A class of words, and combinatorial structures of Josephus permutations and a cyclic tournament

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1 Introduction

Read-Corneil [11] and Klin-Poschel [6] reported that there had been no good condition for the graph isomorphism. Also, Ádám [1] presented a characterization problem of directed graphs with a cyclic property. Some papers (cf.[10]) discussed the Ádám's problem by studying the automorphism groups of graphs.

The *Josephus permutations* $J_n^N(j) := jn \pmod N$ are special permutations in the *set* S_N *of permutations of degree* N but have a long history since the first century A.D. (cf.[5, pages 121-128]).

This paper deals with combinatorial structures in a new class of words, called words of class $\mathcal{D}$ (cf. Definition 2.1.1). The structures shall be applied to a characterization of the Josephus permutations in the set S_N of permutations of degree N and an isomorphism problem of a special graph, called a cyclic tournament.

In Theorem 1 in Chapter 2, §1, any word ω of class $\mathcal{D}$ is uniquely represented by

$$\omega = \mathcal{L}(S, S^*)(01),$$

where S and S^*(cf. Definition 2.1.2) are dual substitutions.

Chapter 2,§2 contains main results in this paper. This section gives a totally order, called ω-order, depending on each word ω of class $\mathcal{D}$. Theorem 2 gives a transformation of the order by the substitutions S and S^*, and determines the order. We shall give a relation between the transformation of the order and perfect shuffles (as for the definition and the history, see [3]).

Chapter 3 gives two applications of results in Chapter 2.

One in §1 is to characterize the Josephus permutations in S_N. Each word of class $\mathcal{D}$ is realized as the up-down symbol(cf.[4]) of each Josephus permutation. This can be regarded as an enumerative aspect of the well-known Euclidean algorithm.

Another one in §2 is to give a criterion wether any cyclic tournament is isomorphic to a given cyclic tournament or not. For this criterion, we shall use only words of class $\mathcal{D}$ and even length.

Let us explain the mathematical terminology in this paper.

The terminology on *words* is used as in [7]. Unless we specify the set A^* of words, the terminology *"words"* are used as it in $\{0,1\}^*$. Set

$$\omega = a_0 a_1 \dots a_n; \quad \omega' = b_0 b_1 \dots b_m, \quad a_i, b_j \in \{0,1\}, i, j = 0, 1, \dots$$

The *intersection number* $< \omega \mid \omega' >$ of ω and ω' is

$$< \omega \mid \omega' >= \sum_{i,j} < a_i \mid b_j >,$$

where $< a_i \mid b_j >= \delta_{a_i b_j}$ (Kroneker delta).

The *subword* $\omega_{i,j}$ of ω is

$$\omega_{i,j} = a_i a_{i+1} \dots a_j.$$

For the particular $\omega_{0,j}$, set

$$\omega_j = \omega_{0,j}.$$

The *dual word* ω^* of ω is

$$\omega^* = \overline{a_n}\,\overline{a_{n-1}} \dots \overline{a_0},$$

where $\overline{a_i}$ is not a_i in $\{0,1\}$

2 A class of words and a binary relation on each word in the class

2.1 Combinatorial structures of a class of words

In this section, let us introduce a new class of words, called of class $\mathcal{D}$(cf. Definition 2.1.1). So each word of class $\mathcal{D}$ shall be uniquely represented by a leaf in the binary tree (cf. Figure 1), which is generated by two simple substitutions (cf. Definition 2.1.2).

Definition 2.1.1 *Let ω be the word*

$$\omega = a_0 a_1 \dots a_n.$$

The word ω is called of class $\mathcal{D}$ if ω satisfies the following:

(*i*) $a_0 = 0, a_n = 1$ *and* $a_i = a_{n-i}$, $i = 1, 2, \dots n-1$;

(*ii*) *there exists ϵ in the set $\{0,1\}$such that for any i and j,* $0 \le i \le j \le n$,

$$< \omega_{i,j} \mid 1 >=< \omega_{j-i} \mid 1 > + \epsilon. \tag{2.1.1}$$

Each of the above conditions (i) and (ii) gives the following properties of words.

Proposition 2.1.1 *Let ω be any word to satisfy the condition (i) of class $\mathcal{D}$. Then ω is the primitive word.*

Proposition 2.1.2 *Let ω be any word to satisfy the condition (ii) of class $\mathcal{D}$. Set*

$$\omega = 0^{k_0}1^{l_0}0^{k_1}1^{l_1}\cdots, \qquad k_i, l_j > 0, i, j = 0, 1, \ldots.$$

Then:

$$(i). \quad \textit{If} \quad k_0 \geq 2, \quad k_i = k_0 - 1 \textit{ or } k_0 \textit{ , and } l_j = 1;$$

$$(ii). \quad \textit{If} \quad k_0 = 1, \quad k_i = 1 \textit{ and, } l_j = l_0 \textit{ or } l_0 + 1, \qquad i, j = 0, 1, \ldots.$$

Corollary *Let ω be the word in Proposition 2.1.2. Then :*

$$\textit{If} \quad k_0 \geq 2, \quad \omega \textit{ is in } \{0, 01\}^*.$$

$$\textit{If} \quad k_0 = 1, \quad \omega \textit{ is in } \{1, 01\}^*.$$

Let us introduce two sbstitutions S and S^*, which are fundamental operations in this paper.

Definition 2.1.2 *The subsitutitons S and S^* are defined on $\{0,1\}^*$ as follows:*

$$S: \begin{cases} 0 \to 0 \\ 1 \to 01, \end{cases} \qquad S^*: \begin{cases} 0 \to 01 \\ 1 \to 1. \end{cases}$$

The inverse substitutions of S and S^* are denoted by $S^{-1}$and $S^{*^{-1}}$, respectively.

The operations S and S^* are each other's dual substitutions in the following sense.

Proposition 2.1.3 *Let ω and ω' be any words. Then:*

$$(i). \quad (S(\omega))^* = S^*(\omega^*) \textit{ and } (S^*(\omega))^* = S(\omega^*);$$

$$(ii). \quad < S(\omega) \mid \omega' > = < \omega \mid S^*(\omega') > .$$

The following lemma is the key to combinatorial structures on words of class $\mathcal{D}$.

Lemma 2.1.4 *Let ω be a word to satisfy the condition (ii) of class $\mathcal{D}$. Let i, j and ϵ in (2.1.1) be any fixed. Set the subscripts i' and j' of $S^{(*)}(\omega)$ corresponding to i and j such that*

$$S^{(*)}(\omega_{ij}) = S^{(*)}(\omega)_{i'j'}, \tag{2.1.2}$$

where $S^{(*)}$ is S or S^*.

Then we have the following:

$$(1) \quad < S^{(*)}(\omega)_{i'j'} \mid 1 >=< S^{(*)}(\omega)_{j'-i'} \mid 1 > +\epsilon;$$

$$(2) \quad < S^{(*)}(\omega)_{i',j'+1} \mid 1 >=< S^{(*)}(\omega)_{j'+1-i'} \mid 1 >$$

$$\textit{if } a_{j+1} = \begin{cases} 1 & \textit{for } S^{(*)} = S \\ 0 & \textit{for } S^{(*)} = S^* \end{cases};$$

$$(3) \quad < S^{(*)}(\omega)_{i'+1,j'} \mid 1 >=< S^{(*)}(\omega)_{j'-i'-1} \mid 1 > +1$$

$$\textit{if } a_i = \begin{cases} 1 & \textit{for } S^{(*)} = S \\ 0 & \textit{for } S^{(*)} = S^* \end{cases};$$

$$(4) \quad < S^{(*)}(\omega)_{i'+1,j'+1} \mid 1 >=< S^{(*)}(\omega)_{j'-i'} \mid 1 > +\epsilon$$

$$\textit{if } a_i = a_{j+1} = \begin{cases} 1 & \textit{for } S^{(*)} = S \\ 0 & \textit{for } S^{(*)} = S^*. \end{cases}$$

Remark. *Let $S^{(*)}$ be S^{-1} or $S^{*^{-1}}$ in Lemma 2.1.4. If only (2.1.2) and $S^{(*)}(\omega)$ are defined, the equation (1) in Lemma 2.1.4 holds.*

Example.

$$\begin{array}{rccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \omega = & 0 & 0 & 1 & 0 & 1 & 0 & 1; \end{array} \qquad \text{word of class } \mathcal{D}.$$

$$\begin{array}{rcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ S(\omega) = & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1. \end{array}$$

$$S(\omega_{23}) = S(\omega)_{24} \qquad < \omega_{23} \mid 1 >=< \omega_1 \mid 1 > +1.$$

$$< S(\omega)_{24} \mid 1 > \;=\; < S(\omega)_2 \mid 1 > +1.$$

$$< S(\omega)_{25} \mid 1 > \;=\; < S(\omega)_3 \mid 1 > .$$

$$< S(\omega)_{34} \mid 1 > \;=\; < S(\omega)_1 \mid 1 > +1.$$

$$< S(\omega)_{35} \mid 1 > \;=\; < S(\omega)_2 \mid 1 > +1.$$

The following theorem is a main result in this section, which characterizes the set of all words of class $\mathcal{D}$ in $\{0,1\}^*$.

Theorem 1 *The word ω is of class $\mathcal{D}$ if and only if the word ω has the representation*

$$\omega = \mathcal{L}(S, S^*)(01), \tag{1.1.3}$$

where $\mathcal{L}(S, S^*)$ is a word in $\{S, S^*\}^*$.

Then, the representation (1.1.3) is unique.

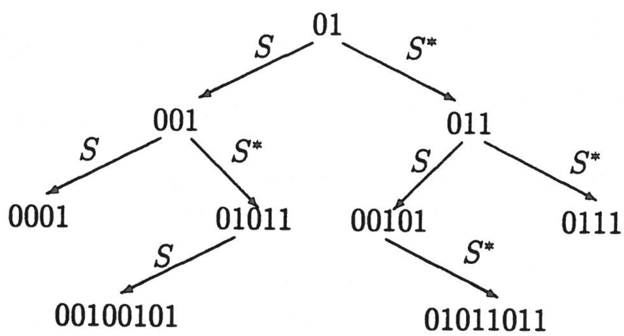


Figure 1

Corollary *Let ω be the word of class $\mathcal{D}$ with (1.1.3). Then the dual word ω^* is given by*

$$\omega^* = \mathcal{L}(S^*, S)(01).$$

Hence the dual word ω^* is of class $\mathcal{D}$.

2.2 A binary relation on each word of class $\mathcal{D}$

In this section, let us introduce a binary relation on each set of subscripts depending on each word of class $\mathcal{D}$. The binary relation becomes a totally order and gives a crucial rule in Chapter 3. By the substitutions S and S^*, the transformation of the relation shall be given in Lemma 2.2.2 and Theorem 2.

In this section, let ω be any fixed word of class $\mathcal{D}$ such that

$$\omega = a_0 a_1 \cdots a_n.$$

To each $a_j, j = 0, 1, \ldots, n$ in the word ω, let an element $< j >$ uniquely correspond. Set

$$P = \{< 0 >, < 1 >, \ldots, < n >\}.$$

Let us introduce a binary relation in the set P.

Definition 2.2.1 *The set P is called the ω-ordered set with the binary relation $\prec_\omega$ (shortly, ω-ordered set) if any $<i>$ and $<j>$ in the set P have the relation*

$$<i> \succ_\omega <j> \ \ or \ \ <i> \prec_\omega <j>$$

,and the relation for $<i>$ and $<j>$, $0 \le i < j \le n$, is given by the following:

$$\begin{array}{ccccccc} <i> & \succ_\omega & <j> & \Longleftrightarrow & <\omega_{i,j-1}\,|\,1> & = & <\omega_{j-i-1}\,|\,1>; \\ <i> & \prec_\omega & <j> & \Longleftrightarrow & <\omega_{i,j-1}\,|\,1> & = & <\omega_{j-i-1}\,|\,1>+1. \end{array}$$

We use the mathematical term "*ordered* " in Definition 2.2.1 by the following.

Proposition 2.2.1 *Let P be the ω-ordered set . Then P is a totally ordered set.*

The following gives a transformation of the order by the substitutions S and S^*, and is the key to determine the order of elements in the set P.

Lemma 2.2.2 *Fix any subscripts i and j of the subword $\omega_{i,j}$ of ω. Let the subscripts i' and j' of the word $S^{(*)}(\omega)$ correspond to the subscripts i and j such that*

$$S^{(*)}(\omega)_{i',j'} = S^{(*)}(\omega_{i,j}),$$

where $S^{(*)}$ is the substitution S or S^*.
Then we have the following:

$$\begin{array}{llllllll} (1) & <i'> & \begin{array}{c}\succ_{S^{(*)}(\omega)} \\ (\prec_{S^{(*)}(\omega)})\end{array} & <j'+1> & if & <i> & \begin{array}{c}\succ_\omega \\ (\prec_\omega)\end{array} & <j+1>, \ respectively; \\ (2) & <i'> & \succ_{S^{(*)}(\omega)} & <j'+2> & if & a_{j+1} & = & \left\{\begin{array}{ll} 1 & for\ S^{(*)} = S \\ 0 & for\ S^{(*)} = S^* \end{array}\right. ; \\ (3) & <i'+1> & \prec_{S^{(*)}(\omega)} & <j'+1> & if & a_i & = & \left\{\begin{array}{ll} 1 & for\ S^{(*)} = S \\ 0 & for\ S^{(*)} = S^* \end{array}\right. ; \\ (4) & <i'+1> & \begin{array}{c}\succ_{S^{(*)}(\omega)} \\ (\prec_{S^{(*)}(\omega)})\end{array} & <j'+2> & & & & \end{array}$$

$$if\ a_i = a_{j+1} = \left\{\begin{array}{ll} 1 & for\ S^{(*)} = S \\ 0 & for\ S^{(*)} = S^* \end{array}\right. \ and\ <i> \begin{array}{c}\succ_\omega \\ (\prec_\omega)\end{array} <j+1>, \ respectively.$$

The iteration $S^{*^m} S$ $(m \ge 0)$ of substitutions is the following:

$$S^{*^m} S : \left\{\begin{array}{lll} 0 & \to & \overbrace{01\cdots1}^{m+1} \\ 1 & \to & \underbrace{01\cdots1}_{m+2}. \end{array}\right.$$

Set

$$S^{*^m} S(\omega) = b_0 b_1 \cdots b_N.$$

The transformation of the ω-order by the substitution $S^{*^m}S$ *perfectly shuffles* the order as follows:

Corollary *In the word $S^{*^m}S(\omega)$, let the letters a_i and a_{j+1} in the word ω be transformed into*

$$S^{*^m}S(a_i) \;=\; b_{i'}b_{i'+1}\cdots b_{i''}$$

and

$$S^{*^m}S(a_{j+1}) \;=\; b_{j'}b_{j'+1}\cdots b_{j''}.$$

Then if in the ω-order,

$$<i> \;\underset{(\prec_\omega)}{\succ_\omega}\; <j+1>,$$

in the $S^{*^m}S(\omega)$-order, we have,respectively, the following:

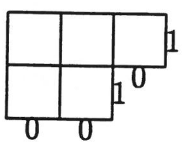
$$\begin{aligned} <i'+k> \;&\underset{(\prec_{S^{(*)}(\omega)})}{\succ_{S^{(*)}(\omega)}}\; <j'+k> \\ &\underset{(\prec_{S^{(*)}(\omega)})}{\succ_{S^{(*)}(\omega)}}\; <i'+k+1> \\ &\underset{(\prec_{S^{(*)}(\omega)})}{\succ_{S^{(*)}(\omega)}}\; <j'+k+1>, \quad k=0,1,\ldots,\min\{i'',j''\}-1. \end{aligned}$$

Using Lemma 2.2.2, we shall give a realization of the ω-ordered set P.

There exists the one-one correspondence between each word μ and the right boundary of each standard Young diagram (as for the definition of the Young diagram, see [8]) as the following example.

Example.

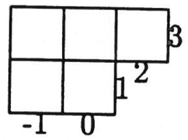
$\mu = 00101 \iff$



We note that in the corresponding Young diagram $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l)$, λ_1 is $<\omega\,|\,0>$.

For the Young diagram, let us label on the successive segments in the right boundary with the numbers $<\mu \mid 0>, <\mu \mid 0> -1, \ldots, -<\mu \mid 1> +1$, starting at the rightmost vertical segment.

Example.



Since the ω-ordered set P is a totally order by Proposition 2.2.1, let us realize the set P on the set

$$Q = \{-<\omega \mid 1> +1, -<\omega \mid 1> +2, \ldots, <\omega \mid 0>\}$$

by the mapping ρ_ω

$$\rho_\omega : \{0, 1, \ldots, n\} \longrightarrow Q$$

(cf.[2]).

Then the substitutions S and S^* transform the values of ρ_ω as follows:

Theorem 2 *For* $i = 0, 1, \ldots, n$,

$$S : \begin{pmatrix} a_i \\ \rho_\omega(i) \end{pmatrix} \longrightarrow \begin{cases} \begin{pmatrix} 0 \\ \rho_\omega(i) + <S(\omega) \mid 1> \end{pmatrix} & \textit{for } a_i = 0, \\ \begin{pmatrix} 0 & 1 \\ \rho_\omega(i) + <S(\omega) \mid 1> & \rho_\omega(i) \end{pmatrix} & \textit{for } a_i = 1. \end{cases}$$

$$S^* : \begin{pmatrix} a_i \\ \rho_\omega(i) \end{pmatrix} \longrightarrow \begin{cases} \begin{pmatrix} 0 & 1 \\ \rho_\omega(i) & \rho_\omega(i) - <S^*(\omega) \mid 1> \end{pmatrix} & \textit{for } a_i = 0, \\ \begin{pmatrix} 1 \\ \rho_\omega(i) \end{pmatrix} & \textit{for } a_i = 1. \end{cases}$$

Using Lemma 2.2.2, we determine the mapping ρ_ω as follows:

Corollary 1

$$\rho_\omega(0) = <\omega \mid 0>,$$

and for $i = 0, 1, \ldots, n-1$,

$$\rho_\omega(i+1) - \rho_\omega(i) = \begin{cases} -<\omega \mid 1> & \text{if } a_i a_{i+1} = 00 \text{ or } 01, \\ <\omega \mid 0> & \text{if } a_i a_{i+1} = 11 \text{ or } 10 \end{cases}.$$

Corollary 2

$$\rho_\omega(i) \equiv (i+1) < \omega \mid 0 > \pmod{< \omega \mid 01 >}, \quad i = 0, 1, \ldots n.$$

3 Combinatorial strucures of Josephus permutations and a cyclic tournament

In this chapter, we shall use the words of class $\mathcal{D}$ to characterize Josephus permutations in the set of permutations and a special cyclic tournament in the set of cyclic tournaments.

3.1 Josephus permutations

In this section, we shall give a characterization of Josephus permutations by the distribution of the ascents and descents. The characterization shall give a one-one correspondence between each Josephus permutation and each word of class $\mathcal{D}$.

Definition 3.1.1 (cf.[4]) *Let us denote the ascents and descents in the permutation $\sigma \in \mathcal{S}_N$ as follows:*

$$\begin{cases} a_i = 1 & \text{for } \sigma(i) < \sigma(i+1) \text{(ascent)}, \\ a_i = 0 & \text{for } \sigma(i) > \sigma(i+1) \text{(descent)} \end{cases} \qquad i = 1, 2, \ldots, N-1.$$

The word

$$\omega = 0a_1 \cdots a_{N-1}$$

is called the up-down symbol of the permutation σ.

Theorem 3 *Any permutation $\sigma \in \mathcal{S}_N$ is the Josephus permutation $J_{n,N}$ if and only if the up-down symbol ω of σ is a word of class $\mathcal{D}$ and*

$$< \omega \mid 0 >= n$$

Corollary *If the word ω of class $\mathcal{D}$ is the up-down symbol of the Josephus permutation $J_{n,N}$, the dual word ω^* is the up-down symbol of $J_{N-n,N}$.*

As well-known, the Euclidean algorithm gives the finite continued fraction. The Euclidean algorithm for integers n and N, $N > n > 0$ is the following:

$$\begin{aligned} N &= b_0 n + c_0, (n > c_0) \\ n &= b_1 c_0 + c_1, (c_0 > c_1) \\ &\vdots \end{aligned}$$

Then the set of numbers $b_0, c_0, b_1, c_1, \ldots$ characterizes the sequence of cardinalities

$$A_j = \mid \{(k+1)n \pmod{N};\ jN < (k+1)n \leq (j+1)N\} \mid, \quad k, j = 0, 1, \ldots.$$

In the case of the coprime integers n and N, this sequence is uniquely given by the up-down symbol of $J_{n,N}$. If this sequence shall be regarded as an enumerative aspect of the Euclidean algorithm or a continued fraction , our word of class $\mathcal{D}$ gives it.

Example.

$$\begin{array}{rcl} J_{3,7} & = & (\ 3\ ,\ 6\ ,\ 2\ ,\ 5\ ,\ 1\ ,\ 4\ ,\ 7) \\ \omega & = & \ \ 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \end{array} .$$

$$\begin{array}{rcl} J_{4,7} & = & (\ 4\ ,\ 1\ ,\ 5\ ,\ 2\ ,\ 6\ ,\ 3\ ,\ 7) \\ \omega^* & = & \ \ 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \end{array} .$$

3.2 A cyclic tournament

In this section, the words of class $\mathcal{D}$ shall be used for characterizing cyclic tournaments isomorphic to a given cyclic tournament.

A (round-robin) tournament Γ consists of vertexes 0, 1,..., t such that each pair of distinct vertexes i and j is joined by one and only one of the oriented arcs $\vec{ij}$ or $\vec{ji}$. If the arc $\vec{ij}$ is in Γ , then we say that i dominates j (symbolically, $i \to j$). Two tournaments are isomorphic if there exists a one-one dominance-preserving correspondence between their vertexes. If the transposition (0, 1,..., t) is a dominance-preserving permutation of the vertexes of a given tournament Γ, Γ is called a cyclic tournament. Then the number of vertexes is odd , that is $t = 2M$. Let the vertexes in Γ be labelled the numbers $\{0, 1,..., 2M\}$ denoted by $V(\Gamma)$. It is clear that all arcs in any cyclic tournament Γ are completely determined by arcs between the vertex 0 and the vertexes k , $k =$ 1,3,..., $2k-1$,..., $2M-1$ (cf.[9]).

Let Γ_0 be a cyclic tournament with $V(\Gamma_0) = \{0, 1,..., 2M\}$ such that $0 \to k$, $k =$ 1,3,..., $2k-1$,..., $2M-1$.

Let Γ be any cyclic tournament with the vertex set $V(\Gamma) = \{0, 1, \ldots, 2M\}$. Divide the set $V(\Gamma)$ into the sequence of vertex blocks $\mathbf{B}_j$, $j = 0, 1,..., L$ such that

(1). $p_0 = 0$ and $p_{L+1} = 2M$,

(2). $\mathbf{B}_j = \{p_j + 1, p_j + 2, \ldots, p_{j+1}\}$ and $\mathbf{B}_L = \{p_L + 1, p_j + 2, \ldots, p_{L+1}, 0\}$,

(3). if $0 \rightleftarrows p_j + 1$, then $0 \rightleftarrows p$ for any p in $\mathbf{B}_j$ and $0 \leftrightarrows p_{j-1} + 1, p_{j+1} + 1$, respectively.

Set

$$| \mathbf{B}_j | - | \mathbf{B}_0 |= a_j, \quad j = 0, 1, \ldots, L \tag{3.2.1}$$

and

$$\omega = a_0 a_1 \cdots a_L. \tag{3.2.2}$$

Then we have

Theorem 4 *Any cyclic tournament Γ is isomorphic to the given cyclic tournament Γ_0 if and only if the sequence ω by (3.2.1) and (3.2.2) is a word of class $\mathcal{D}$ and even length.*

Set

$$\tilde{\omega} = S^{*^{b_0-1}} S(\omega),$$

where $b_0 = \mid \mathrm{B}_0 \mid$.

Then the isomorphism $\varphi : \Gamma \to \Gamma_0$ is given by the realization $\rho_{\tilde{\omega}}$ of the $\tilde{\omega}$-ordered set as follows:

$$\textit{if} \quad 0 \to p_0+1, \quad \varphi(i) \equiv \begin{cases} \rho_{\tilde{\omega}}(i) \pmod{2M+1} & \textit{if}\, 0 \to p_0+1 \\ -\rho_{\tilde{\omega}}(i) \pmod{2M+1} & \textit{if}\, 0 \leftarrow p_0+1, \end{cases} \quad i = 0, 1, \ldots, 2M.$$

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