Plane trees and Shabat polynomials

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Abstract

In his unpublished paper [G] Alexandre Grothendieck has indicated that there exist profound relations between the theory of number fields and that of maps on two-dimensional surfaces. This theme was later explored by George Shabat (Moscow) and his students (see [ASh-1], [ASh-2], [Sh-1], [Sh-2], [ShV], [VSh]).

For the simplest class of maps, that of plane trees, this theory leads to a very interesting class of polynomials which generalize Chebyshev polynomials and which we call Shabat plynomials. Our work consisted in compiling a catalog of Shabat plynomials for all plane trees up to 8 edges (see [BPZ]), as well as for some infinite series of plane trees. In the process, some conjectures were disproved, some other conjectures appeared, and some new methods of computation of Shabat polynomials were developed.

The volume of the catalog does not allow us to reproduce it here. It is available as an internal publication of LaBRI. This paper could be regarded as a kind of introduction to it.

1. Critical points and critical values of complex polynomials

Consider a polynomial $P(z)$ with complex coefficients. It maps a complex plane onto another one. Take a point $w \in \mathbb{C}$ and consider its inverse image $P^{-1}(w) = \{ z \mid P(z) = w \}$.

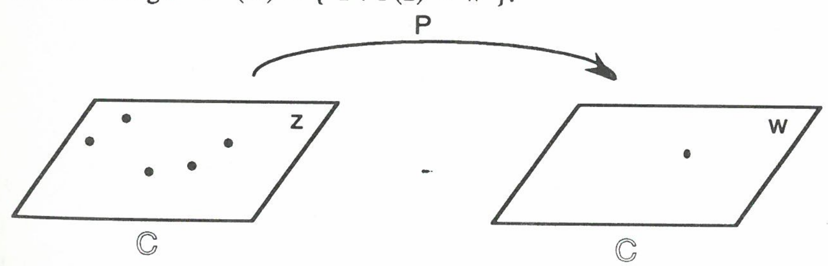


Fig.1. Inverse image of a point

In general, this set consists of n separate points, i.e. solutions of the equation $P(z) = w$, where n is the degree of the polynomial P (See Fig.1). But for some specific values of w this equation could have multiple roots, the inverse image of w thus having less than n points. Multiple roots of $P(z) = w$ are, obviously, the roots of $P'(z) = 0$.

A point $z \in \mathbb{C}$, at which $P'(z) = 0$, is called a *critical point*. A value $w = P(z)$ at a critical point z is called a *critical value*. We say that critical point z has *order* k, if at this point $P'(z) = 0$, $P''(z) = 0, \dots, P^{(k-1)}(z) = 0$, $P^{(k)}(z) \neq 0$ (the least possible order of a critical point is thus equal to 2). The number of critical points, i.e. roots of $P'(z)$, is $n-1$, counting with multiplicity; the number of critical values could be much smaller than that.

Example 1. Polynomial $P(z) = z^n$ has one critical point of order n (namely, $z = 0$), and one critical value $w = 0$.

Example 2. Chebyshev polynomials

$$T_n(z) = \cos(n \arccos(z))$$

have $n-1$ critical points of order 2 (namely, $z = \cos(\frac{k\pi}{n})$, $k = 1, 2, \dots, n-1$), and only two critical values: $w = \pm 1$.

Now, take a straight line segment, joining two points c_1 and c_2 on the w-plane. From now on we will denote it $[c_1,c_2]$, even in the case when its endpoints are not real. The inverse image $P^{-1}([c_1,c_2])$ of this segment is, as a rule, a disjoint union of n separate sets on z-plane, each one being homeomorphic to a segment, but not necessarily straight (see Fig.2). We will still call them segments. To distinguish between the ends of the segments, we mark one of them black, and the other one white.

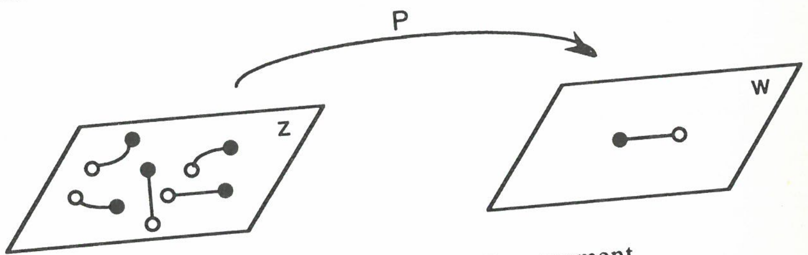


Fig.2. Inverse image of a segment

If, by chance, c_1 or c_2 (or both) are critical values of $P(z)$, and there is no critical values inside the segment $[c_1,c_2]$, then some of the segments on z-plane will glue together, thus forming a graph (see Fig.3). In this graph, vertices of degree 1 are ordinary (non-critical) points, and vertices of degree greater than 1 are critical points, their order coinciding with the number of incident edges.

The most interesting case is when $P(z)$ has only one or two critical values, namely, $\{c_1,c_2\}$.

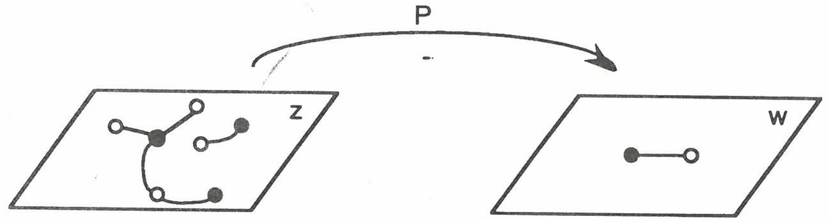


Fig.3. Ends of some segments are critical points

2. Polynomials with one or two critical values

Definition. We call a polynomial $P(z)$ *Shabat polynomial* if there exist two complex numbers c_1 and c_2 such that

$$P'(z) = 0 \quad \Rightarrow \quad P(z) \in \{c_1,c_2\}.$$

The starting fact of the present theory is that the inverse image of the segment $[c_1,c_2]$ under Shabat polynomial is a *tree* drawn on the plane, or a *plane tree*. Precise definitions will be given below. Two examples of Shabat polynomials were given above: $P(z) = z^n$ and $P(z) = T_n(z)$ (Chebyshev polynomial). The inverse image of the segment $[0,1]$ under z^n is a "star-tree" (or a "hedgehog"), and the inverse image of the segment $[-1,1]$ under $T_n(z)$ is a "chain-tree", see Fig.4.

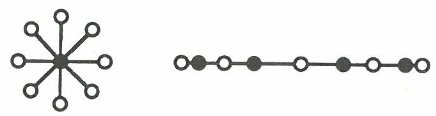


Fig.4. Trees corresponding to z^n and $T_n(z)$

The class of Shabat polynomials is invariant under any non-degenerate linear transformations of z- and w-planes: $z \to c*z + d$ and $w \to C*w + D$, $c,d,C,D \in \mathbb{C}$, $c \neq 0$, $C \neq 0$. By means of the second transformation (of w-plane) we can transform the segment $[c_1,c_2]$ into any other segment. There are two *canonic* choices of the segment on w-plane: $[0,1]$ and $[-1,1]$. It is sometimes more convenient to use one canonic form, sometimes the other, and sometimes just take an *arbitrary* segment $[c_1,c_2]$.

Definition. Two Shabat polynomials $P(z)$ and $Q(z)$ are called *equivalent*, if there exist $c,d,C,D \in \mathbb{C}$, $c \neq 0$, $C \neq 0$, such that

$$P(z) = C*Q(c*z+d) + D.$$

Main Theorem (Riemann, Belyi, Grothendieck, Shabat). There is a bijection between the set of *combinatorial bicolored plane trees*, and the set of equivalence classes of Shabat polynomials.

To make this statement clear, we need some definitions concerning combinatorial trees.

3. Combinatorial bicolored plane trees

A *tree* is a connected graph without circuits. A *plane tree* is a tree which is drawn on (imbedded into) the plane. A "picture" of a tree on the plane determines, for any vertex of the tree, a cyclic permutation on the set of adjacent vertices. An isomorphism of plane trees is an isomorphism of trees that conserves these cyclic permutations (and, hence, the orientation). A class of isomorphism of plane trees is called combinatorial plane tree; see Fig.5.

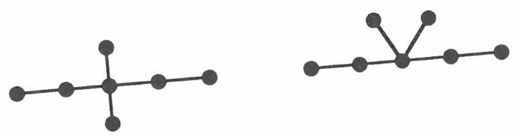


Fig.5. One tree, but two plane trees

Combinatorial plane trees were enumerated in [HPT] using the theorem of Pólya. More explicit formulas for the number of plane trees are given in [L]. See also [T].

Any tree has a natural structure of a bipartite graph: its vertices could be colored in two colors (say, black and white) in such a way that adjacent vertices would have different colors. Having chosen one of two possible colorings, we obtain a *bicolored plane tree*. An isomorphism of bicolored plane trees must conserve not only the structure of plane tree, but also the colors of corresponding vertices; see Fig.6.

Fig.6. One plane tree, but two bicolored plane trees

For the sake of brevity in what follows we will usually call bicolored combinatorial plane trees just trees.

Let n be the number of edges, and let $\alpha_1, \alpha_2, ..., \alpha_p$ (resp. $\beta_1, \beta_2, ..., \beta_q$) be the sequence of degrees of black (resp. white) vertices, ordered in decreasing manner. Every edge joins a black

vertex with a white one; therefore, $\sum_{i=1}^{p}\alpha_i = \sum_{j=1}^{q}\beta_j = n$. On the other hand, the tree has p black vertices and q white vertices, hence $p + q = n + 1$. Thus, $<\alpha,\beta>$ is a pair of partitions of number n, having together $n+1$ parts. We call it the *set of degrees* of the tree, and say also that the tree is of *type* $<\alpha,\beta>$. The following simple lemma shows that the reciprocal statement also holds: *for any pair $<\alpha,\beta>$ of partitions of number n having p and q parts respectively, such that $p + q = n + 1$, there exists at least one bicolored plane tree having this pair as its set of degrees.*

If we exchange black and white colors, set of degrees $<\alpha,\beta>$ will be replaced by $<\beta,\alpha>$. Thus, we usually obtain the bicolored tree of another type. But in case when $\alpha = \beta$ the type remains the same. For example, both trees shown in Fig.6 have the same type $<3,2,1,1;\ 3,2,1,1>$.

4. Geometry of plane trees

Linear transformation $z \to c*z + d$ mentioned in section 2 may change the size of a tree and its position on z-plane, but it does not change its geometric form. This fact, in combination with Main Theorem, leads to a remarkable consequence:

Every plane tree has a unique and canonical geometric form.

On the next page the reader will find several pictures representing the *true* geometric form of the corresponding plane trees. This page represents an excerpt from the catalog [BPZ]. Our main software tool was MAPLE-V.

5. Calculation of Shabat polynomials

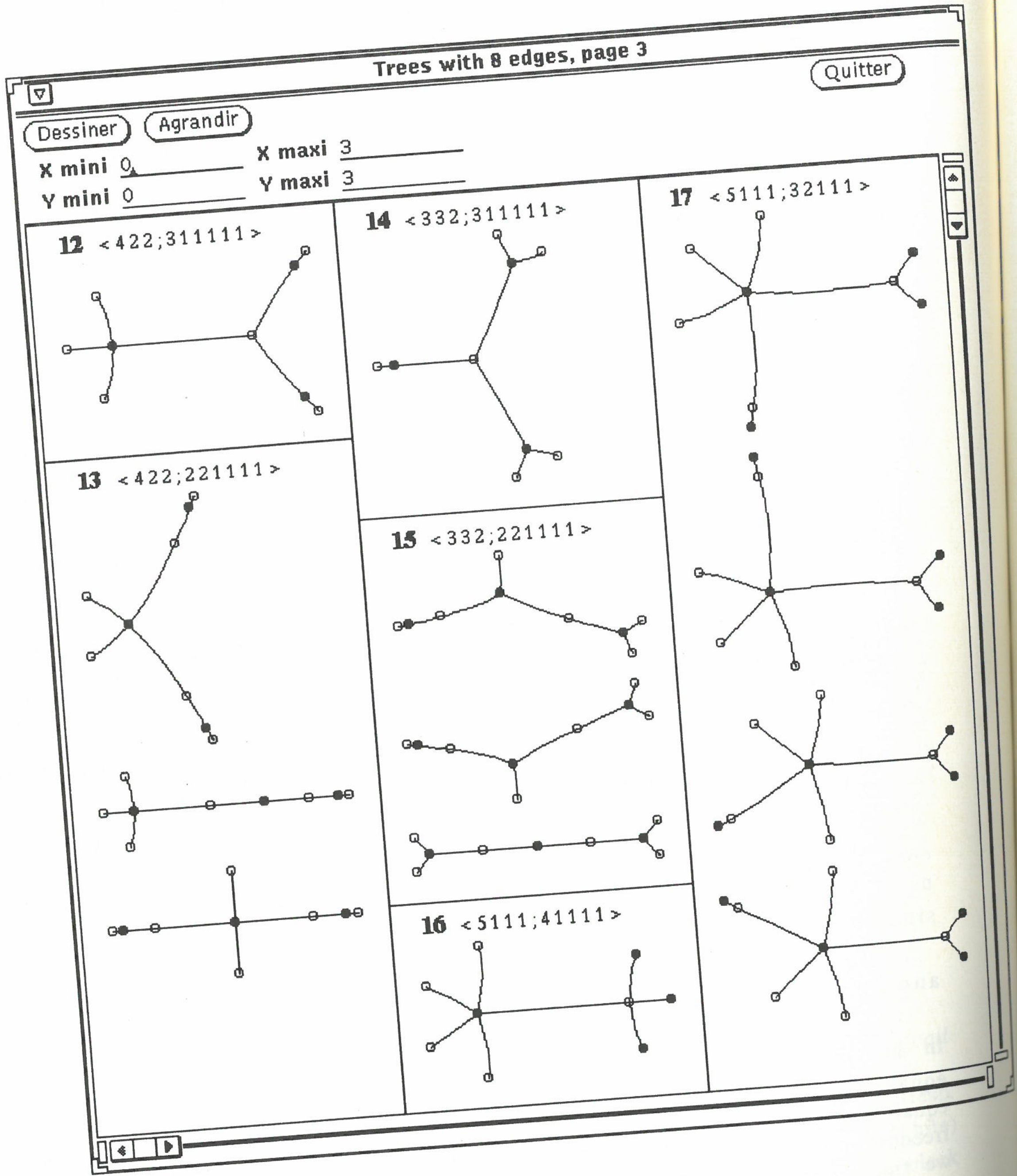
Let the type $<\alpha,\beta> = <\alpha_1,\alpha_2,...,\alpha_p;\ \beta_1,\beta_2,...,\beta_q>$ of a tree be given; we also set $[c_1, c_2] = [0,1]$. Then in order to find the corresponding Shabat polynomial we need to find $n+2$ complex numbers $\lambda,\ a_1,a_2,...,a_p,\ b_1,b_2,...,b_q$ such that we have simultaneously

$$P(z) = \lambda(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2} \dots (z-a_p)^{\alpha_p}$$

and

$$P(z) - 1 = \lambda(z-b_1)^{\beta_1}(z-b_2)^{\beta_2} \dots (z-b_q)^{\beta_q}.$$

In addition, all the values a_i and b_j should be different. The equalities between coefficients provide us with n algebraic equations (the main term being the same). Two more "degrees of freedom" remain at our disposal, and we may use them in the way we find it convenient. We may fix the positions of any two points,



or make some other choice that places a tree in an unambiguous position on complex plane.

This method, taken literally, is far too complicated. Many improvements of various levels of generality are given in [BPZ]. See also section "Examples".

6. Fields of definition, Galois groups

Let $\mathbb{A}$ be the field of **algebraic** numbers, i.e. the field of all (complex) roots of **polynomials** with rational coefficients. The *Galois group* $\Gamma = \mathbf{Gal}(\mathbb{A}|\mathbb{Q})$ is the group of automorphisms of the field $\mathbb{A}$ which do not move the elements of $\mathbb{Q}$. This group acts on the roots of polynomials with rational coefficients by permuting (some of) them. For a specific irreducible polynomial f the *Galois group of* f is the factor-group of Γ by the subgroup of those automorphisms that do not move elements of not only $\mathbb{Q}$ but also of its extension by roots of f. For an irreducible polynomial of degree n it is some subgroup of the symmetric group $\mathbf{S}_n$ that acts transitively on the roots of f.

The previous section makes it clear that *coefficients* of Shabat polynomials could always be made algebraic numbers (if only we have not used our "two degrees of freedom" not properly to make some of the equations transcendental). They belong to some *number field*, i.e. finite extension of the field $\mathbb{Q}$ of rational numbers by roots of some irreducible polynomial f. *The reader should not confuse Shabat polynomial with this latter polynomial:* we repeat once more that we need polynomial f to determine an algebraic number field to which the coefficients of Shabat polynomial P belong.

Action of Galois group Γ, besides rational numbers, also conserves all algebraic relations with rational coefficients. Thus, acting on **coefficients** of a Shabat polynomial, Γ transforms it into other Shabat polynomials. Therefore, the *action of* Γ *on plane trees is defined*. The main goal of the theory is to understand the combinatorial and geometric nature of this action. For example, from the previous section it is clear that the type $<\alpha,\beta>$ is an invariant of this action: trees belonging to the same orbit have the same type. A set of the trees of the same type is called a *family*. Each family is either an orbit of Galois group action, or a union of several orbits.

As another combinatorial invariant we could mention the order of the symmetry of a tree; recall that plane tree may have only cyclic group of symmetries.

The notion of a *field of definition* of a map or a tree is rather complicated and involves the Galois cohomology theory (see [Sh-2]). But in case of a tree it is just the smallest number field to which the coefficients of the corresponding Shabat polynomial belong. The field of definition is the same for the whole orbit. The *Galois group* of a particular orbit could also be defined as the group of automorphisms of the field of definition, that conserve the subfield of rational numbers.

7. Examples

(A) Consider a plane tree of the type <3,2,2; 2,2,1,1,1>.

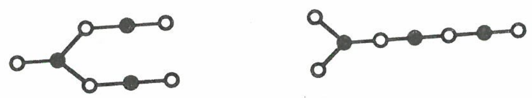


Fig.7. Pair of conjugate trees

Let us place the black vertex of degree 3 at $z=0$; let two other black vertices be the roots of a quadratic polynomial z^2-2z+a (the coefficient -2 in front of z means that we place these two roots in such a way that the middle point of the segment joining them lies at $z=1$). Thus, we may look for a Shabat polynomial having the form

$$P(z) = z^3(z^2-2z+a)^2$$

(this means that we take the end-point c_1 of an image-segment equal to zero, and do not impose any conditions on c_2). The derivative of $P(z)$ is equal to

$$P'(z) = z^2(z^2-2z+a)(7z^2-10z+3a).$$

Two white vertices of degree 2 are the roots of the last factor $Q(z)=7z^2-10z+3a$. These two points must satisfy two conditions: (1) they must be distinct, thus $21a-25 \neq 0$, and (2) the values of $P(z)$ at them must be equal to each other. Compute the remainder after division of $P(z)$ by $Q(z)$: it is equal to $Az+B$, where

$$A = -\frac{16}{7^6}\,(21a-25)(49a^2-476a+400),$$

$$B = -\frac{196}{7^6}\,a(28a-25)(7a-10).$$

Condition (2) leads to the equation $A=0$, hence

$$f(a) = 49a^2-476a+400 = 0, \qquad \text{and} \qquad a = \frac{1}{7}(34 \pm 6\sqrt{21}).$$

A priori it is difficult to decide what sign we should take, plus or minus. So let us take both; then we obtain both trees of the type <3,2,2; 2,2,1,1,1>, see Fig.7. These two trees are *conjugate* to each other; they form an orbit of Galois group action, and their field of definition is $\mathbb{Q}(\sqrt{21})$.

(B) For some time the following question remained open: *whether the set of degrees and the order of the symmetry group completely*

characterize the orbit of Galois group action on trees. We found that the answer to this question is negative. Consider the set of degrees <4,2,1; 2,2,1,1,1>. There are four plane trees of that type, and they are all asymmetric (via rotations), see Fig.8.

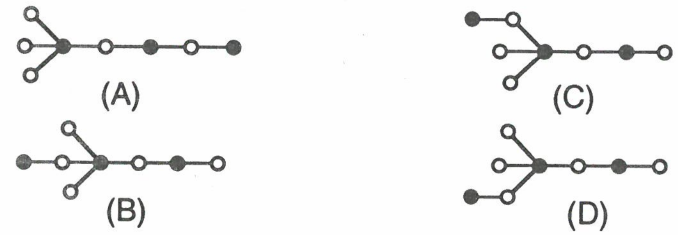


Fig.8. Two orbits of Galois group

However, they do not compose a single orbit but split into two separate orbits. One orbit consists of the trees (A) and (B), with the field of definition $\mathbb{Q}(\sqrt{21})$, the other one, of the trees (C) and (D), with the field of definition $\mathbb{Q}(\sqrt{-7})$.

Other examples of the same kind were also found by Leila Schneps [Sch] and by Nikolaï Adrianov [Sh-3].

(C) The following example is very instructive. Consider the set of degrees <3,2,1,1; 3,2,1,1>. There are six plane trees of this type. Three of them are symmetric (with the symmetry of order 2), and form therefore a separate orbit of Galois group action (the field of definition being $\mathbb{Q}(\sqrt[3]{28})$). But the field of definition of the other three (asymmetric) trees is not cubic but have degree six! The reason is that instead of three *plane trees* we must consider six *bicolored plane trees* with the same set of degrees (see Fig.6, where two of these bicolored plane trees are shown), and they are all conjugate to each other. Thus, the right combinatorial structure corresponding to the algebraic one is that of a *bicolored* plane tree.

(D) The following example was a consequence of our efforts to find a cubic orbit (i.e. an orbit with a cubic field of definition) with the *cyclic Galois group*. Let us consider a "generic" case, when the set of degrees is <m,m,n,n,n,n,n; 7,1,1,...,1>, with m and n positive and non-equal. The corresponding family contains three plane trees of diameter 4 having $2m+5n$ edges, with the folloing cyclic orders of vertex degrees around the center:

$$(m,m,n,n,n,n,n), \quad (m,n,m,n,n,n,n), \quad (m,n,n,m,n,n,n).$$

One of the trees is shown on Fig.9 (with black and white colors reversed).

Place the "center" of the tree at $z=0$; let the vertices of degree m be the roots of z^2-2z+a, and the vertices of degree n, the roots of $z^5+b_4z^4+b_3z^3+b_2z^2+b_1z+b_0$. We are looking for a Shabat polynomial of the form

$$P(z) = (z^2-2z+a)^m(z^5+b_4z^4+b_3z^3+b_2z^2+b_1z+b_0)^n.$$

Derivative $P'(z)$, besides obvious roots, must have a root of multiplicity 6 at $z=0$. This leads to a system of equations on a and b_i. Eliminating b_i one after another from this system, we finally obtain the cubic equation on a: $f(a)=0$, where

$$f(a) = 15n^3*a^3 - 90n^2(m+3n)*a^2 + 60n(m+3n)(m+4n)*a - 8(m+3n)(m+4n)(m+5n).$$

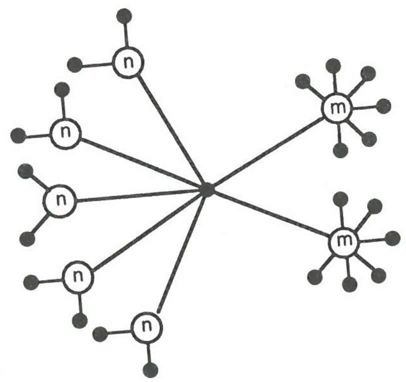


Fig.9. One of the three plane trees of the type
<m,m,n,n,n,n,n; 7,1,...,1>

The discriminant of the polynomial f could be easily calculated (using MAPLE, of course), and factorized:

$$D(f) = 518400n^6(m+3n)^2(m+4n)(2m+3n)(2m+5n)^2.$$

It is well known that a Galois group for an irreducible cubic polynomial is equal to C_3 (cyclic group of order 3) if its discriminant is a complete square, and to S_3 (group of permutations of three elements) otherwise. Therefore, to obtain an orbit with Galois group C_3 we just need the polynomial f being irreducible, and $(m+4n)(2m+3n)$ being a complete square. Degrees $m=37$, $n=2$ produce an example, number of edges of the tree being equal to 84. Obviously, infinite series of examples could be generated likewise.

(E) Let us treat in the same way another cubic family, that of the type <m,m,n,k; 4,1,1,...1>, numbers m,n,k being all different.

This family contains three trees of diameter 4, corresponding to the following cyclic orders of vertex degrees around the center:

$$(m,m,n,k),\quad (m,m,k,n),\quad (m,n,m,k).$$

Making the similar calculations, we get

$$D(f) = -\ 432m^3n^2k^2(m+n)(m+k)(m+n+k)^3(2m+n)^2(2m+n+k)^2,$$

where f is a polynomial generating the field of definition of the orbit.

The discriminant is negative, so this time it could not be a square. Let us ask another question: when is the field of definition a *purely cubic field*, i.e. a field of the form $\mathbb{Q}(\sqrt[3]{a})$ with some integer a? It is known that to have such a field, the discriminant of f must be equal to

$$D(f) = -\ 3*(\text{complete square}).$$

Thus, the product $m(m+n)(m+k)(m+n+k)$ must be a square. To make it a square it is sufficient to have, for example,

$$m = x^2,\quad m+n = y^2,\ m+k = z^2,\ m+n+k = t^2.$$

These equations are not independent: we have

$$x^2+t^2 = y^2+z^2 = 2m+n+k.$$

Thus, the number of edges $2m+n+k$ must have *two different representations as a sum of two squares.*

In the whole history of number theory it is difficult to find a more classic problem than that of representation of a number as a sum of two squares. The main result was announced by Pierre Fermat in a letter to Marin Mersenne (1642): every prime of the form $4k+1$ has a unique representation as a sum of two squares; all the primes of the form $4k+3$ do not have such a representation. It could be readily seen that a product of two different numbers representable as a sum of two squares has a non-unique representation (an elegant proof of this statement belongs to Lewis Carroll). Infinite series of examples could be constructed in this way. For example, take number of edges equal to $5*13 = 65$. Then, $65 = 1+64 = 16+49$, and we may take $m=1$, $n=15$, $k=48$.

8. General conjecture concerning discriminants

The problem of computing the discriminant of a field of definition of a "Grothendieck dessin" (or a closely connected quantity, the discriminant of an irreducible polynomial whose roots generate the field), and the problem of understanding its combinatorial nature (if any), were always considered as very important. Shabat in his letter [Sh-3] called it "the most interesting problem in the domain for the time being". He has also remarked that for all known examples the number of edges of a tree is always among the factors of the discriminant, and wrote: "The others are quite mysterious, and understanding them is definitely necessary for the further progress".

We have computed the discriminants for more than 20 infinite families of trees of diameter 4. And in all these cases the discriminant of the polynomial f in question is completely split into the linear combinations of vertex degrees with integral coefficients.

More accurately, the following observation proved to be true in all cases considered up to now:

Let for a tree of diameter 4 the set of white vertices have k_1 *vertices of degree* m_1, ..., k_r *vertices of degree* m_r, *the central (black) vertex thus having the degree* $K = k_1+\ldots+k_r$. *Then the discriminant in question is split into factors of the form*

$$a_1m_1+\ldots+a_rm_r,$$

where a_i *are integers, and* $0 \leq a_i \leq k_i$, $i=1, \ldots, r$. *Besides, the discriminant has a numeric factor, whose all prime divisors are less than* K.

One of the consequences of this splitting is that the discriminant does not have prime factors that are bigger than the number $k_1m_1+\ldots+k_rm_r$, which is the number of edges of the tree. The latter consequence is important for an (eventual) reduction of the algebraic problem to the fields of positive characteristic.

We conjecture that the splitting of this kind is valid for all trees of diameter 4.

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