The Combinatorics of Symmetric Functions and Permutation Enumeration of the Groups S_n and B_n

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1 Introduction

We study permutation enumeration of the symmetric group, S_n, and the hyperoctahedral group, B_n, via the combinatorics of symmetric functions. Our work is motivated by a recent paper of F. Brenti [2]. Brenti studies the unimodality of the polynomials obtained by enumerating sets of permutations of S_n with respect to the number of excedances. He shows that the resulting polynomials for various classes, including conjugacy classes, are both unimodal and symmetric. By defining a homomorphism on the ring of symmetric functions, Brenti shows that the polynomials obtained by enumerating with respect to excedances over conjugacy classes of S_n arise naturally. Our work discusses the effects of applying this homomorphism on the various bases of the ring of symmetric functions from a combinatorial point of view. We use combinatorial definitions of the transition matrices between the various bases of the ring of symmetric functions to give combinatorial proofs to Brenti's results. The combinatorial proofs enable us to extend his results. For example, by defining a second homomorphism, we give q-analogues of Brenti's results. Here, we would like to give an idea of the types of involutions and combinatorial interpretations contained in our work. We will give two examples of our interpretations, after which we will state some of our other results, without proof. We have also derived analogous results for permutation enumeration of B_n, and we will discuss these briefly.

2 Permutation Enumeration of S_n

2.1 Notation

We use Macdonald's notation [5] for symmetric functions. For λ a partition of n, m_λ denotes the monomial symmetric functions; p_λ denotes the power symmetric functions; e_λ denotes the elementary symmetric functions; h_λ denotes the homogeneous symmetric functions; s_λ denotes the Schur functions; and f_λ denotes the forgotten symmetric functions.

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of S_n, given in one-line notation. Then we have three permutation statistics. If $\sigma_i > i$, then i is an *excedance* of σ. We denote by $e(\sigma) = |\{i : \sigma_i > i\}|$ (the number of excedances of σ). If $\sigma_i > \sigma_{i+1}$, then i is a *descent* of σ. We denote by $d(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$ (the number of descents of σ). An *inversion* occurs if $\sigma_i > \sigma_j$ for $i < j$. We denote the number of inversions of σ by $inv(\sigma) = \sum_{i<j}\chi(\sigma_i > \sigma_j)$. Here, we use the notation that for a statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

Let λ be a partition of n, also denoted $\lambda \vdash n$. We denote the length of the side of the Durfee square by $D(\lambda)$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$, then the number of parts of λ is denoted by $l(\lambda)$, and $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)}$. If we denote a partition as $\lambda = (1^{m_1}, 2^{m_2}, \ldots n^{m_n})$ where m_i is the number of parts of length i, then $z_\lambda = 1^{m_1}2^{m_2}\cdots n^{m_n}m_1!m_2!\cdots m_n!$

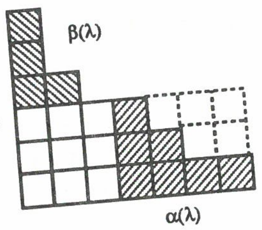
The conjugacy class corresponding to a partition λ is denoted by

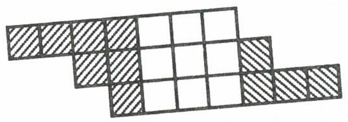
$$S_n(\lambda) = \{\sigma \in S_n : \text{cycle type of } \sigma = \lambda\}$$

We will need to refer to various parts of λ; we introduce some notation similar to that used by Brenti and define the partitions:

- $\alpha(\lambda) = (\alpha_1, \ldots, \alpha_{D(\lambda)})$ where $\alpha_i = \lambda_{l-d(\lambda)+i} - D(\lambda)$, for $i = 1, \ldots, D(\lambda)$.
- $\beta(\lambda) = (\beta_1, \ldots, \beta_{D(\lambda)})$ where $\beta_i = \lambda'_{l-d(\lambda)+i} - D(\lambda)$, for $i = 1, \ldots, D(\lambda)$.
- $\delta(\lambda) = (\alpha_{D(\lambda)} - \alpha_{D(\lambda)-1}, \alpha_{D(\lambda)} - \alpha_{D(\lambda)-2}, \ldots, \alpha_{D(\lambda)} - \alpha_1)$.
- $\gamma(\lambda) = (\alpha_{D(\lambda)} + D(\lambda))^{D(\lambda)} + \beta(\lambda)$

In particular, we will consider the skew shape $\gamma(\lambda)/\delta(\lambda)$. The Ferrers' diagrams associated with these partitions are illustrated below. The dotted portion of the diagram on the left corresponds to the partion $\delta(\lambda)$.

$F_\lambda =$ 

$F_{\gamma(\lambda)/\delta(\lambda)} =$ 

2.2 An Example of a Combinatorial Interpretation

Brenti defines the homomorphism $\xi : \Lambda \to Q[x]$ on the elementary symmetric functions as follows:

$$\xi(e_k) = \frac{(1-x)^{k-1}}{k!} \tag{1}$$

where $\xi(e_0) = 1$. This homomorphism has rather remarkable combinatorial properties and connections with permutation enumeration. Here are a few of the highlights among the results that Brenti proved about the homomorphism.

(i) $n!\xi(h_n) = \frac{A_n(x)}{x} = \sum_{\sigma \in S_n} x^{d(\sigma)}$, where $A_n(x)$ is the n-th Eulerian polynomial.

(ii) $\frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{\sigma \in S_n(\lambda)} x^{e(\sigma)}$

(iii) The leading coefficient of $\xi(s_\lambda)$ is $(-1)^{|\beta(\lambda)|}\frac{f^{\gamma(\lambda)/\delta(\lambda)}}{|\lambda|!}$.

Brenti also finds an expression for the leading coefficient of $\xi(m_\lambda)$ and $\xi(f_\lambda)$. Note that (ii) shows that $\sum_{\sigma \in S_n(\lambda)} x^{e(\sigma)}$ is unimodal and symmetric, which answered a question of Stanley.

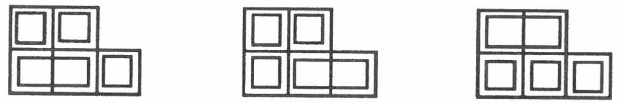
A Combinatorial Interpretation of $n!\xi(h_n)$ and a q-analogue

Brenti established his results by mostly algebraic means. We have been able to give combinatorial proofs of all his results using the combinatorial interpretations for the entries of the transition matrices between bases of symmetric functions. As an example, we give a combinatorial interpretation to (i) , after which we will give a q-analogue of the result.

Before we can give our interpretation, we need to define combinatorial objects called μ-brick tabloids of shape λ. We introduce them here by way of an example; details can be found in [3].

For μ a partition of n, we create bricks of length equal to the length of the parts of μ. For example, if $\mu = (1,1,1,2)$, then we have the following μ-bricks:

Let $\lambda = (2,3)$. Then the μ-brick tabloids of shape λ are:



Basically, in constructing a μ-brick tabloid, two rules must be followed:

1. μ-bricks can not overlap, and
2. each brick must lie within a single row.

In our combinatorial interpretations, we will be concerned with the number of μ-brick tabloids of shape λ, denoted by $B_{\mu,\lambda}$. In the example above, $B_{(1,1,1,2),(2,3)} = 3$. We denote by $\mathcal{B}_{\mu,\lambda}$ the set of all μ-brick tabloids of shape λ.

Now we will give a bijective proof of the expression Brenti derives for $n!\xi(h_n)$, providing a set of objects which describe the polynomial. Our proof depends on the fact that we can express h_λ combinatorially in terms of e_μ as given in [3]:

$$h_\lambda = \sum_{\mu \vdash n}(-1)^{n-l(\mu)}B_{\mu,\lambda}\, e_\mu \tag{2}$$

where $B_{\mu,\lambda}$ denotes the number of μ brick tabloids of shape λ.

Multiplying the special case of (2) where $\lambda = (n)$ by n! and applying the homomorphism to both sides we have

$$n!\xi(h_n) = \sum_{\mu \vdash n}(-1)^{n-l(\mu)}B_{\mu,(n)}n!\,\xi(e_\mu)$$

$$= \sum_{\mu\vdash n}(-1)^{n-l(\mu)}B_{\mu,(n)}n!\prod_{i=1}^{l(\mu)}\frac{(1-x)^{\mu_i}}{\mu_i!}$$

$$= \sum_{\mu\vdash n}\sum_{T\in\mathcal{B}_{\mu,(n)}}\binom{n}{\mu_1,\mu_2,\ldots}(x-1)^{n-l(\mu)}. \qquad (3)$$

Equating (i) and expression (3) we see that we must prove that

$$\sum_{\sigma\in S_n}x^{d(\sigma)} = \sum_{\mu\vdash n}\sum_{T\in\mathcal{B}_{\mu,(n)}}\binom{n}{\mu_1,\mu_2,\ldots}(x-1)^{n-l(\mu)}. \qquad (4)$$

We begin by showing that the right hand side of (4) is equivalent to a sum of signed, weighted combinatorial objects, i.e.

$$\sum_{\mu\vdash n}\sum_{T\in\mathcal{B}_{\mu,(n)}}\binom{n}{\mu_1,\mu_2,\ldots}(x-1)^{n-l(\mu)} = \sum_{o\in\mathcal{O}_h}sgn(o)w(o). \qquad (5)$$

Then we will describe a sign reversing weight preserving involution on these objects. The fixed points of the involution will clearly express $\sum_{\sigma\in S_n}x^{d(\sigma)}$.

Let $\mu\vdash n$. The elements in $\mathcal{O}_h$ are μ-brick tabloids of shape (n) (a single row). The bricks are each filled with a decreasing sequence of integers from the set $\{1,\ldots,n\}$ such that each integer appears exactly once in the tabloid. We define the weight of each of the cells, $w(c_i), 1\le i\le n$, in a tabloid as follows:

$$w(c_i) = \begin{cases} 1 & \text{if } c_i \text{ is at the end of a brick} \\ 1 \text{ or } x & \text{otherwise}\end{cases}$$

The weight of an object, $w(o)$, is then defined as

$$w(o) = \prod_{i=1}^{n}w(c_i).$$

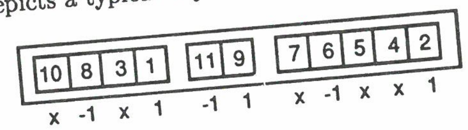
We define the sign of cell $sgn(c_i), 1\le i\le n$ as

$$sgn(c_i) = \begin{cases} -1 & \text{if } w(c_i)=1 \text{ and } c_i \\ & \text{is not at the end of a brick} \\ +1 & \text{otherwise}\end{cases}$$

The sign of an object, $sgn(o)$, is then defined as

$$sgn(o) = \prod_{i=1}^{n}sgn(c_i).$$

The following figure depicts a typical object:



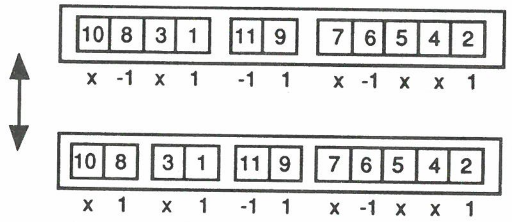
Clearly, (5) holds. Examining the left hand side of (5), we see by the second summation shows that the objects are μ-brick tabloids of shape (n). The multinomial coefficient fills each brick with a different sequence of integers from the set $\{1,2,\ldots,n\}$. By convention, we choose the integers to be decreasing in each brick. The term $(x-1)^{n-l(\mu)}$ gives a weight of either an +x or a +1 to each of $n-l(\mu)$ cells, where the cells weighted by by x have sign +1 and the cells weighted +1 have sign -1 . The remaining $l(\mu)$ cells have weight +1 and sign +1; we use the convention that the last cell in each brick shall carry this weight and sign.

Next we define an involution on objects in $\mathcal{O}_h$. For our involution, we check from the left of the tableau until we find the leftmost occurrence of one of the two conditions described below and perform the corresponding operation.

1. If there is a decrease between the last element of one brick and the first element in the next brick, we join the two bricks together and change the sign of the last cell of the first brick from +1 to -1.

2. If we see a -1 as a sign on a cell we cut the brick after that cell into two bricks and we change the sign of the cell from -1 to +1.

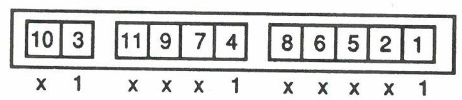
An example of the involution is given in Figure 2.2. The sign associated with an object is reversed by changing a +1 into a -1 and vice versa; hence the involution is sign reversing. And, because we do not change the entries otherwise, the involution preserves the number of x's in the tableau.



The fixed points of our involution will consist of those μ-brick tableaux of shape (n) which

1. have weights of x on all the cells except the last cell in each brick. (The last cells are weighted with +1),
2. have sign +1 (there are no longer any cells with negative signs), and
3. have fillings such that the elements increase between bricks and decrease within bricks.

The following is an example of a fixed point:



We can see that the elements along the row of a fixed point constitute a permutation of S_n. For each filling, or permutation, we have weighted with an x every place that the values decrease (at a descent), and we have weighted with a $+1$ the places of the increases (where there is no descent). Thus, we have enumerated the permutations of S_n with respect to descents.

By slightly altering the homomorphism given by Brenti, we give a q-analogue of $n!\xi(h_n)$. We define a ring homomorphism $\bar{\xi}:\Lambda \to (Q[q])[x]$ by letting

$$\bar{\xi}(e_k) = \frac{(1-x)^{k-1} q^{\binom{k}{2}}}{[k]!}. \tag{6}$$

We define $\bar{\xi}(e_0) = 1$. Here $[k] = 1 + q + \cdots + q^{k-1}$ and $[k]! = [k][k-1]\cdots[1]$.

We have the following.

Theorem 2.1 *Let $n \in P$ and let $\bar{\xi}:\Lambda \to (Q[q])[x]$ be the ring homomorphism defined by (6). Then,*

$$[n]!\bar{\xi}(h_n) = \sum_{\sigma \in S_n} x^{d(\sigma)} q^{inv(\sigma)}. \tag{7}$$

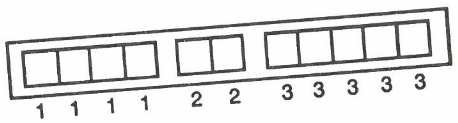
Proof. We again express h_n in terms of the e_μ [3]:

$$h_n = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)}\, e_\mu.$$

Multiplying by $[n]!$ and applying $\bar{\xi}$ to both sides we have

$$\begin{aligned} [n]!\bar{\xi}(h_n) &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} [n]!\, \bar{\xi}(e_\mu) \\ &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} [n]! \prod_{i=1}^{l(\mu)} \frac{(1-x)^{\mu_i} q^{\binom{\mu_i}{2}}}{[\mu_i]!} \\ &= \sum_{\mu \vdash n} \sum_{T \in B_{\mu,(n)}} \begin{bmatrix} n \\ \mu_1, \mu_2, \ldots \end{bmatrix} q^{\sum_i \binom{\mu_i}{2}} (x-1)^{n-l(\mu)} \end{aligned} \tag{8}$$

We now give an interpretation to expression (8). The objects, o, in the sum are single row tabloids filled with μ-bricks. For any given o, let $B_1, \ldots, B_l$ denote the bricks which occur in o in order from left to right. Let $b_i = |B_i|$ for $i = 1, \ldots, l$ so that $b_1, \ldots, b_l$ is a rearrangement of $\mu_1, \ldots \mu_l$. We associate 1's, 2's, etc. with the bricks from left to right as pictured below.

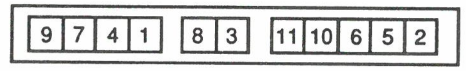


Let $\mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$ denote the rearrangements of b_1 1's, b_2 2's, $\ldots$, b_l l's. Then, for each rearrangement $r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$, we create a permutation of n, $\sigma(r)$, by numbering

from right to left, first the 1's, then the 2's, and so on. We then take the inverse of the permutation, $\sigma^{-1}(r)$:

$$\begin{aligned} r &= 1\ 3\ 2\ 1\ 3\ 3\ 1\ 2\ 1\ 3\ 3 \\ \sigma(r) &= 4\ 11\ 6\ 3\ 10\ 9\ 2\ 5\ 1\ 8\ 7 \\ \sigma^{-1}(r) &= 9\ 7\ 4\ 1\ \ 8\ 3\ \ 11\ 10\ 6\ 5\ 2 \end{aligned}$$

By the way we have constructed the permutation $\sigma^{-1}(r)$, we have sequences of decreasing integers which fit into the μ-bricks:



By a theorem of Carlitz,

$$\begin{bmatrix} n \\ \mu_1, \mu_2, \ldots, \mu_l \end{bmatrix} = \sum_{r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})} q^{inv(r)},$$

and by the way we constructed $\sigma(r)$, it is easy to see that

$$inv(\sigma^{-1}(r)) = inv(\sigma(r)) = inv(r) + \binom{b_1}{2} + \binom{b_2}{2} + \cdots \binom{b_l}{2}.$$

Hence, we can interpret the right hand side of (8) as $\sum_{o \in \mathcal{O}_{qh}} sgn(o) w(o)$. Let v_j denote the integer in cell c_i. With one exception, the objects in $\mathcal{O}_{qh}$ are the same as the objects in $\mathcal{O}_h$. The difference is that each cell of $o \in \mathcal{O}_{qh}$ has an additional weight of q^p where p denotes the number of integers which appear to the right of the cell which are smaller than v_j.

The involution we use on the objects in $\mathcal{O}_{qh}$ is the same as that used in our interpretation of $n!\xi(h)$. Note that we do not rearrange the fillings of the μ-bricks, and so consequently the q weight is not changed. Thus the fixed points will count the permutations of S_n with respect to the statistic $x^{d(\sigma)} q^{inv(\sigma)}$. □

2.3 Other Results

Our work also involves combinatorial proofs of the following.

A Combinatorial Interpretation of $n!\xi(h_\lambda)$ and a q-analogue

We must define a new permutation statistic which will be used in our interpretation of $n!\xi(h_\lambda)$. This statistic involves both a permutation and a partition.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n and let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of n, written in one line notation. For the permutation statistic, we break σ into pieces of lengths $\lambda_1, \lambda_2, \ldots, \lambda_l$. We then count only the descents, $\sigma_i > \sigma_{i+1}$ such that both i and $i+1$ occur within one of these pieces, and we denote the sum by $d_\lambda(\sigma)$. For example, if $\sigma = 8\,6\,2\,7\,4\,3\,1\,5$ and $\lambda = (1,3,4)$, then we break σ into pieces [8][6 2 7][4 3 1 5]. In this case, $d_\lambda(\sigma) = 3$. Note in this example the descents of σ occurring at positions 1 and 5 are not counted because positions 1 and 2 and positions 5 and 6 do not occur within a single piece. We have the following theorems.

Theorem 2.2 *Let $\xi : \Lambda \rightarrow Q[x]$ be the ring homomorphism defined by (1) and let λ be a partition of n. Then*

$$n!\xi(h_\lambda) = \sum_{\sigma \in S_n} x^{d_\lambda(\sigma)}.$$

Theorem 2.3 *Let $n \in P$, let λ be a partition of n, and let $\bar{\xi} : \Lambda \rightarrow (Q[q])[x]$ be the ring homomorphism defined by (6). Then,*

$$[n]!\bar{\xi}(h_\lambda) = \sum_{\sigma \in S_n} x^{d_\lambda(\sigma)} q^{inv(\sigma)}.$$

A Combinatorial Interpretation of $\frac{n!}{z_\lambda}\xi(p_\lambda)$ and a q-analogue

We use weighted brick tableaux (see [3]) to give a combinatorial proof of (ii), namely,

$$\frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{\sigma \in S_n(\lambda)} x^{e(\sigma)}$$

in much the same spirit as our combinatorial proof of (i). Moreover, we can also give a q-analogue of (ii).

We introduce the following permutation statistics which are very similar to the excedance count and the inversion number, which will be used in our expression of $[n]!\bar{\xi}(p_\lambda)$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n and let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of n written in one line notation. We break up σ, and we consider the pieces to be cycles. We count the excedances occurring in the permutation given in cycle notation, however we do not count excedances occurring between the last and first element in a cycle. We denote this number by $\bar{e}_\lambda(\sigma)$. For example, if $\sigma = 35824617$ and $\lambda = (1,3,4)$, then we break σ into cycles $(3)(582)(4617)$. In this case, $\bar{e}_\lambda(\sigma) = 3$ (Note that $e((3)(582)(4617)) = 4$).

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of n. We define $\overline{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j)$. Then we have the following.

Theorem 2.4 *Let $n \in P$ and let $\bar{\xi} : \Lambda \rightarrow (Q[q])[x]$ be the ring homomorphism defined by (6). Then,*

$$[n]!\bar{\xi}(p_\lambda) = \sum_{\sigma \in S_n} x^{\bar{e}_\lambda(\sigma)+l(\lambda)-\sum_{i=1}^{l(\lambda)} f_i(\sigma)} q^{\overline{inv}(\sigma)} \prod_{i=1}^{l(\lambda)} f_i(\sigma)(x^{f_i(\sigma)} - (x-1)^{f_i(\sigma)})$$

where $f_i(\sigma)$ is the length of the last strictly increasing sequence in the i^{th} part of σ (σ is divided into into parts of size $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$).

The Schur Functions under ξ and a q-analogue

By adapting a combinatorial proof of the Jacobi-Trudi identity given by Eğecioğlu and Remmel [4], we also give a combinatorial proof of Brenti's expression of the leading coefficient

of $\xi(s_\lambda)$. Using our proof we then extend Brenti's results and give an expression for the coefficients of $(1-x)^{n-l(\lambda)}/n!$ in $\xi(s_\lambda)$.

Before we give the expression, we give some definitions and notation. Consider a Ferrers' diagram of shape λ. A rim hook of λ is a sequence of cells, h, along the north-east boundary of the diagram such that any two consecutive cells in h share an edge and such that the removal of the cells in h leaves a legal diagram. We define a k-border rim hook tabloid of shape ν, $H_\nu = (h_1, h_2, \ldots, h_k)$, as a filling of the Ferrers' diagram of shape ν with rim hooks $h_1, h_2, \ldots, h_k$ such that

1. h_1 is a rim hook of the Ferrers' diagram of shape ν, and for $1 < i < k$, h_i must be a rim hook of the Ferrers' diagram of shape $\nu - (h_1, \ldots, h_{i-1})$ where $\nu - (h_1, \ldots, h_{i-1})$ denotes the diagram of shape ν with the cells of hooks $h_1, \ldots, h_{i-1}$ removed.
2. h_i starts above h_j for $i < j$ in the sense that if the first square of h_1 (reading from top to bottom) is (i_1, j_1) and the first square of h_2 is (i_2, j_2), then $i_1 - j_1 > i_2 - j_2$.

An example of a 4-border rim hook tabloid of shape $(1^2, 3^3, 4, 6^2)$ is given below.

Let $|h_i|$, $1 \leq i \leq k$, denote the length of hook h_i, i.e. the number of cells that h_i occupies. Define the sign of a hook h_i as

$$sgn(h_i) = (-1)^{r(h_i)-1}$$

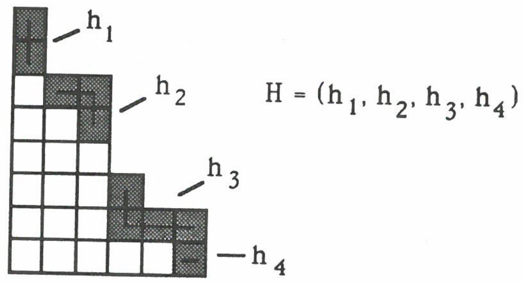
where $r(h_i)$ denotes the number of rows that h_i occupies. We define the sign of a k-border rim hook tabloid H_ν as

$$sgn(H_\nu) = \prod_{i=1}^{k} sgn(h_i)$$

Let $\nu_{\overline{H}}$ correspond to the shape $\nu - (h_1, \ldots, h_k)$. We define

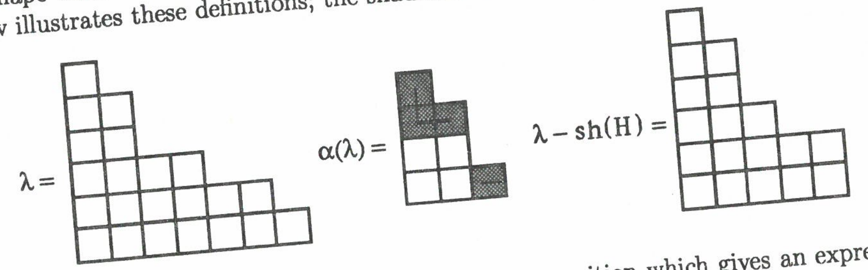
$$sh(H_\nu) = \frac{\nu}{\nu_{\overline{H}}}.$$

Note that $sh(H_\nu)$ may correspond to disjoint pieces as in the figure below.



In the figure, the shaded cells correspond to $sh(H_\nu)$ and the unshaded cells correspond to the shape $\nu_{\overline{H}}$. Denote by $\mathcal{B}_\nu^k$ the set of all k-border rim hook tabloids of shape ν.

In the proposition to follow, the shape of the r-border rim hook tabloid will correspond to the shape $\alpha(\lambda)$ for λ a partition. We define $\lambda - sh(H_{\alpha(\lambda)})$ to be the Ferrers' diagram of the shape which remains after the k-border rim hooks are removed from $\alpha(\lambda)$. An example below illustrates these definitions; the shaded cells correspond to $sh(H_{\alpha(\lambda)})$.



The previous definitions lead to the following proposition which gives an expression for the coefficients of $(1-x)^{n-l(\mu)}/n!$.

Proposition 2.1 *Let $n \in P$, let λ be a partition of n, and let ξ be defined by (1). Then*

$$\xi(s_\lambda) = \frac{1}{n!} \sum_{k=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} C_{k,\lambda'}(1-x)^{n-k}$$

where

$$C_{k,\lambda'} = \sum_{H_{\alpha(\lambda)}=(h_1,h_2,\ldots h_r)\in B^r_{\alpha(\lambda)}} w(H_{\alpha(\lambda)}) \binom{n}{|h_1|,\ldots|h_r|} (-1)^{\alpha(\nu)} f^{\gamma(\nu)/\delta(\nu)}$$

where $r = l(\mu) - D(\lambda)$ and $\nu = \lambda - sh(H_{\alpha(\lambda)})$.

We have the following expression for $[n]!\bar{\xi}(s_\lambda)$. The expression comes directly from the q-analogue of $n!\bar{\xi}(h_\lambda)$.

Theorem 2.5 *Let $n \in P$, let λ be a partition of n, and let $\bar{\xi} : \Lambda \to Q[x][q]$ be the ring homomorphism defined by (6). Then,*

$$[n]!\bar{\xi}(s_\lambda) = \sum_{\sigma\in S_n} q^{inv(\sigma)} r_\lambda(\sigma)$$

where $r_\lambda(\sigma) = \sum_{\mu\vdash n} K^{-1}_{\mu,\lambda} x^{d_\mu(\sigma)}$.

Here, $K^{-1}_{\mu,\lambda}$ is the inverse Kostka number and can be expressed as the sum of the signs of all special rim hook tabloid of shape λ and type μ, see [4].

We also give combinatorial expressions for the objects which appear in the sums of the polynomials which result when the homomorphism is applied to the monomial and the forgotten symmetric functions.

3 Permutation Enumeration of B_n

3.1 Notation

Recall that the conjugacy classes of B_n are indexed by a pair of partitions, (λ, μ), where the parts of λ correspond to the lengths of the cycles of even parity and the parts of μ correspond

to the lengths of the cycles of odd parity. We denote

$$B_n(\lambda,\mu) = \{\sigma \in B_n : \text{cycle type of } \sigma = (\lambda,\mu)\}$$

We define the following linear order Γ.

$$1 <_\Gamma 2 <_\Gamma \cdots <_\Gamma n <_\Gamma \cdots <_\Gamma -n <_\Gamma \cdots <_\Gamma -2 <_\Gamma -1 \tag{9}$$

Define Θ to be a partial order such that $i \equiv \bar{i}$ and which otherwise is the usual linear order on integers:

$$1 \equiv -1 <_\Theta 2 \equiv -2 <_\Theta \cdots <_\Theta n \equiv -n \tag{10}$$

Essentially, Θ acts to consider negative and positive as equivalent. Then, for $\sigma \in B_n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_i \in \{\pm1, \pm2, \ldots, \pm n\}$, be a permutation in one-line notation. For such a permutation, we define the number of *descents* of σ by

$$d_B(\sigma) = |\{i : \sigma_i >_\Gamma \sigma_{i+1}, 1 \le i \le n\}|$$

where $\sigma_{n+1} = n+1$ (see [6]).

For $\sigma \in B_n$, let $\sigma = (\sigma_{1_1}\sigma_{1_2}\cdots\sigma_{1_{l(1)}})(\sigma_{2_1}\sigma_{2_2}\cdots\sigma_{2_{l(2)}})\cdots(\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_{l(k)}})$ be the permutation in cycle notation. Then, for such a permutation, a decedance occurs at the j^{th} position in the i^{th} cycle if $\sigma_{i_j} >_\Gamma \sigma_{i_{j+1}}$, for $1 \le j < l(i)$ or if $\sigma_{i_{l(i)}} >_\Gamma \sigma_{i_1}$, for $j = l(i)$. If k denotes the number of cycles of σ, we define the number of *decedances* of σ to be

$$de_B(\sigma) = \sum_{i=1}^{k} \left[|\{j : \sigma_{i_j} >_\Gamma \sigma_{i_{j+1}}, 1 \le j < l(i)\}| + \chi(\sigma_{i_{l(i)}} >_\Gamma \sigma_{i_1}) \right]$$

Next, we define an inversion statistic which is the same as the S_n inversion statistic, except that the partial order is that defined by (10). A B_n- *inversion* occurs if $\sigma_i >_\Theta \sigma_j$ for $i < j$. We denote the number of inversions of σ by

$$inv_B(\sigma) = \sum_{i<j} \chi(\sigma_i >_\Theta \sigma_j)$$

In the representation theory of B_n, the characteristic map sends the class functions into $\oplus_{k=0}^{n} \Lambda_k(\mathrm{x})\otimes\Lambda_{n-k}(\bar{\mathrm{x}})$, denoted $\Lambda_{B_n}(x,\bar{x})$ (see [7]). We denote $\Lambda_B = \bigoplus_{n\ge0}\Lambda_{B_n}$. There are ten bases of $\Lambda_{B_n}(x,\bar{x})$ all of which are given in λ-ring notation with the exception of the analogue of the power symmetric functions. For a pair of partitions, (λ,μ), such that $|\lambda| + |\mu| = n$, the analogue of the Schur functions is $s_\lambda(X+\overline{X})s_\mu(X-\overline{X})$, the analogue of the power symmetric functions is $p_\lambda(x)p_\mu(\overline{x})$, the analogues of the homogeneous and the elementary symmetric functions are $h_\lambda(X+\overline{X})h_\mu(X-\overline{X})$, $h_\lambda(X+\overline{X})e_\mu(X-\overline{X})$, $e_\lambda(X+\overline{X})h_\mu(X-\overline{X})$, $e_\lambda(X+\overline{X})e_\mu(X-\overline{X})$, and the analogues of the monomial and forgotten symmetric functions are $m_\lambda(X+\overline{X})m_\mu(X-\overline{X})$, $m_\lambda(X+\overline{X})f_\mu(X-\overline{X})$, $f_\lambda(X+\overline{X})m_\mu(X-\overline{X})$, and $f_\lambda(X+\overline{X})f_\mu(X-\overline{X})$.

As in the case of S_n, studying permutation enumeration of B_n requires the use of the transition matrices between these bases of $\Lambda_{B_n}(x,\bar{x})$. Most of the transition matrices are just comprised of pairs of the transition matrices as defined between the bases of $\Lambda_n(x)$. However, the matrices between $p_\lambda(x)p_\mu(\overline{x})$ and the other bases are more interesting, and in these cases, it was necessary to develop the combinatorial definitions of these matrices ourselves [1].

3.2 B_n Results

We define a homomorphism, $\zeta : \Lambda_B \to Q[x]$, on $e_k(X+\overline{X})$ and $e_k(X-\overline{X})$ as follows:

$$\zeta(e_k(X+\overline{X})) = \frac{(1-x)^{k-1} + x(x-1)^{k-1}}{2^k k!} \tag{11}$$

$$\zeta(e_k(X-\overline{X})) = \frac{(1-x)^{k-1} - x(1-x)^{k-1}}{2^k k!} \tag{12}$$

Then we combinatorially prove the following.

Theorem 3.1 *Let $n \in P$ and let $\zeta : \Lambda_B \to Q[x]$ be the ring homomorphism defined by (11) and (12). Then*

$$\frac{2^n n!}{z_\lambda z_\mu}\zeta(p_\lambda(x)p_\mu(\overline{x})) = \sum_{\sigma \in B_n(\lambda,\mu)} x^{de_B}(\sigma)$$

$$2^n n! \zeta(h_n(X+\overline{X})) = \sum_{\sigma \in B_n} x^{d_B}(\sigma)$$

and

$$2^n n! \zeta(h_n(X-\overline{X})) = (1-x)^n.$$

Note that the descedance statistic has not been defined before, and that here it arises naturally. A bijection shows that

$$\sum_{\sigma \in B_n((n),\emptyset)} x^{de_B}(\sigma) = \sum_{\sigma \in B_n(\emptyset,(n))} x^{de_B}(\sigma) = 2^{n-1} \sum_{\sigma \in S_{n-1}} x^d(\sigma).$$

where $d(\sigma)$ is the S_n descent statistic. Hence, all the polynomials in Theorem 3.1 are symmetric and unimodal.

We also give expressions for $2^n n! \zeta(h_\lambda(X+\overline{X}))$ and $2^n n! \zeta(h_\mu(X-\overline{X}))$. Furthermore, we prove that the leading term in $\zeta(s_\lambda(X+\overline{X})s_\mu(X-\overline{X}))$ is of degree n and has coefficient

$$\frac{(-1)^{|\mu|}}{2^n |\lambda|! |\mu|!} f^{\lambda'} f^{\mu'}.$$

By altering the homomorphism ζ, we give q-analogues to the previous results. Namely, we define $\overline{\zeta} : \Lambda_B \to Q[x]$ as follows:

$$\overline{\zeta}(e_k(X+\overline{X})) = \frac{q^{\binom{k}{2}}\left((1-x)^{k-1} + x(x-1)^{k-1}\right)}{2^k [k]!} \tag{13}$$

$$\overline{\zeta}(e_k(X-\overline{X})) = \frac{q^{\binom{k}{2}}\left((1-x)^{k-1} - x(1-x)^{k-1}\right)}{2^k [k]!} \tag{14}$$

Then we prove

Theorem 3.2 *Let $n \in P$ and let $\overline{\zeta} : \Lambda_B \to Q[x]$ be the ring homomorphism defined by (13) and (14). Then*

$$2^n [n]! \overline{\zeta}(h_n(X+\overline{X})) = \sum_{\sigma \in B_n} x^{d_B}(\sigma) q^{inv_B(\sigma)}$$

and

$$2^n [n]! \overline{\zeta}(h_n(X-\overline{X})) = \sum_{\sigma \in \tilde{B}_n} (-x)^{d_B}(\sigma) q^{inv_B(\sigma)}$$

where $\tilde{B}_n = \{\sigma_1\sigma_2\cdots\sigma_n : \sigma_i \in \{i, -i\}\}$.

We give an expression for $\frac{2^n[n]!}{z_\lambda z_\mu}\overline{\zeta}(p_\lambda(x)p_\mu(\overline{x}))$ which involves the B_n-descent and B_n-inversion statistics as well as statistics which correspond to the lengths of the last strictly decreasing sequences of elements in cycles of $\sigma \in B_n$ which are induced by the fixed pair of partitions (λ, μ). We also give an analogue of Theorem 2.5.

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