Extremal problems for the Möbius function in the lattice of faces of the n-dimensional octahedron

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Abstract

In the vein of recent work of Sagan, Yeh and Ziegler, we study extremal problems connected with the Möbius function of certain families of subsets from O_n, the lattice of faces of the n-dimensional octahedron. In particular, we find that for lower order ideals $\mathcal{F}$ in O_n, $|\mu(\mathcal{F})|$ attains a maximum by taking the lower two-thirds of the poset. Currently we are proving the conjecture for intervals of rank-selections, the maximum attained by taking the interval of ranks from two-fifths through four-fifths of O_n, and are formulating the conjecture for the arbitrary rank selection case.

0 Notation

Let P be a partially ordered set (*poset*) that has unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ (i.e. P is *bounded*). All the posets we study will be finite and graded, so we have a rank function associated with a given poset. In particular, let O_n denote the lattice of faces of the n-dimensional octahedron. We can represent O_n as the poset of all signed subsets of $\{1, \ldots, n\}$ ordered by inclusion with the element $\hat{1}$ adjoined. The *rank* of any element $S \in O_n$ is given by $|S|$, where $|\cdot|$ denotes cardinality. Observe that $\hat{1}$ has rank $n+1$ in O_n.

Recall from [11, p. 101] that given a poset P, its dual P^* is the poset satisfying

$$x \leq y \text{ in } P^* \iff y \leq x \text{ in } P.$$

We have $Q_n = O_n^*$, where Q_n is the lattice of faces of the n-dimensional cube. We can represent the elements of Q_n as ordered n-tuples of $+1$'s, -1's and $*$'s in $\mathbf{R}^n$, where

$$* = \{t \mid 0 \leq t \leq 1\}.$$

The elements of this poset are ordered lexicographically using the convention $-1 \leq *$ and $+1 \leq *$. The rank of any element $S \in Q_n$ is given by the number of $*$'s in S plus one.

For μ the Möbius function of a poset P we let $\mu(P)$ denote the value of $\mu_P(\hat{0}, \hat{1})$, for x an element of P we let $\mu(x)$ denote $\mu_P(\hat{0}, x)$, and for any family $\mathcal{F}$ of elements of P we let $\mu(\mathcal{F})$ equal $\mu_{\hat{\mathcal{F}}}(\hat{0}, \hat{1})$. (Here $\hat{\mathcal{F}}$ is $\mathcal{F}$ adjoined with $\hat{0}$ and $\hat{1}$, if necessary).

Given a non-negative integer n, we let

$$[n] = \{1, 2, \ldots, n\}.$$

For $\mathcal{F}$ a family in P and $R \subseteq [\text{ rank } \hat{1} - 1]$, we define the *rank-selected subposet*

$$\mathcal{F}(R) = \{S \in \mathcal{F} : \quad \text{rank } S \in R\}.$$

In particular, the i^{th} *rank level* of $\mathcal{F}$ is given by

$$\mathcal{F}(i) = \mathcal{F}(\{i\}) = \{A \in \mathcal{F} : |A| = i\}.$$

Also, we use the shorthand

$$\mathcal{F}[k] = \mathcal{F}([k]).$$

Finally, a *lower order ideal* $\mathcal{A}$ is a subset $\mathcal{A}$ of P such that if $x \in \mathcal{A}$ and $y \leq x$ then $y \in \mathcal{A}$.

1 Introduction

Given P a bounded poset with a fixed number of elements, Stanley [11, Exercise 3.41a] posed the question of finding the maximum value of $|\mu(P)|$. In [12] Ziegler answers this question for both bounded posets and graded posets, and determines the extremal configuration in each situation.

Recent work of Sagan, Yeh and Ziegler [9] has approached extremal problems involving the Möbius function from a slightly different angle. They study the maximum value attained by the Möbius function μ taken over certain subsets of a particular poset. More specifically, if $\mathcal{F}$ is a family of subsets contained in the Boolean algebra B_n, then the $\max_{\mathcal{F}} |\mu(\mathcal{F})|$ has been found for three categories of families:

(i.) all lower order ideals
(ii.) all intervals of ranks
(iii.) all rank-selections.

The maxima are obtained by taking the lower half, middle third, and every other rank of B_n, respectively. The lower order ideal case was first solved by Eckhoff [6] and Scheid [1], and viewed in the context of the reduced Euler characteristic by Björner and Kalai [4]. Niven [7] and de Bruijn [5] have previously solved the arbitrary rank-selection case, while the interval of ranks case is a new result.

In this paper, we will address analogous extremal problems for O_n, the lattice of faces of the n-dimensional octahedron. More specifically, by extending the techniques developed for B_n in [9] to O_n, we find the extremal configuration for lower order ideals is the lower two-thirds of the poset O_n. In section 3, we indicate the work we have completed-to-date for the interval of ranks result. Finally, in section 4 we describe our future work.

2 Lower Order Ideals

In this section we will be concerned with maximizing $|\mu(\mathcal{F})|$ as $\mathcal{F}$ ranges over all lower order ideals in O_n. Once we have derived some facts about the Möbius function of certain subsets of O_n, we will be able to re-establish the well-known result that O_n is Eulerian. This important property enables us to "dualize" the problem-at-hand (whenever necessary) from O_n to an equivalent problem in Q_n.

We state the main result of this section:

Theorem 2.1 *If $\mathcal{F}$ is a lower order ideal in O_n, then*

$$|\mu(\mathcal{F})| \leq |\sum_{k=0}^{\lfloor \frac{2n}{3} \rfloor} \binom{n}{k} (-2)^k |,$$

with equality occurring if and only if

$$\mathcal{F} = O_n[k] \ \ with\ k = \lfloor \tfrac{2n}{3} \rfloor.$$

(Here $\lfloor \cdot \rfloor$ denotes the greatest integer function.)

Before proving Theorem 2.1, we will first specialize $\mathcal{F}$ to be rank-selected lower order ideals, i.e., lower order ideals of the form $O_n[k]$. We show in Lemma 2.8 that $|\mu(O_n[k])|$ is maximized if we take k to be $\lfloor \frac{2n}{3} \rfloor$, i.e. the lower two-thirds of O_n. Once we generalize $\mathcal{F}$ to be any lower ideal in O_n, we will see the ideal $O_n[\lfloor \frac{2n}{3} \rfloor]$ is also the maximal configuration for Theorem 2.1.

We begin by establishing some elementary properties of O_n.

Proposition 2.2 *Let x be an element of $O_n - \hat{1}$ with rank k. Then $\mu(x) = (-1)^k$.*

Proof. Recall

$$O_n - \hat{1} \cong \underbrace{V \times \cdots \times V}_{n},$$

where

$$V = \quad .$$

Applying the product theorem for the Möbius function [11, Prop. 3.8.2] gives the result. □

We are able to express $\mu(O_n[k])$ in two ways: in terms of a summation formula (Corollary 2.3) or a recurrence (Corollary 2.5).

Corollary 2.3 *The following summation formula holds for $\mu(O_n[k])$:*

$$\mu(O_n[k]) = -\Big(\sum_{j=0}^{k} \binom{n}{j}(-2)^j\Big), \quad n \geq 1,\ 0 \leq k \leq n.$$

Proof. Use Proposition 2.2 and the fact there are $\binom{n}{j}(2)^j$ elements of rank j in O_n. □

Corollary 2.4 $\mu(O_n) = (-1)^{n+1}$, $n \geq 1$.

Proof. It is enough to observe $\mu(O_n) = \mu(O_n[n])$ (recall $\hat{1}$ has rank $n+1$ in O_n). The result then immediately follows once we apply the binomial theorem to the summation expression for $\mu(O_n[n])$ given in Corollary 2.3. □

Corollary 2.5 *The $\mu(O_n[k])$'s satisfy the recurrence*

$$\mu(O_n[k]) = -2\mu(O_{n-1}[k-1]) + \mu(O_{n-1}[k]),$$

where $n \geq 2$, $0 < k < n$, with boundary conditions

$$\mu(O_n[0]) = -1 \ \textit{for}\ n \geq 0$$

and

$$\mu(O_n[n]) = (-1)^{n+1} \ \textit{for}\ n \geq 1.$$

We have two different proofs of this recurrence. The first applies the interpretation of the Möbius function as counting certain chains in a poset. The second is a direct application of the summation formula for $\mu(O_n[k])$ and induction.

We are now ready to prove

Corollary 2.6 *O_n is an Eulerian poset.*

Proof. Since O_n is a finite graded poset with $\hat{0}$ and $\hat{1}$, it is enough to show

$$\mu_{O_n}(x, y) = (-1)^{\ell(x,y)} \quad \text{for all } x \le y \text{ in } O_n \tag{1}$$

in order to conclude O_n is Eulerian. Once we recognize that for $y = \hat{1}$

$$[x, \hat{1}] \cong O_{\ell(x,\hat{1})-1},$$

and for $y \neq \hat{1}$

$$[x, y] \cong B_{\ell(x,y)},$$

where $B_{\ell(x,y)}$ is the Boolean algebra on $\ell(x, y)$ elements, it becomes a straightforward exercise to construct the isomorphisms and to show (1) holds. □

Since O_n is an Eulerian poset, we can take advantage of its duality properties. In particular, we are now able to apply the following result from [11, Prop. 3.14.5]:

Proposition 2.7 *Let P be an Eulerian poset of rank n, and let Q be any subposet of P containing $\hat{0}$ and $\hat{1}$. Set $\overline{Q} = (P \setminus Q) \cup \{\hat{0}, \hat{1}\}$. Then*

$$\mu_Q(\hat{0}, \hat{1}) = (-1)^{n-1}\mu_{\overline{Q}}(\hat{0}, \hat{1}).$$

This proposition implies that for any lower order ideal $\mathcal{A}$ with $\hat{0}$ and $\hat{1}$ in O_n,

$$\mu_{O_n}(\mathcal{A}) = \mu_{Q_n}(\overline{\mathcal{A}}^*)$$

up to a sign. (Here $\overline{\mathcal{A}} = (O_n \setminus \mathcal{A}) \cup \{\hat{0}, \hat{1}\}$).

In order to state a special case of the theorem, we make two more definitions. We say a sequence $a_0, a_1, \ldots, a_n$ of real numbers is *unimodal* if for some k, $0 \le k \le n$, we have

$$a_0 \le \ldots \le a_k \ge \ldots \ge a_n.$$

Similarly, a sequence is *strictly unimodal* if we replace the inequalities by strict inequalities in the definition of unimodal. We now state

Lemma 2.8 *For fixed $n \geq 2$, $|\mu(O_n[k])|$ is strictly unimodal with unique maximum occurring when $k = \lfloor \frac{2n}{3} \rfloor$.*

Proof. We proceed by induction on n. The lemma is easily checked to be true for $n = 2, 3$, and 4. Let k^* correspond to the index k for which $|\mu(O_n[k])|$ is a maximum. By Corollary 2.5, $|\mu(O_n[k])|$ satisfies the recurrence

$$|\mu(O_n[k])| = 2|\mu(O_{n-1}[k-1])| + |\mu(O_{n-1}[k])|. \tag{2}$$

The main theorem in [8] gives conditions for a triangular array of non-negative integers satisfying a recurrence to be log concave. Applying [8, Theorem 1], we immediately see $|\mu(O_{n+1}[k])|$ is log concave in k, hence unimodal. This fact, coupled with the recurrence in Corollary 2.5 and the induction hypothesis for the n^{th} row of $O_n[k]$'s, enables us to pinpoint the index k corresponding to the maximum $|\mu(O_{n+1}[k])|$ to one of two possibilities. A case-by-case argument using the equivalence class modulo 3 of n and applying the summation formula for $\mu(O_n[k])$ gives the result.

We indicate the proof for $n \equiv 1(3)$ below. For ease in notation, let

$$\begin{aligned} a_k &= |\mu(O_n[k])| \\ b_k &= |\mu(O_{n+1}[k])|. \end{aligned}$$

Since $n + 1 \equiv 2(3)$, we wish to show

$$b_{k^*+1} > b_{k^*}.$$

Applying the recurrence (2) to the above, it suffices to show

$$2a_{k^*} + a_{k^*+1} > 2a_{k^*-1} + a_{k^*},$$

i.e.

$$a_{k^*} > 2a_{k^*-1} - a_{k^*+1}.$$

The summation formulas given in Corollary 2.3 and the fact $n \equiv 1(3)$ enable us to rewrite the above as

$$\begin{aligned} \sum_{j=0}^{k^*} \binom{n}{j}(-2)^j &> -2\sum_{j=0}^{k^*-1} \binom{n}{j}(-2)^j + \sum_{j=0}^{k^*+1} \binom{n}{j}(-2)^j \\ &= -\sum_{j=0}^{k^*-1} \binom{n}{j}(-2)^j + \sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j. \end{aligned} \tag{3}$$

In terms of the a_k's, equation (3) says

$$a_{k^*} > a_{k^*-1} + \sum_{j=k^*}^{k^*+1} \binom{n}{j} (-2)^j.$$

By the induction hypothesis we have $a_{k^*} > a_{k^*-1}$. If we can show $\sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j$ is negative, then we will be finished. Write n as $3m+1$. Then $k^* = \lfloor \frac{2n}{3} \rfloor = 2m$. Thus

$$\begin{aligned} \sum_{j=k^*}^{k^*+1} \binom{n}{j}(-2)^j &= \binom{3m+1}{2m}(-2)^{2m} + \binom{3m+1}{2m+1}(-2)^{2m+1} \\ &= \frac{(3m+1)!}{(2m)!\, m!}(-2)^{2m}\left[\frac{-1}{(m+1)(2m+1)}\right], \end{aligned}$$

which is negative, as desired.

The argument for the remaining two cases proceeds similarly. The exception is that the $n \equiv 0(3)$ case requires the induction hypothesis for both the n^{th} and $n-1^{st}$ rows, plus the result

$$a_{k^*-1} = a_{k^*+1} \quad \text{for } n \equiv 2(3), \tag{4}$$

which is found in the course of proving the lemma for the $n \equiv 2(3)$ case. □

Suppose we are given elements S all of the same rank in a poset. Since we are working with lower order ideals, we would naturally like to be able to estimate the number of elements in the poset covered by S. (Recall the Möbius function can be defined inductively in terms of covering relations.) More formally, we define the *shadow* of a subset S of rank r in O_n by

$$\Delta(S) = \{B \in O_n(r-1) : \ B \subseteq A \text{ for some } A \in S\}.$$

We then have

Lemma 2.9 *(Shadow Lemma for O_n) If $S \subseteq O_n(r)$, where $r \geq \frac{2n+2}{3}$, then $|\Delta(S)| \geq |S|$ with equality only when $n \equiv 2(3)$ and $S = O_n(\frac{2n+2}{3})$.*

Proof. For the first half of the proof we utilize an edge-counting argument. Consider the bipartite graph G formed in the Hasse diagram of O_n by S and $\Delta(S)$. Each vertex $A \in S$ has degree r, so the graph G has exactly $r|S|$ edges. Also, every vertex $B \in \Delta(S)$ has degree at most $2(n-r+1)$, giving an edge count of at most $2(n-r+1)|\Delta(S)|$. Thus when $r > \frac{2n+2}{3}$, the first part of the lemma follows.

If $n \equiv 2(3)$ and $r = \frac{2n+2}{3}$, then the above argument works as long as some vertex in $\Delta(S)$ does not have degree $\frac{2n+2}{3}$. If every vertex $B \in \Delta(S)$ has this degree, then in O_n the vertices of $\Delta(S)$ are only adjacent to vertices of S (and vice-versa). Hence if $S \subset O_n(r)$ (strict containment), this would contradict shellability of the chain complex of O_n. [2] [3] □

We similarly prove a shadow lemma for Q_n, the lattice of faces of the n-dimensional cube. We shall simply state it here:

Lemma 2.10 *(Shadow Lemma for Q_n) If $S \subseteq Q_n(r)$ where $r \geq \frac{n+4}{3}$, then $|\Delta(S)| \geq |S|$ with equality only when $n \equiv 2(3)$ and $S = Q_n(\frac{n+4}{3})$.*

Now we are ready to give a proof of Theorem 2.1. Let $\mathcal{F} \subseteq O_n$ be a lower order ideal with maximum $|\mu(\mathcal{F})|$ and let k be the maximum rank of all the elements in $\mathcal{F}$. We will first derive some expressions that will enable us to compare $\mu(\mathcal{F})$ with $\mu(\mathcal{F}[k-1])$ and $\mu(\mathcal{F}[k-2])$, yielding an upper bound for k. The Shadow Lemma for O_n and the following proposition enable us to do this:

Proposition 2.11 *[11, Lemma 3.14.4] Let P be a bounded poset and $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$. If $T \subseteq \max \bar{P}$ then*

$$\mu(P) = \mu(P \setminus T) - \sum_{x \in T} \mu(\hat{0}, x). \tag{5}$$

This result follows by counting the chains in P not containing elements of T and the chains in P containing elements of T. Proposition 2.11 is useful in the sense that it enables us to see how the Möbius function of a poset changes if we "peel off" some (or all) of its top elements.

Applying equation (5) to $P = \mathcal{F}$, $T = \mathcal{F}(k)$, and recalling $\mu(\hat{0}, x) = (-1)^k$ for $x \in O_n \setminus \hat{1}$ of rank k gives

$$\mu(\mathcal{F}) = \mu(\mathcal{F}[k-1]) - (-1)^k |\mathcal{F}(k)|,$$

i.e.

$$\mu(\mathcal{F}[k-1]) = \mu(\mathcal{F}) + (-1)^k |\mathcal{F}(k)|. \tag{6}$$

Similarly applying equation (5) to $P = \mathcal{F}[k-1]$, $T = \mathcal{F}(k-1)$, substituting for $\mu(\mathcal{F}[k-1])$ in equation (6) and solving for $\mu(\mathcal{F}[k-2])$ gives

$$\mu(\mathcal{F}[k-2]) = \mu(\mathcal{F}) - (-1)^k (|\mathcal{F}(k-1))| - |\mathcal{F}(k)|). \tag{7}$$

Since $\mathcal{F}$ is an ideal, $\Delta\mathcal{F}(k) \subseteq \mathcal{F}(k-1)$, implying

$$|\mathcal{F}(k-1))| - |\Delta\mathcal{F}(k)| \geq 0.$$

Suppose $k > \lceil\frac{2n}{3}\rceil$. By the Shadow Lemma 2.9 we know

$$|\Delta\mathcal{F}(k)| > |\mathcal{F}(k)|.$$

Hence

$$|\mathcal{F}(k-1)| - |\mathcal{F}(k)| > 0.$$

After considering all the possibilities for the sign of $\mu(\mathcal{F})$ and the parity of k, we see one of equations (6), (7) implies $|\mu(\mathcal{F})|$ is not a maximum, contradicting the fact that $\mathcal{F} \subseteq O_n$ is an ideal with maximum $|\mu(\mathcal{F})|$. Hence $k \leq \lceil\frac{2n}{3}\rceil$. In particular,

$$k = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0(3) \\ \frac{2n+1}{3}, & \text{if } n \equiv 1(3) \\ \frac{2n+2}{3}, & \text{if } n \equiv 2(3). \end{cases}$$

As we have previously remarked after Proposition 2.7, $\mu_{O_n}(\mathcal{F})$ equals $\mu_{Q_n}(\bar{\mathcal{F}}^*)$ up to a sign. We will now work with $\bar{\mathcal{F}}^*$ in Q_n to extract further information about the structure of $\mathcal{F}$. Define

$$\ell = \min\{|B| : \ B \in O_n \setminus \hat{\mathcal{F}}\}.$$

Using the Shadow Lemma for Q_n, we proceed as in the first part of the proof of this theorem. Let $\mathcal{G} = (\bar{\mathcal{F}}^*)$ and let $\bar{\ell}$ be the maximum size of a set in $\mathcal{G}$. As before, we apply equation (5) first to $P = \mathcal{G}, T = \mathcal{G}(\bar{\ell})$ and then to $P = \mathcal{G}[\bar{\ell}-1], T = \mathcal{G}(\bar{\ell}-1)$ to obtain equations resembling those of equations (6) and (7):

$$\mu(\mathcal{G}[\bar{\ell}-1]) = \mu(\mathcal{G}) + (-1)^{\bar{\ell}}|\mathcal{G}(\bar{\ell})| \tag{8}$$

$$\mu(\mathcal{G}[\bar{\ell}-2]) = \mu(\mathcal{G}) - (-1)^{\bar{\ell}}(|\mathcal{G}(\bar{\ell}-1)| - |\mathcal{G}(\bar{\ell})|). \tag{9}$$

(Here we are using the fact Q_n is Eulerian, so if $x \in Q_n$ of rank $\bar{\ell}$, then $\mu_{Q_n}(\hat{0}, x) = (-1)^{\bar{\ell}}$.) Since $\mathcal{G}$ is an ideal, $\Delta\mathcal{G}(\bar{\ell}) \subseteq \mathcal{G}(\bar{\ell}-1)$, implying

$$|\mathcal{G}(\bar{\ell}-1)| - |\Delta\mathcal{G}(\bar{\ell})| \geq 0.$$

As before, we first suppose $\bar{\ell} > \frac{n+4}{3}$ and apply the Shadow Lemma for Q_n. Once we consider all the possibilities for the sign of $\mu(\mathcal{G})$ and the parity of $\bar{\ell}$, we see that one of equations (8), (9) implies $|\mu(\mathcal{G})|$ is not a maximum. Hence we must have $\bar{\ell} \leq \frac{n+4}{3}$. This says

$$\bar{\ell} \leq \begin{cases} \frac{n+3}{3}, & \text{for } n \equiv 0(3) \\ \frac{n+2}{3}, & \text{for } n \equiv 1(3) \\ \frac{n+4}{3}, & \text{for } n \equiv 2(3). \end{cases}$$

Via the map $r \longmapsto (n+1) - r$, we convert this rank information from Q_n to O_n to conclude

$$\ell \geq \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0(3) \\ \frac{2n+1}{3}, & \text{if } n \equiv 1(3) \\ \frac{2n-1}{3}, & \text{if } n \equiv 2(3). \end{cases}$$

To finish this argument, we reason in the following manner: for each equivalence class of n modulo 3, the bounds for k and ℓ will enable us to find rank-selected lower order ideal configurations containing $\mathcal{F}$ and contained in $\mathcal{F}$, respectively. As before, we will apply equation (5) to obtain expressions for the Möbius function of these two lower order ideal configurations, apply a parity and sign argument, and then the rank selection Lemma 2.8 to derive the required result.

By definition of k and ℓ, we have $k \geq \ell - 1$. We first consider the case $n \equiv 0(3)$. We have

$$k \geq \ell - 1 \geq \tfrac{2n}{3} - 1,$$

and from before

$$k \leq \tfrac{2n}{3},$$

implying

$$k = \tfrac{2n}{3} \text{ or } \tfrac{2n}{3} - 1.$$

For convenience, let $\ell^* = \frac{2n}{3}$. Then we have

$$O_n([\ell^* - 1]) \subseteq \mathcal{F} \subseteq O_n([\ell^*]),$$

where the first containment follows from the definition of ℓ and the second from the definition of k. Note that ℓ^* is always even, so by equation (5) we have

$$\mu(O_n[\ell^* - 1]) = \mu(\mathcal{F}) + |\mathcal{F}(\ell^*)|. \tag{10}$$

Letting $P = O_n[\ell^*]$, $T = O_n(\ell^*) \setminus \mathcal{F}(\ell^*)$ in equation (5) gives

$$\mu(O_n[\ell^*]) = \mu(\mathcal{F}) - |O_n(\ell^*) \setminus \mathcal{F}(\ell^*)|. \tag{11}$$

If $\mu(\mathcal{F}) > 0$, then equation (10) and the maximality of $|\mu(\mathcal{F})|$ imply $\mathcal{F}(\ell^*) = \emptyset$. Hence $k = \ell^* - 1$, implying

$$\begin{aligned} \mathcal{F} &= O_n[\ell^* - 1] \\ &= O_n[\lfloor \tfrac{2n}{3} \rfloor - 1], \end{aligned}$$

contradicting the rank selection Lemma 2.8. If $\mu(\mathcal{F}) < 0$, then equation (11) plus the maximality of $|\mu(\mathcal{F})|$ imply $O_n(\ell^*) = \mathcal{F}(\ell^*)$. Thus $k = \ell^*$, implying

$$\begin{aligned} \mathcal{F} &= O_n[\ell^*] \\ &= O_n[\lfloor \tfrac{2n}{3} \rfloor], \end{aligned}$$

as desired.

The cases $n \equiv 1(3)$ and $n \equiv 2(3)$ proceed similarly to that of $n \equiv 0(3)$. The bounds for k and l, plus the definitions of k and l allow us to narrow down k to one of two (or three, if $n \equiv 2(3)$) possibilities. As before, we complete the argument by applying Proposition 2.11 and the rank-selection Lemma 2.8. We thus conclude the extremal configuration occuring in Lemma 2.8 coincides with the extremal configuration for Theorem 1. □

3 Interval of Ranks

In this section we are interested in maximizing $|\mu(\mathcal{F})|$, where $\mathcal{F}$ runs over all intervals of ranks in O_n (i.e. $\mathcal{F}$ of the form $O_n[i, j]$). Recall by [10, Corollary 3.3] that if P is a finite poset whose chain complex can be lexicographically shelled and $R \subseteq$ ranks of P, then

$$\mu(P(R)) = (-1)^{|R|+1}\{\text{number of maximal chains with descent set } R\}.$$

In particular, if we let $\beta_n(R)$ denote $|\mu(O_n(R))|$, then for O_n we have

$$\begin{aligned} \beta_n(R) = \{&\text{number of permutations in the hyperoctahedral group } \mathcal{B}_n \\ &\text{with descent set } R\}. \end{aligned}$$

Thus, the question at hand is equivalent to maximizing the number of permutations in the hyperoctahedral group $\mathcal{B}_n$ with descent set $[i, j]$. The data for O_n strongly supports the following

Conjecture 3.1 *For fixed $n > 0$, $n \neq 2$, $\beta_n(R)$ achieves a unique maximum when*

$$R = [\lfloor \tfrac{2n+5}{5} \rfloor, \lfloor \tfrac{4n+1}{5} \rfloor].$$

For $n = 2$, the maxima occur when

$$R = [1,1] \text{ or } [2,2].$$

To prove Conjecture 3.1, first we fix $n > 0$ and form a triangular array of the $\beta_n[i,j]$'s. Its k^{th} row consists of the values

$$\beta_n[1, n-k+1], \beta_n[2, n-k+2], \ldots, \beta_n[k, n], \quad k = 1, \ldots, n.$$

We look at each row of this array (equivalently, we fix the length r of the interval of descent) and find the maximum value of $\beta_n[i, i+r]$. We have almost completed the proof of

Conjecture 3.2 *For fixed $0 \leq r \leq n-1$, the sequence*

$$\beta_n[1, 1+r],\ \beta_n[2, 2+r],\ \ldots,\ \beta_n[n-r, n]$$

is almost strictly unimodal. Its maxima occur at

$$\begin{array}{ll} \beta_n[\lfloor \frac{2n-2r+1}{3} \rfloor, \lfloor \frac{2n+r+1}{3} \rfloor] & \text{for } n \not\equiv 2(3) \text{ or } r \neq 0 \\ \beta_n[\lfloor \frac{2n+1}{3} \rfloor, \lfloor \frac{2n+1}{3} \rfloor] = \beta_n[\lfloor \frac{2n+4}{3} \rfloor, \lfloor \frac{2n+4}{3} \rfloor] & \text{for } n \equiv 2(3) \text{ and } r = 0. \end{array}$$

(Here by *almost strictly unimodal* we mean a sequence that is strictly unimodal or is of the form

$$a_1 < a_2 < \ldots < a_k = a_{k+1} > a_{k+2} > \ldots > a_n.)$$

We next look at the subsequence corresponding to the maxima in each row that we found in Conjecture 3.2. The maximum of this subsequence will give the overall maximum in the main Conjecture 3.1. For completeness, we state this last

Conjecture 3.3 *For fixed $n > 0$, the sequence*

$$\beta_n[\emptyset], \beta_n[\lfloor \tfrac{2n+1}{3} \rfloor, \lfloor \tfrac{2n+1}{3} \rfloor], \beta_n[\lfloor \tfrac{2n-1}{3} \rfloor, \lfloor \tfrac{2n+2}{3} \rfloor], \beta_n[\lfloor \tfrac{2n-3}{3} \rfloor, \lfloor \tfrac{2n+3}{3} \rfloor], \ldots, \beta_n[1, n]$$

is strictly unimodal with maximum

$$\beta_n[\lfloor \tfrac{2n+5}{5} \rfloor, \lfloor \tfrac{4n+1}{5} \rfloor].$$

We expect the proof of Conjecture 3.3 to utilize the same basic techniques as in the proof of Conjecture 3.2.

4 Future Research

A natural question to ask is what sort of posets P will the extremal configuration which maximizes $|\mu(\mathcal{F})|$ over all lower order ideals $\mathcal{F}$ in P correspond to a rank-selected lower order ideal. We conjecture this is the case for Eulerian posets whose adjacent ranks are biregular and the (cardinality of its) ranks form a unimodal sequence.

We are in the process of forming the conjecture for the family of arbitrary rank selections. Once this case is complete, our research will address analogous questions for $L_n(q)$, the lattice of subspaces, and $I_n(q)$, the poset of isotropic subspaces.

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