+1)^{m_{k+1}} \dots n^{m_n}$ be a partition of n and $\lambda = 1^i 2^j$ k+1-j,n-i-k-j-1 be a double hook with inner hook of lenght k. Then*

$$\chi^{\lambda}_{\gamma} = (-1)^k \sum_{\substack{\gamma' \vdash i+k+1 \\ \gamma' \lhd \gamma}} sgn(\gamma') \binom{m_{k+1}}{m'_{k+1}} \dots \binom{m_n}{m'_n}$$

where the sum is over all partitions $\gamma' = (k+1)^{m'_{k+1}} \dots n^{m'_n}$ of i+k+1 satisfying $\gamma' \lhd \gamma$.

Proof Using Murnaghan's method, the contribution of the fixed points of γ in the character's polynomial of λ is given by lemma 2.1 in which we replace n=i+2j+m+t by 0. This contribution is always zero unless $j = 0$ and $m = 1$ in which case we obtain $(-1)^{i+1}$. In other words, the contribution of the fixed points is always zero unless λ is reduced to a hook. This forces us to remove the border strip of lenght i+k+1 that contains the inner hook of λ with border strips of lenght at least $k+1$ that will constitute each γ'. This explains the index set and the binomial coefficients in the sum. To explain the sign contribution in the expression, suppose that the border strip of lenght $i+k+1$ is a vertical strip (which imposes that $k=0$), then only the term $sgn(\gamma')$ will appear. If it is not a vertical strip, the only alteration to sgn(γ') is provided by the removal of the border strip that contains the inner hook of λ. Such border strip of lenght say $l = i'+k+1$ will have height $i'+j$ and the hook that will remain after the removal will have height j. So the contribution of the removal of that border strip will have sign $(-1)^{i'+j}(-1)^j$ which is the signature of the associated cyclic permutation of lenght $i'+k+1$ times $(-1)^k$ and the proposition is proved. ■

Remark 2.1 Observe that the sum in proposition 2.1 does not depend on the variable j and that in fact, proposition 2.1 is equivalent to the following generating function of double hooks with inner hook of lenght k evaluated on classes of type γ restricted as in proposition 2.1:

$$\sum_{i=0}^{n-2k-2} \chi_\gamma^{1^i 2^j k+1-j, n-i-j-k-1} y^{i+k+1} = (-1)^k \prod_{i=k+1}^{n} (1-(-y^i))^{m_i}$$

where $\gamma = (k+1)^{m_{k+1}} \dots n^{m_n}$

Lemma 2.4 *Let $\gamma = (k+1)^{m_{k+1}} \dots n^{m_n}$ be a partition of n and $0 \le i < k$, then*

$$\chi_\gamma^{1^i n-i} = (-1)^i$$

Proof This is a straightforward application of either Murnaghan-Nakayama's rule or Murnaghan's method. ■

Theorem 2.2 *Let $\gamma = (k+1)^{m_{k+1}} \dots n^{m_n}$ be a partition of n and $0 \le k \le n-1$, then*

$$(\gamma, 1^k n-k, 1^k n-k) = \frac{(-1)^k \binom{n-k}{k} |C_{1^k n-k}|}{k!(n-2k+1)|C_\gamma|} \sum_{\gamma' \triangleleft \gamma} \frac{sgn(\gamma')|C_{\gamma'}||C_{\gamma-\gamma'}|}{\binom{|\gamma'|-1}{k}\binom{n-|\gamma'|-1}{k}}$$

where the sum is over all partitions γ' satisfying $\gamma' \triangleleft \gamma$ as defined in 1.1.

Proof We verify immediately that theorem 2.2 gives $(\gamma, 1^k n-k, 1^k n-k) = 0$ if $sgn(\gamma) = -1$. We may thus assume that C_γ contains only even permutations. By equation 2.2 and theorem 2.1, the only non zero characters here are indexed by hooks or double hooks and we have:

$$(\gamma, 1^k n-k, 1^k n-k) = \frac{\binom{n}{k}(n-k-1)!}{k!(n-k)} \Bigg[(1+(-1)^{n-1}) \sum_{i=0}^{k-1} \frac{\left(\chi_{1^k n-k}^{1^i n-i}\right)^2 \chi_\gamma^{1^i n-i}}{f^{1^i n-i}} + \sum_{i=0}^{n-2k-2} \sum_{j=0}^{k-1} \frac{\left(\chi_{1^k n-k}^{1^i 2^j k-j+1, n-i-j-k-1}\right)^2 \chi_\gamma^{1^i 2^j k-j+1, n-i-j-k-1}}{f^{1^i 2^j k-j+1, n-i-j-k-1}} \Bigg] \tag{2.4}$$

The first sum in 2.4, is multiplied by $(1+(-1)^{n-1})$ because we use the fact that when $\tilde{\lambda}$ is the conjugate partition of λ, then $\chi_\gamma^{\tilde{\lambda}} = sgn(\gamma)\chi_\gamma^\lambda$. Thus each irreducible character in this sum has a conjugate and can be counted twice or annihilated. Then using identities 1.3, 1.5, lemmas 2.2 and 2.4 and proposition 2.1, we transform equation 2.4 into:

$$(\gamma, 1^k n-k, 1^k n-k) = \frac{\binom{n}{k}(n-k-1)!}{k!(n-k)} \left[(1+(-1)^{n-1}) \sum_{i=0}^{k-1} \frac{(-1)^i \binom{k-1}{i}^2}{\binom{n-1}{i}} + \right.$$
$$\left. \sum_{i=0}^{n-2k-2} \sum_{\substack{\gamma' \\ |\gamma'|=i+k+1}} (-1)^k sgn(\gamma') \prod_{r=k+1}^{n} \binom{m_r}{m'_r} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{f^{1^i 2^j k-j+1, n-i-j-k-1}} \right] \tag{2.5}$$

Now recall the following two binomial identities ([2], equation 7.1 and [4], 5.93)

$$\sum_{i=0}^{k-1} \frac{(-1)^i \binom{k-1}{i}\binom{s-1}{i}}{\binom{n-1}{i}} = \frac{\binom{n-k}{k-1}}{\binom{n-1}{k-1}} \tag{2.6}$$

$$\sum_{j=0}^{k-1} \frac{\binom{k-1}{j}\binom{i+j+1}{j}}{\binom{n-i-k-2}{j}} = \frac{\binom{n-k}{k-1}}{\binom{n-i-k-2}{k-1}} \tag{2.7}$$

and observe using (2.7) and lemma 2.1:

$$\begin{aligned}\sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{f^{1^i 2^j k-j+1, n-i-j-k-1}} &= \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2 \binom{i+j+1}{j}(k+i+1)}{\binom{k-1}{j}\binom{n}{k}\binom{n-k-1}{i}\binom{n-i-k-2}{j}(n-i-2k-1)} \\ &= \frac{k+i+1}{\binom{n}{k}\binom{n-k-1}{i}(n-i-2k-1)} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}\binom{i+j+1}{j}}{\binom{n-i-k-2}{j}} \\ &= \frac{(n-k)\binom{n-k}{k}}{(n-2k+1)\binom{n}{i+k+1}\binom{k+i}{k}\binom{n-i-k-2}{k}}\end{aligned} \tag{2.8}$$

Observe also that for partitions $\gamma' = 1^{m'_{k+1}} \dots n^{m'_n} \lhd \gamma = 1^{m_1} \dots n^{m_n}$, we have:

$$\prod_{r=1}^{n} \frac{\binom{m_r}{m'_r}}{\binom{n}{|\gamma'|}} = \frac{|C_{\gamma'}||C_{\gamma-\gamma'}|}{|C_\gamma|} \tag{2.9}$$

Then we use (2.6) to transform equation 2.5 into:

$$(\gamma, 1^k n-k, 1^k n-k) = \frac{\binom{n}{k}(n-k-1)!}{k!(n-k)} \left[(1+(-1)^{n-1})\frac{\binom{n-k}{k-1}}{\binom{n-1}{k-1}} + \right.$$

$$\left. \frac{\binom{n-k}{k}(n-k)}{(n-2k+1)} \sum_{i=0}^{n-2k-2} \sum_{\substack{\gamma' \\ |\gamma'|=i+k+1}} \frac{(-1)^k sgn(\gamma') \prod_{r=k+1}^{n} \binom{m_r}{m'_r}}{\binom{n}{|\gamma'|}\binom{|\gamma'|-1}{k}\binom{n-|\gamma'|-1}{k}} \right]$$

$$= \frac{\binom{n-k}{k}|C_{1^k n-k}|}{k!(n-2k+1)} \left[\frac{(1+(-1)^{n-1})}{\binom{n-1}{k}} \right] +$$

$$\frac{(-1)^k \binom{n-k}{k}|C_{1^k n-k}|}{k!(n-2k+1)|C_\gamma|} \sum_{\substack{\gamma' \lhd \gamma \\ \gamma' \neq \emptyset, \gamma}} \frac{sgn(\gamma')|C_{\gamma'}||C_{\gamma-\gamma'}|}{\binom{|\gamma'|-1}{k}\binom{n-|\gamma'|-1}{k}} \tag{2.10}$$

The double sum in the right hand side of 2.10 is the result we are looking for except that it excludes the two cases $\gamma' = \emptyset, \gamma' = \gamma$, but if we use the following fact:

$$\frac{(-1)^k}{|C_\gamma|} \sum_{\gamma'=\emptyset,\gamma} \frac{sgn(\gamma')|C_{\gamma'}||C_{\gamma-\gamma'}|}{\binom{|\gamma'|-1}{k}\binom{n-|\gamma'|-1}{k}} = \frac{(-1)^k}{|C_\gamma|} \left[\frac{|C_\gamma|}{(-1)^k \binom{n-1}{k}} + \frac{(-1)^{n-1}|C_\gamma|}{\binom{n-1}{k}(-1)^k} \right]$$

$$= \frac{(1+(-1)^{n-1})}{\binom{n-1}{k}}$$

we see that the sum in the right hand side of (2.10) now covers all partitions $\gamma' \lhd \gamma$ and the theorem is proved. ■

Remark 2.2 We obtain from theorem 2.2 the formulas:

$$((n), 1^k n-k, 1^k, n-k) = \frac{(1+(-1)^{n-1})n(n-k-1)!\binom{n-k}{k}}{k!(n-2k+1)(n-k)} \tag{2.11}$$

$$((k+1)^2, 1^k k+2, 1^k k+2) = (k+1)^3 \tag{2.13}$$

Observe also that $\gamma \neq (n)$ in theorem 2.2 implies the inequality $k < n-k$. Moreover, theorem 2.2 is a generalization of [3], corollary 1, [1], corollary 4.8 and [8], theorem 3.1 (when $k = 2$).

Remark 2.3 An immediate consequence of theorem 2.1 is that if we have $k < s$ and $\gamma = (s+1)^{m_{s+1}} \dots n^{m_n}$, then the product $\chi^\lambda_{1^k n-k}\chi^\lambda_{1^s n-s}\chi^\lambda_\gamma$ is non zero only if λ is a hook. This observation has the following consequence:

Corollary 2.1 Let $0 \le k < s \le n/2$ and $\gamma = (s+1)^{m_{s+1}} \dots n^{m_n}$, then

$$(\gamma, 1^k n-k, 1^s n-s) = \begin{cases} \frac{2n(n-k-1)!(n-s-1)!}{k!s!(n-k-s+1)!} & \text{if } sgn(\gamma) = (-1)^{k+s} \\ 0 & \text{otherwise} \end{cases}$$

Proof We use remark 2.3, lemma 2.4 and formula 1.5 and we transform formula 2.1 into the following:

$$(\gamma, 1^k n-k, 1^s n-s) =$$
$$\frac{n!}{k!(n-k)s!(n-s)} \sum_{i=0}^{k-1} \frac{(-1)^i \binom{k-1}{i}\binom{s-1}{i}}{\binom{n-1}{i}} (1 + sgn(1^k n-k) sgn(1^s n-s) sgn(\gamma)) \tag{2.12}$$

and we deduce corollary 2.1 by using identity (2.6). ■

The coefficients obtained in corollary 2.1 do not depend on the partition γ. This means that all conjugacy classes with cycles of lenght at least $s+1$ have the same coefficient in the decomposition of $K_{1^k n-k} * K_{1^s n-s}$. This fact is reminiscent of the observation that the product $K_{(n)} * K_{(1,n-1)}$ has constant coefficients in its decomposition over all odd conjugacy classes (see [3]).

Theorem 2.2 only permits the evaluation of coefficients of conjugacy classes K_γ with relatively large cycle lenghts but it is possible to find coefficients of conjugacy classes K_γ with partitions γ containing fixed points and large cycles by using the following conjecture:

Conjecture 2.1 *Let* $0 \le k \le n-1$ *and* $\gamma = \gamma^* + 1^{m_1}$, *where* $\gamma^* = 2^{m_2} \ldots n^{m_n}$ *is the part of* γ *that does not contain fixed points. Then:*

$$(\gamma, 1^k n-k, 1^k n-k) = m_1! \sum_{i=0}^{k} \frac{\binom{n-2k+i-1}{m_1-i}}{i!} (\gamma^*, 1^{k-i} n-k+i-m_1, 1^{k-i} n-k+i-m_1)$$

This conjecture generalizes a result of Walkup ([10], theorem 1) and has been proved by the author for $0 \le k \le 3$.

3. The product $K_{(k,n-k)} * K_{(s,n-s)}$

Let $k \le n-k$ and write the character formula (2.1) for the product $K_{(k,n-k)} * K_{(k,n-k)}$:

$$(\gamma, (k,n-k), (k,n-k)) = \frac{|C_{(k,n-k)}|^2}{n!} \sum_{\lambda \vdash n} \frac{\chi^\lambda_{(k,n-k)} \chi^\lambda_{(k,n-k)} \chi^\lambda_\gamma}{f^\lambda} \tag{3.1}$$

The following result contributes to the evaluation of the characters in formula 3.1.

Proposition 3.1 *Let* $\lambda \vdash n$, $\gamma = (k+1)^{m_{k+1}} \ldots n^{m_n}$ *and* $0 \le k \le n-k$. *The product* $\chi^\lambda_{(k,n-k)} \chi^\lambda_\gamma$ *is non zero only if* λ *has one of the following two shapes:*

i) $\lambda = 1^i n-i$, $\quad 0 \le i \le k-1$ or $n-k \le i \le n-1$

ii) $\lambda = 1^i 2^j$k-j+1,t; $\quad$ *i.e.* λ *is a double hook with inner hook of lenght k.*

Proof Since $\chi^\lambda_{(k,n-k)}$ is non zero only if a hook remains after removal of a border strip of lenght k, the partition λ must be a double hook, possibly degenerated into a hook. Moreover, this double hook $\lambda = 1^i 2^j m, t$ must satisfy at least one of the following conditions:

$a)\ m = k - i - j$
$b)\ t = k - j$
$c)\ m = k - j + 1$

This is because the border strip of lenght k that is removed starts either at the top (condition a), at the right (condition b) or is the inner hook (condition c) of the Ferrers's diagram λ. When the first or the second constraints are satisfied, then $\chi_\gamma^\lambda = 0$ unless λ is a hook, i.e. $j = 0$ and $m = 1$. This permits shape i). Shape ii) is obtained by the third constraint. ■

The main result of this section is the following:

Theorem 3.1 *Let* $\gamma = (k+1)^{m_{k+1}} \ldots n^{m_n}$ *and* $0 \leq k \leq n-k$, *then*

$$(\gamma, (k, n-k), (k, n-k)) =$$
$$\frac{(-1)^k |C_{(k,n-k)}|}{k\ |C_\gamma|} \sum_{\gamma' \triangleleft \gamma} \frac{sgn(\gamma')|C_{\gamma'}||C_{\gamma-\gamma'}|}{\binom{|\gamma'|-1}{k}(n-|\gamma'|-k)} \sum_{j=0}^{k-1} \frac{\binom{|\gamma'|-k+j}{j}}{\binom{n-|\gamma'|-1}{j}\binom{k-1}{j}}$$

To prove theorem 3.1, we establish the following lemmas.

Lemma 3.1 *Let* $0 \leq k \leq n-k$, *then*

$$i)\ \chi_{(k,n-k)}^{1^r n-r} = \begin{cases} (-1)^r, & \text{if } 0 \leq r < k \\ (-1)^{r+1}, & \text{if } n-k \leq r < n \\ 0 & \text{otherwise} \end{cases}$$

$$ii)\ \chi_{k,n-k}^{1^i 2^j k-j+1, n-i-k-j-1} = (-1)^{i+j+1} \qquad i,j \geq 0$$

Proof These two identities follow from Murnaghan-Nakayama's rule. ■

Lemma 3.2 *Let* k, n *be two positive integers, then we have:*

$$\sum_{j=0}^{k-1} \frac{(-1)^j \binom{n-k+j}{j}}{\binom{k-1}{j}} = \frac{k}{n+1}\left[1 + (-1)^{k-1}\binom{n}{k}\right]$$

Proof Set $f(n,k) := \sum_{j=0}^{k-1} \frac{(-1)^j \binom{n-k+j}{j}}{\binom{k-1}{j}}$. Then using the binomial identity ([2], formula 4.1)

$$\frac{1}{\binom{k-1}{j}} = \frac{k}{k+1}\left[\frac{1}{\binom{k}{j}} + \frac{1}{\binom{k}{j+1}}\right] \tag{3.2}$$

we have:

$$f(n,k) = \frac{k}{k+1}\sum_{j=0}^{k-1}(-1)^j\left[\frac{\binom{n-k+j}{j}}{\binom{k}{j}} + \frac{\binom{n-k+j}{j}}{\binom{k}{j+1}}\right]$$
$$= \frac{k}{k+1}\left[f(n+1,k+1) + (-1)^{k-1}\binom{n}{k} + \sum_{j=0}^{k-1}(-1)^j\left[\frac{\binom{n-k+j+1}{j+1}}{\binom{k}{j+1}} - \frac{\binom{n-k+j}{j+1}}{\binom{k}{j+1}}\right]\right]$$
$$= \frac{k}{k+1}\left[f(n+1,k+1) + (-1)^{k-1}\binom{n}{k} - f(n+1,k+1) + f(n,k+1)\right]$$
$$= \frac{k}{k+1}\left[(-1)^{k-1}\binom{n}{k} + f(n,k+1)\right] \tag{3.3}$$

On the other hand, we also have:

$$\frac{k}{n+1}\left[1+(-1)^{k-1}\binom{n}{k}\right] + \frac{k}{k+1}(-1)^k\binom{n}{k} = \frac{k}{n+1}\left[1+(-1)^k\binom{n}{k+1}\right] \tag{3.4}$$

Thus both sides of lemma 3.2 satisfy the same recurrence with the same initial value k=1 and their equality follows. ■

Proof of theorem 3.1 We proceed similarly to the proof of theorem 2.2 and we assume that $sgn(\gamma) = 1$. Using proposition 3.1, we transform equation 3.1 into:

$$(\gamma,(k,n-k),(k,n-k)) = \frac{n!}{k^2(n-k)^2}\left[\sum_{i=0}^{n-1}\frac{\left(\chi_{(k,n-k)}^{1^i n-i}\right)^2\chi_\gamma^{1^i n-i}}{f^{1^i n-i}} + \right.$$
$$\left.\sum_{i=0}^{n-2k-2}\sum_{j=0}^{k-1}\frac{\left(\chi_{(k,n-k)}^{1^i 2^j k-j+1,n-i-j-k-1}\right)^2\chi_\gamma^{1^i 2^j k-j+1,n-i-j-k-1}}{f^{1^i 2^j k-j+1,n-i-j-k-1}}\right] \tag{3.5}$$

Then using identity 1.3, lemma 2.4, lemma 3.1 and proposition 2.1, we have:

$$(\gamma,(k,n-k),(k,n-k)) = \frac{n!}{k^2(n-k)^2}\left[2\sum_{i=0}^{k-1}\frac{(-1)^i}{\binom{n-1}{i}} + \right.$$
$$\left.(-1)^k\sum_{i=0}^{n-2k-2}\sum_{\gamma'\vdash i+k+1} sgn(\gamma')\prod_{r=k+1}^{n}\binom{m_r}{m'_r}\sum_{j=0}^{k-1}\frac{1}{f^{1^i 2^j k-j+1,n-i-k-j-1}}\right] \tag{3.6}$$

Using lemma 2.1 and the binomial identity([2], formula 2.1):

$$\sum_{i=0}^{k-1}\frac{(-1)^i}{\binom{n-1}{i}} = \frac{n}{n+1}\left[1+\frac{(-1)^{k-1}}{\binom{n}{k}}\right] \tag{3.7}$$

we get:

$$(\gamma,(k,n-k),(k,n-k)) = \frac{n!}{k^2(n-k)^2}\left[\frac{2n}{(n+1)}\left[1+\frac{(-1)^{k-1}}{\binom{n}{k}}\right]+\right.$$

$$\left.(-1)^k \sum_{i=0}^{n-2k-2} \sum_{\gamma' \vdash k+i+1} sgn(\gamma') \prod_{r=k+1}^{n} \frac{\binom{m_r}{m'_r}(k+i+1)}{\binom{n}{k}\binom{n-k-1}{i}(n-2k-i-1)} \sum_{j=0}^{k-1} \frac{\binom{i+j+1}{j}}{\binom{k-1}{j}\binom{n-i-k-2}{j}}\right] \quad (3.8)$$

where the second sum in the right hand side of 3.8 is over all partitions $\gamma' = (k+1)^{m'_{k+1}} \ldots n^{m'_n}$ of $k+i+1$. This sum does not contain the two partitions $\gamma' = \emptyset, \gamma$. But if we set $|\gamma'| = k+i+1$ and we use 3.7 and lemma 3.2, we observe the two identities:

$$\frac{(k+i+1)}{\binom{n}{k}\binom{n-k-1}{i}} = \frac{(n-k)}{\binom{n}{i+k+1}\binom{k+i}{k}} \quad (3.9)$$

$$\frac{(-1)^k(n-k)}{\binom{k+i}{k}(n-2k-i-1)} \sum_{\gamma'=\emptyset,\gamma} sgn(\gamma') \sum_{j=0}^{k-1} \frac{\binom{i+j+1}{j}}{\binom{k-1}{j}\binom{n-i-k-2}{j}}$$

$$= (-1)^k(n-k) \sum_{\gamma'=\emptyset,\gamma} \frac{1}{\binom{|\gamma'|-1}{k}(n-|\gamma'|-k)} \sum_{j=0}^{k-1} \frac{\binom{|\gamma'|-k+j}{j}}{\binom{k-1}{j}\binom{n-|\gamma'|-1}{j}}$$

$$= \frac{2n}{(n+1)}\left(1+\frac{(-1)^{k-1}}{\binom{n}{k}}\right) \quad (3.10)$$

So that the right hand side of identity 3.8 now contains all partitions $\gamma' \triangleleft \gamma$ and we use identities 2.9 and 3.9 to get:

$$(\gamma,(k,n-k),(k,n-k)) =$$

$$\frac{(-1)^k|C_{(k,n-k)}|}{k\;|C_\gamma|} \sum_{\gamma' \triangleleft \gamma} \frac{sgn(\gamma')|C_{\gamma'}||C_{\gamma-\gamma'}|}{\binom{|\gamma'|-1}{k}(n-|\gamma'|-k)} \sum_{j=0}^{k-1} \frac{\binom{|\gamma'|-k+j}{j}}{\binom{n-|\gamma'|-1}{j}\binom{k-1}{j}}$$

which proves our theorem. ∎

Corollary 3.1 For $0 \le k \le n-k$, we have:

$$((n),(k,n-k),(k,n-k)) = (1+(-1)^{n-1})\left(\frac{n}{n+1}\right)\left[\binom{n}{k}+(-1)^{k-1}\right]$$

Proof This is immediate from theorem 3.1 ∎

Remark 3.1 No closed form similar to formula 2.7 seems to be known for the binomial expression

$$\sum_{j=0}^{k-1} \frac{\binom{i+j+1}{j}}{\binom{k-1}{j}\binom{n-i-k-2}{j}}$$

contained in 3.8. For example , if we apply theorem 3.1 to compute the structure constants $a(k) := ((k+1)^2, (k, k+2), (k, k+2))$, we obtain:

$$a(k) = \frac{2n!}{k^2(n-k)^2}\left[\frac{n}{n+1}\left(1+\frac{(-1)^{k+1}}{\binom{n}{k}}\right)+\frac{(k+1)}{\binom{n}{k}}\sum_{j=0}^{k-1}\frac{(j+1)}{\binom{k-1}{j}\binom{k}{j}}\right] \tag{3.11}$$

and the more simple binomial expression in 3.11 does not seem expressible in closed form. The first nine terms of the sequence $a(k)$ starting at k=1 are:

$$8, 27, 384, 12100, 736128, 70990416, 9939419136, 1896254551296, 472882821120000$$

No rational function seems to generate the sequence a(k) or the binomial expression in 3.11.

Remark 3.2 (J. Remmel [8]) An immediate consequence of proposition 3.1 is that if $k \neq s$ then the product $\chi^{\lambda}_{1^k n-k}\chi^{\lambda}_{1^s n-s}\chi^{\lambda}_{\gamma}$ is non zero only if λ is a hook. We thus obtain the following result:

Corollary 3.2 Let $0 \leq k < s \leq n/2$ and $\gamma = (s+1)^{m_{s+1}} \ldots n^{m_n}$, then

$$(\gamma, (k, n-k), (s, n-s)) = \begin{cases} \frac{2n!n}{k(n-k)s(n-s)(n+1)}\left[1+\frac{(-1)^{k-1}}{\binom{n}{k}}\right] & \text{if } sgn(\gamma) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof We use remark 3.2, lemma 2.4 and lemma 3.1 and we transform formula 2.1 into the following:

$$(\gamma, (k, n-k), (s, n-s)) = \frac{n!}{k(n-k)s(n-s)}\sum_{i=0}^{k-1}\frac{(-1)^i}{\binom{n-1}{i}}(1+sgn(k, n-k)sgn(s, n-s)sgn(\gamma)) \tag{3.12}$$

then we use formula 3.7 to obtain our result. ■

Again, we observe that the coefficients obtained in corollary 3.2 do not depend on the partition γ so that we have the same coefficient for all the conjugacy classes K_{γ} with cycles of lenght at least s+1 in the decomposition of the product $K_{(k,n-k)} * K_{(s,n-s)}$

We terminate with another product whose decomposition has the same property of being distributed evenly over a restricted set of classes.

Corollary 3.3 Let $0 \leq k < s \leq n/2$ and $\gamma = (s+1)^{m_{s+1}} \dots n^{m_n}$, then

$$(\gamma, (1^k, n-k), (s, n-s)) = \begin{cases} \frac{2n\binom{n}{k}(n-k-1)!}{s(n-s)(n-k+1)} & \text{if } sgn(\gamma) = (-1)^{k-1} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (s-1)_{s-k}(\gamma, 1^s n-s, (k, n-k)) & \text{if } sgn(\gamma) = (-1)^{k-1} = (-1)^{s-1} \\ 0 & \text{otherwise} \end{cases}$$

Proof For the first identity, we apply theorem 2.1, lemma 2.2, lemma 3.1 and lemma 2.1 to the character formula 2.1. We obtain:

$$(\gamma, (1^k, n-k), (s, n-s)) =$$

$$\frac{\binom{n}{k}(n-k-1)!}{s(n-s)} \sum_{i=0}^{n-1} \frac{\binom{k-1}{i}}{\binom{n-1}{i}} (1 + sgn(1^k n-k) sgn(s, n-s) sgn(\gamma)) \tag{3.13}$$

and the first identity follows. The second identity is obtained from the observation:

$$|C_{1^k n-k}||C_{(s,n-s)}| = (s-1)_{s-k}|C_{1^s n-s}||C_{(k,n-k)}| \tag{3.14}$$

where $(s-1)_{s-k}$ is a descending factorial.

References

[1] G. Boccara, *Nombre de représentations d'une permutation comme produit de deux cycles de longueurs données,* Discrete Mathematics **29** (1980), 105–134.

[2] H. W. Gould, *Combinatorial identities. A Standardized Set of Tables Listing 500 Binomial Coefficient Summations,* H. W. Gould, Morgantown, W. Va., 1980.

[3] A. Goupil, *On Products of Conjugacy Classes of the Symmetric Group,* Discrete Mathematics **79** (1988), 49–57.

[4] R. L. Graham; D. E. Knuth; O. Patashnik, *Concrete Mathematics,* Addison-Wesley, 1989.

[5] A. Kerber; K. J. Thürlings, *Symmetrieklassen von Funktionen und ihre Abzählungstheorie (Vol II: Hinzunahme darstellungstheoretischer Begriffsbildungen),* Bayreuth Mathematische Schriften, 1983.

[6] I. G. Macdonald, *Symmetric Functions and Hall Polynomials,* Oxford University Press, New-York, 1979.

[7] F. D. Murnaghan, *The Characters of the Symmetric Group,* Proc. Nat. Acad. Sci., USA., **37** (1951), 55–58.

[8] J. Remmel, *Personnal communication,* oct. 1991.

[9] R. P. Stanley, *Factorization of Permutations into n-Cycles,* Discrete Mathematics **37** (1981), 255–262.

[10] D. W. Walkup, *How Many Ways Can a Permutation Be Factored into Two n-cycles,* Discrete Mathematics **28** (1979), 315–319.