Algebras of multiindexed infinite matrices and the transform approach to the Umbral Calculus

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1 Introduction

In the last two decades there has been a growing interest for the study of formal calculus of operators and Umbral Calculus. One of the main objectives has been the construction of rigorous theories where the formal methods used since the past century may be explained in a unified way. Such theories may also be considered as foundations for several topics, such as combinatorial enumeration, polynomial sequences, combinatorial identities, difference equations and special functions.

The work of Rota and his collaborators in the seventies has had a decisive role in the renewal of the interest on the Umbral Calculus. See Rota, Kahaner, and Odlyzko [8], and Roman and Rota [7]. There are now several approaches to the construction of theories of Umbral Calculi. The common ingredients are formal power series, polynomial sequences, linear functionals, formal differential operators, Hopf algebras, and several kinds of duality. See for example Garsia and Joni [3], Cigler [2], Joyal [5], Roman [6], Barnabei, Brini, and Nicoletti [1], and Ueno [10].

In this paper we present a theory of Umbral Calculi based on the study of algebras of multiindexed infinite matrices over a field. These are large algebraic structures which contain isomorphic images of algebras of formal Laurent series and groups of linear operators on spaces of formal series.

Using certain algebraic analogues of the Laplace and Borel transforms which act on spaces of formal series, we construct transformations on spaces of linear operators. The images under such transformations of certain algebras and groups of multimatrices constitute our sets of umbral operators.

This approach was sketched in our paper [12], where we studied first some algebras of linear operators on spaces of formal Laurent series, provided with an indefinite inner product, and then, using duality and the Borel transform we obtained some groups of umbral operators and several results about polynomial families of binomial type in several variables. One advantage of this approach is that we can separate the study of the core of the theory, which is the study of algebraic structures of general interest such as algebras and groups of linear operators on spaces of formal Laurent series, from the study of particular instances of the Umbral Calculi. In this way we can identify the fundamental ideas and results, and compare their consequences in different versions of the Umbral Calculus. In particular, we consider the usual binomial Umbral Calculus and the Newtonian Umbral Calculus, where divided differences play a basic role. The Newtonian Calculus has been studied by Roman [6], Hirschhorn and Raphael [4], and Verde-Star [13].

2 Multimatrices and formal Laurent series

Let r be a fixed positive integer. The elements of $\mathbb{Z}^r$ will be called multiindices and will be denoted in the form $n = (n_1, n_2, \ldots, n_r)$. We use letters like m, n and k to denote multiindices .

We suppose that $\mathbb{Z}^r$ is equipped with a partial order $\leq$ compatible with the abelian group structure, which makes $\mathbb{Z}^r$ an order complete lattice and such that the order intervals

$$[k, m] = \{n \in \mathbb{Z}^r : k \leq n \leq m\}, \qquad k, m \in \mathbb{Z}^r \tag{2.1}$$

are finite sets or empty. The set $C = \{n \in \mathbb{Z}^r : n \geq 0\}$ is the positive cone of the partial order $\leq$, and $k \leq n$ is equivalent to $n - k \in C$. The usual order corresponds to the cone $\mathbb{N}^r$. In [12] we used a family of orders that are useful to obtain general Lagrange inversion formulas. Some of the material that follows is based on [12], where the reader can find more details and some proofs that we omit here.

Let R be a commutative ring with unity. A multiindexed matrix A, or multimatrix, over R is an array $[a_{k,n}]$ of elements of R, where k and n run over all the multiindices. Addition of multimatrices is defined in the obvious way, but multiplication is not well defined for all pairs of multimatrices. In some cases there appear infinite sums of elements of R and we would need conditions to assure convergence. We prefer to work with rings of multimatrices for which all the sums needed to perform multiplication of multimatrices are finite.

We define the set $\mathcal{L}$ of lower multimatrices as the collection of all multimatrices $A = [a_{k,n}]$ for which the set $\{k - n : a_{k,n} \neq 0\}$ is minorized in $\mathbb{Z}^r$. The greatest lower bound of such set is denoted by $v(A)$ and called the index of A, or the vertex of the support of A.

It is clear that $\mathcal{L}$ is closed under addition. For multiplication we have the following. If $A = [a_{k,n}]$ and $B = [b_{k,n}]$ are in $\mathcal{L}$ then the matrix product $AB = [c_{k,n}]$ is defined as usual by

$$c_{k,n} = \sum_m a_{k,m} b_{m,n}. \tag{2.2}$$

Note that this sum is always finite, since the summand may be nonzero only when m belongs to the order interval $[n + v(B), k - v(A)]$. From this we see that AB is also in $\mathcal{L}$ and $v(AB) \geq v(A) + v(B)$.

The ring $\mathcal{L}$ contains several classes of multimatrices that have certain regularities in their structure. An important example is the ring $\mathcal{F}$ of Toeplitz multimatrices which consist of elements $A = [a_{k,n}]$ such that $a_{k+m,n+m} = a_{k,n}$ for all multiindices k, n and m. This means that A is constant along the 'diagonals'.

The ring $\mathcal{F}$ may be identified with the set $\mathcal{S}$ of all formal Laurent series of the form $f(z) = \sum_n f_n z^n$, where the coefficients are in R and

$$z^n = z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}, \qquad n \in \mathbb{Z}^r,$$

and such that there exists some multiindex $v(f)$ such that $f_n = 0$ whenever $n \not\geq v(f)$. The map that sends $f(z)$ to the multimatrix $S_f = [a_{k,n}]$, where $a_{k,n} = f_{k-n}$, is a ring isomorphism. It is the regular representation of the formal Laurent series in $\mathcal{L}$. Note that $S_f\, a(z) = f(z)a(z)$ for any series $a(z)$ in $\mathcal{S}$.

If we consider the coefficients of a formal Laurent series as an infinite 'column vector' then the elements of $\mathcal{L}$ act by multiplication on the left on

$\mathcal{S}$. We shall identify the multimatrices with their corresponding operators on $\mathcal{S}$.

For any multimatrix A in $\mathcal{L}$ and any multiindex m we identify the m-th column of A with the series

$$f_m(z) = \sum_n a_{n,m}\, z^n. \tag{2.3}$$

Then we can write

$$f_m(z) = A\, z^m, \qquad m \in \mathbb{Z}^r. \tag{2.4}$$

Let us note that, for the multimatrix S_a we have $S_a\, z^n = z^n a(z)$.

The rows of a multimatrix B may be considered as reversed formal Laurent series as follows. If k is a multiindex the k-th row of B corresponds to

$$g_k(z) = \sum_n b_{k,n}\, z^{n^*}, \tag{2.5}$$

where $n^* = -n-e$ and $e = (1,1,\ldots,1)$. Therefore the entries of the product $C = BA$ are given by

$$c_{k,m} = \sum_n b_{k,n} a_{n,m} = \mathrm{Res}(g_k\, f_m), \tag{2.6}$$

where

$$\mathrm{Res}(h) = \text{coeff. of } z^{-e} \text{ in } h \tag{2.7}$$

is the residue of h, for any series h. This is the motivation to define the indefinite inner product

$$\langle a(z), b(z)\rangle = \ \mathrm{Res}\ a(z)b(z), \qquad a, b \in \mathcal{S}. \tag{2.8}$$

We define an involution $*$ in the ring of multimatrices $\mathcal{L}$ as follows. If $A = [a_{k,n}]$ then

$$A^* = [a_{n^*,k^*}], \qquad k, n \in \mathbb{Z}^r. \tag{2.9}$$

The map $A \to A^*$ is obtained by reflection with respect to the 'diagonal' determined by the equation $k = n^*$, and it sends rows to columns and vicev-ersa.

It is easy to see that the set of formal Laurent series

$$\mathcal{H} = \{f(z) = \sum_{n\geq 0} f_n z^n : f_0 = 1\} \tag{2.10}$$

is a group under multiplication.

We define the set

$$\mathcal{G} = \{g = (g_1, g_2, \ldots, g_r) \in \mathcal{S}^r : g_i(z)/z_i \in \mathcal{H}, \quad 1 \le i \le r\}. \tag{2.11}$$

Then, for every g in $\mathcal{G}$ and every multiindex n, $g^n(z)/z^n$ is in $\mathcal{H}$, and hence we can define an operator T_g on $\mathcal{S}$ as follows

$$T_g f(z) = \sum_n f_n g^n(z) = f \circ g(z) = f(g(z)). \tag{2.12}$$

Note that the n-th column of T_g is $T_g\ z^n = g^n(z)$.

The representations of the composition operators T_g in $\mathcal{L}$ form a group of multimatrices that we identify with $\mathcal{G}$. The operation in $\mathcal{G}$ is substitution. The ring $\mathcal{L}$ also contains the matrix representations of linear differential operators of infinite order, with coefficients in $\mathcal{S}$.

The basic relationship between multiplication and composition operators is

$$T_g S_a = S_{a \circ g} T_g, \qquad g \in \mathcal{G},\ a \in \mathcal{S}, \tag{2.13}$$

and is verified by applying both sides to an arbitrary monomial z^n.

For any $f = (f_1, f_2, \ldots, f_r)$ in $\mathcal{S}^r$ the Jacobian matrix $Df = [D_i f_j]$ is a square matrix of order r over $\mathcal{S}$, and its determinant, denoted by Jf is in $\mathcal{S}$.

Now we present some of the basic results. The proofs are in [12].

Proposition 2.1 *For any g in $\mathcal{G}$ we have*

$$DT_g = R_{Dg} T_g D \quad \text{and } JT_g = S_{Jg} T_g J, \tag{2.14}$$

where D denotes the Jacobian matrix map from $\mathcal{S}^r$ to the matrices of order r over $\mathcal{S}$, R_{Dg} denotes multiplication on the right by Dg, and J is the Jacobian determinant map from $\mathcal{S}^r$ to $\mathcal{S}$.

Proposition 2.2 *For any f in $\mathcal{S}^r$ we have* Res $(Jf) = 0$, *and if g is in $\mathcal{G}$ then Jg is in $\mathcal{H}$.*

Proposition 2.3 *For any g in $\mathcal{G}$ we have*

$$T_g^* S_{Jg} T_g = I. \tag{2.15}$$

This identity is called the change of variables theorem because it is equivalent to the following result.

Proposition 2.4 *If g is an element of $\mathcal{G}$ and a and b are in $\mathcal{S}$ then*

$$\langle a, b\rangle = \langle a \circ g, b \circ g \; Jg\rangle. \tag{2.16}$$

Several forms of the Lagrange inversion formula can be obtained from the above propositions.

Let us define the set of multimatrices

$$\mathcal{M} = \{S_a \in \mathcal{L} : a_{v(a)} \text{ is invertible in } R\}. \tag{2.17}$$

It is clear that $\mathcal{M}$ is a group under multiplication.

Define $\mathcal{M}_0$ as the subset of $\mathcal{M}$ of all multimatrices S_a in $\mathcal{M}$ such that $v(a) = 0$. A simple computation shows that $\mathcal{M}_0$ is a subgroup of $\mathcal{M}$.

The group $\mathcal{MG} = \{S_aT_g\}$ is called the general Sheffer group of multimatrices . From (2.13) we see that $\mathcal{MG} = \mathcal{GM}$.

The n-th column of S_aT_g is the series $f_n(z) = a(z)g^n(z)$. Therefore, for any multiindex k we have the relation $f_{n+k}(z) = g^k(z)f_n(z)$. In particular, if $k = e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the i-th position, we get $f_{n+e_i}(z) = g_i(z)f_n(z)$ for $1 \leq i \leq r$. These equations describe a recurrence relation for the columns of S_aT_g.

Let A be the multimatrix whose k-th row is the reverse of the series Jg/ag^{k+e}. Then

$$\langle ag^n, \frac{Jg}{ag^{k+e}}\rangle = \text{Res}\,(g^{n-k-e}Jg) = \delta_{n,k}. \tag{2.18}$$

Here we used Prop. 2.2. Therefore A is the inverse of S_aT_g.

Note that the rows of A also satisfy a recurrence relation similar to the one satisfied by the columns of S_aT_g. This fact is a consequence of the equation $AS_aT_g = I$, and occurs in general for any pair of inverse multimatrices ; if one of them satisfies a recurrence relation by columns, the other one must satisfy a recurrence relation by rows, and vice versa.

Let $g = (g_1, g_2, \ldots, g_r)$ be an element of $\mathcal{G}$ such that each component series g_i is a polynomial and such that $Jg = 1$. Let $h = (h_1, h_2, \ldots, h_r)$ be the inverse of g under substitution, that is, $T_gT_h = I$. The jacobian conjecture of Keller says that in this case the components of h would be polynomials.

Since $Jg = 1$, Prop. 2.3 gives us $T_g^* T_g = I$ and hence $T_g^* = T_h$. This means that $h_i(z) = T_h z_i = T_g^* z_i$ and hence $h_i(z)$ is the reversed series of the row of T_g that corresponds to the multiindex

$$e_i^* = (-1, -1, \ldots, -1, -2, -1, \ldots, -1),$$

where the -2 is in the i-th coordinate. Therefore, in order to prove the jacobian conjecture one must show that each one of the rows of T_g corresponding to multiindices e_i^*, $1 \le i \le r$, has only a finite number of nonzero terms.

The multimatrix T_g satisfies a recurrence relation by columns, described by

$$T_g z^{n+e_i} = g_i(z) T_g z^n, \qquad n \in \mathbb{Z}^r, \quad 1 \le i \le r. \tag{2.19}$$

Since the $g_i(z)$ are polynomials, each column of T_g corresponding to a multiindex $n \ge 0$ has only finitely many nonzero entries.

If we consider T_g as a function defined on $\mathbb{Z}^r \times \mathbb{Z}^r$ which is the solution of a recurrence relation is several variables, with certain boundary conditions, then our problem consists in proving that the presence of a large region of $\mathbb{Z}^r \times \mathbb{Z}^r$ where the solution vanishes implies that the solution also vanishes on a relatively small, but unbounded region, contained in a set of r rows.

From the above discussion it is quite clear that the problem reduces to showing that certain coefficients in some 'nonpositive' powers of $g = (g_1, g_2, \ldots, g_r)$ must be zero.

In this partial difference approach to the jacobian conjecture the main difficulty is to translate the hypothesis $Jg = 1$ into properties of the recurrence relations.

3 Borel transforms and Umbral Calculi

In this section we consider $\mathbb{Z}^r$ with its usual partial order, corresponding to the cone $\mathbb{N}^r$.

Let $*$ denote an involution on $\mathbb{Z}^r$. A generalized Borel transform is a map B defined on some subset of the ring $\mathcal{S}$ of formal Laurent series by

$$B z^n = b_{n^*} z^{n^*}, \qquad n \in \mathbb{Z}^r, \tag{3.1}$$

and extended by linearity, where the coefficients b_k are elements of the ring R. Each choice of the involution $*$ and the family of coefficients b_k determines a particular instance of the transform B. There are two kinds of transforms, the regular transforms and the truncated transforms. For the first kind the coefficients b_k are invertible elements in R for all the multiindices k, and B is an invertible operator. In the truncated case b_k is invertible for $k \in \mathbb{N}^r$ and $b_k = 0$ for $k \notin \mathbb{N}^r$. In this case B may only have a one sided inverse.

We will use the involution $n^* = -n - e$ for $n \in \mathbb{Z}^r$. If $b_k = 1$ for all k then $B : \mathcal{S} \to \mathcal{S}'$ where $\mathcal{S}'$ is the ring of reversed formal Laurent series of the form

$$a(z) = \sum_{n \geq v(a)} a_n z^{n^*} \tag{3.2}$$

and B is an isomorphism of rings.

Let B be a truncated Borel transform. Then $B : \mathcal{S} \to \mathcal{P}$ where $\mathcal{P}$ is the usual space of polynomials in the variables $z_1, z_2, \ldots, z_r$ with coefficients in R. Since b_k is invertible for $k \in \mathbb{N}^r$ we can define the map $B' : \mathcal{P} \to \mathcal{S}$ as follows

$$B' z^n = b_n^{-1} z^{n^*}, \qquad n \in \mathbb{N}^r. \tag{3.3}$$

Note that $B'B = I$ on $\mathcal{P}$, that is, B' is a left inverse of B. The map B induces a transformation $A \to A^{\#}$, called the operator transform, from $\mathcal{L}$, considered as a ring of operators on $\mathcal{S}$, to the set of linear operators on $\mathcal{P}$, and defined as follows

$$A^{\#} = BA^*B'. \tag{3.4}$$

The operator transform is an antihomomorphism of rings, due to the presence of the involution $*$ of $\mathcal{L}$ in (3.4).

Let $x_1, x_2, \ldots, x_r$ be commuting indeterminates that also commute with the variables z_i. We define the formal power series in x and z

$$K(x, z) = \sum_{n \in \mathbb{N}^r} b_n x^n z^n. \tag{3.5}$$

$K(x, z)$ is called the kernel function of the operator transform. It is symmetric in x and z, and may be seen as a formal series in x with polynomial coefficients in z, or the other way around.

Proposition 3.1 *If A is an element of the ring $\mathcal{L}$ then*

$$A_x K(x, z) = A_z^{\#} K(x, z), \tag{3.6}$$

where the subindices of the operators indicate the variable with respect to which they are acting.

The proof is a direct computation.

Suppose now that the ring R is the set of complex numbers. If $b_k = 1/k!$ for $k \in \mathbb{N}^r$, where $k! = k_1!k_2!\cdots k_r!$, and $b_k = 0$ for $k \notin \mathbb{N}^r$. This is the Borel transform that we use in [12], it is related to the usual umbral calculus of families of polynomials of binomial type. The kernel function is $e^{x\dot{z}}$. The operator transform has the following properties

$$(S_{z^n})^{\#} = D^n, \qquad \text{and} \qquad (D^n)^{\#} = S_{z^n}, \qquad n \geq 0. \tag{3.7}$$

Therefore it sends multiplication operators into linear differential operators with constant coefficients and vice versa.

The image of the set of multiplication operators S_a such that $v(a) \geq 0$ is the set of shift invariant operators, which are of the form

$$a(D) = \sum_{n \geq 0} a_n D^n, \qquad a \in \mathcal{S},\ v(a) \geq 0. \tag{3.8}$$

The group of composition operators $\{T_g : g \in \mathcal{G}\}$ is mapped onto the group of normalized umbral operators $\{U_g = T_g^{\#} : g \in \mathcal{G}\}$. These have the following property

$$U_g^{-1} a(D) U_g = a \circ g(D), \qquad a \in \mathcal{S},\ v(a) \geq 0,\ g \in \mathcal{G}. \tag{3.9}$$

Each g in the group $\mathcal{G}$ determines a family of polynomials

$$p_k(z) = U_g^{-1} z^k, \qquad k \geq 0, \tag{3.10}$$

called the basic polynomial family of g. It is a family of binomial type. In [12] we obtained several explicit expressions for the p_k. For exemple,

$$p_n(z) = n! B(g^{n^*}(z) J g(z)), \qquad n \geq o. \tag{3.11}$$

The image under the operator transform of the group $\mathcal{MG}$ is the group of normalized Sheffer operators.

Let us consider now the case where $b_k = 1$ for $k \geq 0$ and $b_k = 0$ for $k \notin \mathbb{N}^r$. For the sake of simplicity we consider the one variable case, that is, $r = 1$. Here the kernel function is

$$K(x, z) = \sum_{n \geq 0} x^n z^n, \tag{3.12}$$

and $(S_z)^\# = L$, $D^\# = R$, are the left and right shift operators on the space of polynomials, defined by $Lz^k = z^{k-1}$ if $k \geq 1$ and $Lz^0 = 0$, $Rz^k = (k+1)z^{k+1}$ for all $k \geq 0$.

For any number x and any polynomial p we have

$$L\ K(x, L)p(z) = \frac{p(x) - p(z)}{x - z}. \tag{3.13}$$

This last expression is a divided difference of p. It is a symmetric polynomial in x and z. The reader is referred to [13], where we studied the polynomial sequences generated by the umbral operators in this case, using the generating function approach .

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