Experimental Algebraic Combinatorics 1:

Looking for Idempotents.

François Bergeron

Extended abstract

This is the first of a series of papers on experimental algebraic combinatorics. In these papers, I plan to illustrate the use of computer algebra for the investigation of algebraic structures such as finitely presented groups or algebras, group algebras, Lie algebras. The first kind of problem we will consider is the search for *idempotents* in a group algebra. In this abstract, we will mainly illustrate, with the group S_3, the kind of questions that we are going to address in general. These questions and observations are often motivated by classical representation theory or by the results obtained in [6] and in a series of papers on the decomposition of the descent algebra of finite Coxeter groups (see [3], [4], [8], [9], [10], and [11]), but they also give indications that deeper problems are being addressed.

Denote $\mathbf{C}[W]$ the group algebra of a finite group W of order N. We are going to investigate idempotents $e \in \mathbf{C}[W]$. Let

$$e = \sum_{w \in W} \alpha_w \, w$$

be such an idempotent. Then $e^2 = e$ corresponds to a set of N quadratic equations

$$\alpha_w = \sum_{u\, v = w} \alpha_u \, \alpha_v,$$

one for each $w \in W$, in the N unknowns α_w. One can theoreticly solve these equations to find all idempotents of $\mathbf{C}[W]$. In the case $W = S_3$, let

$$e = a_1\,[1,2,3] + a_2\,[1,3,2] + a_3\,[2,1,3] + a_4\,[2,3,1] + a_5\,[3,1,2] + a_6\,[3,2,1], \qquad (1)$$

then $e^2 = e$ is equivalent to

$$\begin{aligned}
a_1 &= {a_1}^2 + {a_2}^2 + {a_3}^2 + 2\,a_4\,a_5 + {a_6}^2,\\
a_2 &= 2\,a_1\,a_2 + a_3\,a_4 + a_3\,a_5 + a_4\,a_6 + a_5\,a_6,\\
a_3 &= 2\,a_1\,a_3 + a_2\,a_4 + a_2\,a_5 + a_4\,a_6 + a_5\,a_6,\\
a_4 &= 2\,a_1\,a_4 + a_2\,a_3 + a_2\,a_6 + a_3\,a_6 + {a_5}^2,\\
a_5 &= 2\,a_1\,a_5 + a_2\,a_3 + a_2\,a_6 + a_3\,a_6 + {a_4}^2,\\
a_6 &= 2\,a_1\,a_6 + a_2\,a_4 + a_2\,a_5 + a_3\,a_4 + a_3\,a_5,
\end{aligned} \qquad (2)$$

If we order the elements of W, we can denote any element of the group algebra as the vector of its coordinates in the resulting ordered basis. Thus $(a_1, a_2, a_3, a_4, a_5, a_6)$ will stand for e appearing in (1), for the lexicographic order on permutations. In the sequel of this paper, we will always order permutations in this manner.

Using the grobner basis approach to solve system (2), one finds some *isolated* solutions giving well known idempotents that correspond to irreducible characters of S_3

$$\begin{aligned} \chi_{1^3} &= (1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6), \\ \chi_{21} &= (2/3,\ 0,\ 0,\ -1/3,\ -1/3,\ 0), \\ \chi_3 &= (1/6,\ -1/6,\ -1/6,\ 1/6,\ 1/6,\ -1/6), \end{aligned} \tag{3}$$

and their idempotent linear compositions

$$\begin{aligned} &(1,\ 0,\ 0,\ 0,\ 0,\ 0), \\ &(0,\ 0,\ 0,\ 0,\ 0,\ 0), \\ &(1/3,\ 0,\ 0,\ 1/3,\ 1/3,\ 0), \\ &(5/6,\ -1/6,\ -1/6,\ -1/6,\ -1/6,\ -1/6), \\ &(5/6,\ 1/6,\ 1/6,\ -1/6,\ -1/6,\ 1/6) \end{aligned}$$

and two families of idempotents characterized by the simple systems of equations

$$\begin{aligned} &2\,a_1 - 1 = 0, \\ &2\,(a_2 + a_3 + a_6) + 1 = 0, \\ &a_4 + a_5 = 0, ? \\ &2\,{a_4}^2 - 2\,{a_3}^2 - 2\,a_3\,a_6 - a_3 - a_6 - 2\,{a_6}^2 = 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned} &3\,a_1 - 1 = 0, \\ &a_2 + a_3 + a_6 = 0, \\ &3\,(a_4 + a_5) + 1 = 0, \\ &9\,{a_6}^2 + 9\,a_3\,a_6 + 9\,{a_3}^2 - 9\,{a_4}^2 - 3\,a_4 - 1 = 0. \end{aligned} \tag{5}$$

Thus we obtain the two families,

$$\begin{aligned} &\Big(1/2,\ -x - y + 1/2,\ x,\ \Delta,\ -\Delta,\ y\Big), \\ &\Big(1/2,\ x + y - 1/2,\ -x,\ \Delta,\ -\Delta,\ -y\Big), \\ &\Big(1/2,\ x + y - 1/2,\ -x,\ -\Delta,\ \Delta,\ -y\Big), \\ &\Big(1/2,\ -x - y + 1/2,\ x,\ -\Delta,\ \Delta,\ y\Big), \end{aligned} \tag{6}$$

$$\begin{aligned} &\Big(1/3,\ -x - y + 1/3,\ x - 1/6,\ \Delta - 1/6,\ -\Delta - 1/6,\ y - 1/6\Big), \\ &\Big(1/3,\ x + y - 1/3,\ -x + 1/6,\ \Delta - 1/6,\ -\Delta - 1/6,\ -y + 1/6\Big), \\ &\Big(2/3,\ x + y + 1/3,\ -x + 1/6,\ -\Delta + 1/6,\ \Delta + 1/6,\ -y + 1/6\Big), \\ &\Big(2/3,\ -x - y + 1/3,\ x - 1/6,\ -\Delta + 1/6,\ \Delta + 1/6,\ y - 1/6\Big), \end{aligned} \tag{7}$$

where

$$\Delta = \pm\sqrt{x^2 + x\,y + y^2 - \frac{x + y}{2}}.$$

The idempotents in (6) and (7) come in groups of four because any idempotent e generally gives rise to four idempotents e, $\epsilon(e)$, $1 - e$, and $1 - \epsilon(e)$, where ϵ is the automorphism of the group algebra corresponding to the multiplication of each permutation by its sign. Hence it suffices to explicitly construct the first one via (4) and (5).

Problem 1 *Find all idempotents of the group algebra* $\mathbf{C}[W]$ *of a finite group* W.

Our interest in such idempotents is motivated by the fact that they correspond to projections

$$\pi_e : \mathbf{C}[W] \twoheadrightarrow \mathbf{C}[W]\,e,$$

where π_e is multiplication on the right by e. Clearly, the ideal $H_e := \mathbf{C}[W]\,e$ is a left W-module. The character χ_e of this representation of W, considered as an element of the group algebra, is

$$\chi_e = \frac{1}{N}\sum_{w\in W} w^{-1}\,e\,w. \tag{8}$$

In view of the well known decomposition of the group algebra in term of irreducible representations V_λ

$$\mathbf{C}[S_n] = \bigoplus_{\lambda\vdash n} f_\lambda\, V_\lambda,$$

with the f_λ's given by the hook formula, (8) implies some of the equations of systems (4) and (5), since the only possible characters that can appear are those that are linear combinations

$$\chi_e = \alpha_{1^3}\,\chi_{1^3} + \alpha_{21}\,\chi_{21} + \alpha_3\,\chi_3,$$

of the irreducible characters of S_3 with the constraints that $0 \le \alpha_\lambda \le f_\lambda$, i.e.

$$\begin{aligned} &\alpha_{1^3} \in \{0, 1\}, \\ &\alpha_{21} \in \{0, 1, 2\}, \\ &\alpha_3 \in \{0, 1\}. \end{aligned}$$

For instance (8) implies that the value of

$$a_1 = \frac{1}{6}(\alpha_{1^3} + 2\,\alpha_{21} + \alpha_3),$$

in (2), has to be equal to one of the values $k/6$, with $k = 0, \ldots, 6$.

Problem 2 *Find a nice set of equations for the idempotents of a group algebra.*

Another classical observation is that a family $(e_k)_{k\in K}$ of idempotents such that

$$\sum_{k\in K} e_k = 1, \tag{9}$$

and

$$e_k\, e_j = \begin{cases} e_k & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

corresponds to a direct sum decomposition of the left regular representation $\mathbf{C}[W]$

$$\mathbf{C}[W] = \bigoplus_{k\in K} H_{e_k}. \tag{11}$$

If condition (9) holds, we say that the e_k's form an *orthogonal family* of idempotents. If condition (9) holds as well, we say that the family is *complete*. It is clear that the sum of two orthogonal idempotents is again an idempotent. If a family of orthogonal idempotents is such that all other idempotents can be obtained by summation and/or specialization of their parameters, we say that they form a *fundamental* family of idempotents. For S_3 the four idempotents

$$\begin{aligned}
\chi_{1^3} &= (1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6\,), \\
\chi_3 &= (1/6,\ -1/6,\ -1/6,\ 1/6,\ 1/6,\ -1/6\,), \\
\chi_{21}^{+}(x,y) &= \Big(1/3,\ -x-y+1/3,\ \ x-1/6,\ \ \Delta-1/6,\ -\Delta-1/6,\ \ y-1/6\Big), \\
\chi_{21}^{-}(x,y) &= \Big(1/3,\ \ x+y-1/3,\ \ -x+1/6,\ -\Delta-1/6,\ \ \Delta-1/6,\ -y+1/6\Big),
\end{aligned} \tag{12}$$

(with Δ as in (7)) form such a family, since they are easily checked to be orthogonal, and one can check, by inspection of (3), (4) and (5), that every other idempotents of the group algebra of S_3 can be obtained from them. For example, the first idempotent of (6) is obtained as $\chi_{1^3} + \chi_{21}^{+}(x,y)$. Both idempotents $\chi_{21}^{+}(x,y)$ and $\chi_{21}^{-}(x,y)$ have the same character χ_{21}, and we have

$$\chi_{21} = \chi_{21}^{+}(x,y) + \chi_{21}^{-}(x,y).$$

Thus we obtain a refinement of the usual decomposition of the group algebra $\mathbf{C}[S_3]$ corresponding to the decomposition of 1 in irreducible characters

$$\begin{aligned}
1 &= \chi_{1^3} + \chi_3 + \chi_{21} \\
&= \chi_{1^3} + \chi_3 + \chi_{21}^{+}(x,y) + \chi_{21}^{-}(x,y).
\end{aligned} \tag{13}$$

Problem 3 *Are there always fundamental families of idempotents? If so find all of them.*

Observe that two idempotents $e, f \in \mathbf{C}[W]$ project onto the same left W-module ($\mathbf{C}[W]\,e = \mathbf{C}[W]\,f$) if

$$\begin{aligned} e\,f &= f \\ f\,e &= e. \end{aligned} \tag{14}$$

We will then say that f is *associated* to e. Moreover, if we fix one idempotent e, the set of idempotents

$$I_e = \{f \ \ e\,f = f \text{ and } f\,e = e\},$$

is an affine subspace of $\mathbf{C}[W]$. More precisely

$$I_e = \{e + \eta \ \ \eta \in N_e\},$$

where N_e is a subspace of nilpotent elements of $\mathbf{C}[W]$ with the following properties

1) $e\,\eta = \eta$, for all $\eta \in N_e$,
2) $\eta\, e = 0$, for all $\eta \in N_e$,
3) $\eta_1\eta_2 = 0$ for all $\eta_1, \eta_2 \in N_e$.

Among the idempotents of the group algebra for the symmetric group S_4, all those associated to the lie idempotent (see [10])

$$\chi_{21}^{+}(0, 1/2) = (\,1/3,\ -1/6,\ -1/6,\ -1/6,\ -1/6,\ 1/3\,),$$

can be computed by solving the linear system corresponding to (11). One finds that they are

$$\chi_{21}^{+}(x, 1/2) = (\,1/3,\ -x-1/6,\ x-1/6,\ x-1/6,\ -x-1/6,\ 1/3\,).$$

For S_4, all those associated to

$$\frac{1}{4}(\,1,-1,0,-1,0,1,0,0,0,-1,0,1,0,0,0,0,0,1,0,0,0,0,0,-1\,),$$

are

$$\begin{aligned}
\theta_4(s,t,u,v) = \frac{1}{4}\Big\{&(\,1,-1,0,-1,0,1,0,0,0,-1,0,1,0,0,0,0,0,1,0,0,0,0,0,-1\,) \\
&+ s\,(\,0,1,-1,1,-2,0,0,1,0,0,-1,-1,0,1,-1,2,-1,-1,1,0,0,1,0,0\,) \\
&+ t\,(\,0,0,0,1,2,-1,-1,-1,0,1,2,-1,0,-2,2,-2,1,0,-2,0,0,0,1,0\,) \\
&+ u\,(\,0,0,0,0,1,-1,0,0,-1,1,0,0,0,0,1,-1,0,0,-1,1,0,0,0,0\,) \\
&+ v\,(\,0,1,0,0,-2,0,0,1,0,0,-2,0,-1,2,-1,2,-1,-1,1,0,1,0,0,0\,)\Big\}
\end{aligned} \tag{15}$$

We will see in a moment that these are all the so-called *Lie idempotents* for S_4 in the sense of [10]. As in [6], we set

$$\pi_n(q) = \frac{1}{n}\sum_{\sigma\in S_n} r_{\mathrm{maj}(\sigma)}(q)\,\sigma, \tag{16}$$

where $r_k(q)$ is the remainder of q^k modulo $\varphi_n(q)$, the $n^{\rm th}$ cyclotomic polynomial, and $\mathrm{maj}(\sigma)$ is the *major index* of σ

$$\mathrm{maj}(\sigma) = \sum \{i \mid \sigma_i > \sigma_{i+1}\}.$$

It is shown in [6] theorem 2.3 that for the δ_i's defined by

$$\pi_n(q) = \delta + \eta_1\, q + \ldots \eta_f\, q^f,$$

one has the following Lie idempotents

$$\delta + c_1\, \eta_1 + \ldots c_f\, \eta_f, \tag{17}$$

for all values of the c_k's (since the η_i's are in N_δ). In the case $n = 4$, (14) gives a one parameter family of Lie idempotents

$$\pi_4(q) = \frac{1}{4}\,(1, -q, -1, -q, -1, q, q, 1, -1, -q, -1, q, q, 1, -q, 1, -1, q, q, 1, -q, 1, -q, -1\,). \tag{18}$$

It is easy to verify that $\theta_4(1, -q, 1, -q) = \pi_4(q)$. Thus we conclude that all idempotents of (12) are Lie idempotents. It is interesting to observe that (12) gives a strictly larger family of Lie idempotents than those given by (15).

Problem 4 *For a given idempotent e, what is the dimension of N_e?*

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