A Formula for Two-Row Macdonald Functions

Naihuan Jing and Tadeusz Józefiak

Let F be the field of rational functions in two independent indeterminates q and t over the field of rational numbers. We consider the ring Λ_F of symmetric functions over F in infinitely many variables $x_1, x_2, \cdots$. If p_n is the power sum symmetric function of degree n and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ for a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$, then the set $\{p_\lambda\}$ forms a linear basis for Λ_F.

We define the scalar product on Λ_F by setting

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \prod_{i \geq 1} i^{m_i} m_i! \prod_{j \geq 1} \frac{1 - q^{\lambda_j}}{1 - t^{\lambda_j}}$$

where m_i is the multiplicity of i in the partition λ and $\delta_{\lambda\mu}$ is the Kronecker symbol.

There exists a distinguished orthogonal basis $\{P_\lambda = P_\lambda(q,t)\}$ of Λ_F with respect to the scalar product determined uniquely by the following property [2] [3]:

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu$$

where m_λ is the monomial symmetric function associated with λ, "$<$" is the dominance ordering on partitions, and $u_{\lambda\mu} \in F$.

We are interested in another basis $\{Q_\lambda = Q_\lambda(q,t)\}$ of Λ_F such that

$$\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}.$$

The functions Q_λ (or P_λ) are usually called Macdonald symmetric functions. If $\lambda = (n)$ or (r,s) then we write Q_n, $Q_{r,s}$ for the corresponding Macdonald function. The Q_n can be explicitly expressed in terms of $x_1, x_2, \cdots$. In fact, the generating function of the Q_n has the following form:

$$\sum_{n=0}^{\infty} Q_n z^n = \exp\left(\sum_{n \geq 1} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n z^n \right).$$

The goal of this note is to announce the following result.

THEOREM. *If λ is a partition with two parts r and s, $r \geq s \geq 0$, then*

$$Q_{r,s} = \sum_{i=0}^{s} a_i^p Q_{r+i} Q_{s-i} \qquad (1)$$

where $p = r - s$, $a_0^p = 1$, and for $i > 0$

$$a_i^p = \frac{(t-1)\cdots(t-q^{i-1})}{(1-q)\cdots(1-q^i)} \cdot \frac{(1-q^{p+1})\cdots(1-q^{p+i-1})(1-q^{p+2i})}{(1-q^{p+1}t)\cdots(1-q^{p+i}t)}.$$

If we let $q = t^\alpha$, α a parameter, then

$$Q_\lambda(\alpha) = \lim_{t \to 1} Q_\lambda(t^\alpha, t)$$

is known as the Jack symmetric function associated with λ. Hence the Theorem implies the following formula.

COROLLARY. *If λ is a partition with two parts r and s, $r \geq s \geq 0$, then*

$$Q_{r,s}(\alpha) = \sum_{i=0}^{s} a_i^p(\alpha) Q_{r+i}(\alpha) Q_{s-i}(\alpha) \tag{2}$$

where $p = r - s$ and

$$a_i^p(\alpha) = (-1)^i \frac{(1-\alpha)\cdots(1-(i-1)\alpha)}{i!} \cdot \frac{(p+1)\cdots(p+i-1)(p+2i)}{(1+(p+1)\alpha)\cdots(1+(p+i)\alpha)}.$$

Notice that the normalization of Jack functions is different from that of Stanley in [4].

We can express formula (1) in terms of basic hypergeometric series as follows:

$$Q_{r,s} = {}_4\Phi_3 \left(\begin{matrix} q^p, & t^{-1}, & q^{p/2+1}, & -q^{p/2+1}, \\ & q^{p+1}t, & q^{p/2}, & -q^{p/2} \end{matrix} ; \ q, tR \right) Q_r Q_s$$

where $p = r - s$ and R is the raising operator on monomials $Q_i Q_j$, i.e. $R(Q_i Q_j) = Q_{i+1} Q_{j-1}$. We assume that $Q_i = 0$ for $i < 0$.

Formula (2) can be rewritten as

$$Q_{r,s}(\alpha) = {}_3F_2 \left(\begin{matrix} p, & -1/\alpha, & p/2+1 \\ & p+1+1/\alpha, & p/2 \end{matrix} ; \ R \right) Q_r(\alpha) Q_s(\alpha)$$

where $p = r - s$ and ${}_3F_2$ is a hypergeometric series.

References

[1] I.G. Macdonald, "Symmetric Functions and Hall Polynomials," Oxford University Press, Oxford, 1979.

[2] I.G. Macdonald, *A new class of symmetric functions*, Publ. I.R.M.A. Strasbourg, Actes 20^e Séminaire Lotharingien, 1988, 131-171.

[3] I.G. Macdonald, "2nd edition of [1]," to appear.

[4] R.P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), 76-115.

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824