$$b_0 = \sum_{n\geq 1}\Bigg[x^n q^n \sum_{k=0}^{n-1}\Bigg[\frac{q^{\binom{k}{2}}}{(q)_k\left(yq^{k+1}\right)^2_{n-k}}\Bigg\{q^{\binom{k}{2}}\frac{1-2q^k}{(q)_k}\Bigg\{2k+\sum_{m=1}^{k}\frac{2q^m}{1-q^m}+\sum_{m=k+1}^{n}\frac{2yq^m}{1-yq^m}\Bigg\} - \frac{q^{\binom{k}{2}}}{(q)_k}$$

$$+ 2y\sum_{m=1}^{k}\Bigg\{\frac{(-1)^{k+m}q^{\binom{m+1}{2}}}{(q)_{m-1}\left(yq^m\right)_{k-m+1}}\Bigg(k+m+\sum_{i=1}^{m-1}\frac{2q^i}{1-q^i}+\sum_{i=m}^{k}\Big(\frac{q^i}{1-q^i}+\frac{yq^i}{1-yq^i}\Big)+\sum_{i=k+1}^{n}\frac{2yq^i}{1-yq^i}\Bigg)\Bigg\}\Bigg\}\Bigg]\Bigg],$$

$$b_1 = \sum_{n\geq 1}\Bigg[x^n q^n \sum_{k=0}^{n-1}\Bigg[\frac{q^{\binom{k}{2}}}{(q)_k\left(yq^{k+1}\right)_{n-k-1}\left(yq^{k+1}\right)_{n-k}}$$

$$\Bigg\{q^{\binom{k}{2}}\frac{1-2q^k}{(q)_k}\Bigg\{2k+\sum_{m=1}^{k}\frac{2q^m}{1-q^m}+\sum_{m=k+1}^{n}\frac{2yq^m}{1-yq^m}-\frac{yq^n}{1-yq^n}\Bigg\} - \frac{q^{\binom{k}{2}}}{(q)_k}$$

$$+\sum_{m=1}^{k}\Bigg\{\frac{2y(-1)^{k+m}q^{\binom{m+1}{2}}}{(q)_{m-1}\left(yq^m\right)_{k-m+1}}\Bigg(k+m+\sum_{i=1}^{m-1}\frac{2q^i}{1-q^i}+\sum_{i=m}^{k}\Big(\frac{q^i}{1-q^i}+\frac{yq^i}{1-yq^i}\Big)+\sum_{i=k+1}^{n}\frac{2yq^i}{1-yq^i}-\frac{yq^n}{1-yq^n}\Bigg)\Bigg\}\Bigg\}\Bigg]\Bigg].$$

Combinatorial Properties of the Kazhdan-Lusztig and R-polynomials for S_n

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Extended Abstract

In their fundamental paper [3] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [2], Chap. 7). These polynomials are intimately related to the Bruhat order of W and to the algebraic geometry of Schubert varieties, and have proven to be of fundamental importance in representation theory.

In order to prove the existence of these polynomials Kazhdan and Lusztig defined another family of polynomials (see [3], §2) which are intimately related to the multiplicative structure of the Hecke algebra associated to W. These polynomials are now known as the R-polynomials of W (see, e.g., [2], §7.5) and their importance stems mainly from the fact that their knowledge is equivalent to that of the Kazhdan-Lusztig polynomials.

Given a set T we will let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\text{def}}{=} S([n])$. If $\sigma \in S(T)$ and $T \stackrel{\text{def}}{=} \{t_1,\ldots,t_r\}_< \subseteq \mathbf{P}$ then we write $\sigma = \sigma_1\ldots\sigma_r$ to mean that $\sigma(t_i) = \sigma_i$, for $i = 1,\ldots,r$. If $\sigma \in S_n$ then we will also write σ in *disjoint cycle form* (see, e.g., [5], p.17) and we will usually omit to write the 1-cycles of σ. For example, if $\sigma = 365492187$ then we also write $\sigma = (9,7,1,3,5)(2,6)$. Given $\sigma,\tau \in S_n$ we let $\sigma\tau \stackrel{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$.

We will follow [5], Chap. 3, for notation and terminology concerning partially ordered sets. In particular, we say that a finite graded poset P with $\hat{0}$ and $\hat{1}$ is *Eulerian* if $\mu(x,y) = (-1)^{\rho(y)-\rho(x)}$ for all $x,y \in P$, $x \leq y$, where $\rho : P \to \mathbf{N}$ is the rank function of P. Recall (see, e.g., [5], §3.14, p. 138, or [6], §2, p. 190) that to any Eulerian poset P as above there are associated two polynomials, denoted $f(P;q)$ and $g(P;q)$, defined inductively as follows:

i) if $|P| = 1$ then $f(P;q) \stackrel{\text{def}}{=} g(P;q) \stackrel{\text{def}}{=} 1$;

[1] Part of this work was carried out while the author was visiting the Mittag-Leffler Institute and the Institute of Mathematics of the Hebrew University of Jerusalem

ii) if P has rank $n+1 \geq 1$ and $f(P;q) = \sum_{i \geq 0} k_i q^i$ then

$$g(P;q) \stackrel{\text{def}}{=} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (k_i - k_{i-1}) q^i \,, \tag{1}$$

(where $k_{-1} \stackrel{\text{def}}{=} 0$);

iii) if P has rank $n+1 \geq 1$ then

$$f(P;q) \stackrel{\text{def}}{=} \sum_{a \in P \setminus \{\hat{1}\}} g([\hat{0}, a]; q)\,(q-1)^{n-\rho(a)} \,. \tag{2}$$

The polynomials $f(P;q)$ and $g(P;q)$ were introduced in [6] and are two very subtle invariants of the Eulerian poset P (see [5], §3.14, and [6], §§2,3, for further information). We call $g(P;q)$ the *g-polynomial* of P, and $(h_0, \ldots, h_n)$, where $h_i \stackrel{\text{def}}{=} [q^{n-i}](f(P;q))$, for $i = 0, \ldots, n$, the *h-vector* of P.

We will follow [2] for general Coxeter groups notation and terminology. Given a Coxeter system (W,S) and $\sigma \in W$ we denote by $l(\sigma)$ the length of σ in W, with respect to S, and we let

$$D(\sigma) \stackrel{\text{def}}{=} \{ s \in S : \; l(\sigma\, s) < l(\sigma) \} \,,$$

and

$$d(\sigma) = |D(\sigma)| \,.$$

We call $D(\sigma)$ the *descent set* of σ and say that σ has $d(\sigma)$ *descents*. We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{ \sigma s \sigma^{-1} : \sigma \in W, \; s \in S \}$. If $A \subseteq W$ and $x \in W$ we let $A\,x \stackrel{\text{def}}{=} \{ wx : w \in A \}$. We will always assume that W is partially ordered by (strong) *Bruhat order*. Recall (see, e.g., [2], §5.9) that this means that $x \leq y$ if and only if there exist $r \in \mathbf{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \ldots t_1\, x = y$ and $l(t_i \ldots t_1\, x) > l(t_{i-1} \ldots t_1 x)$ for $i = 1, \ldots, r$. It is well known (see, e.g., [1], Corollary 1) that intervals of W (and their duals) are Eulerian posets.

We denote by $\mathcal{H}(W)$ the *Hecke algebra* associated to W. Recall (see, e.g., [2], Chap. 7) that this is the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w : \; w \in W\}$ as a basis and multiplication such that

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w, & \text{if } l(ws) < l(w), \end{cases} \tag{3}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see, [2], Proposition 7.4).

Proposition 1.1 *Let $y \in W$. Then*

$$(T_{y^{-1}})^{-1} = q^{-l(y)} \sum_{x \leq y} (-1)^{l(y)-l(x)} R_{x,y}(q)\, T_x \,,$$

where $R_{x,y}(q) \in \mathbf{Z}[q]$.

The polynomials $R_{x,y}$ defined by the previous proposition are called the *R-polynomials* of W. It is easy to see that $\deg(R_{x,y}) = l(y) - l(x)$, and that $R_{x,x}(q) = 1$, for all $x, y \in W$, $x \leq y$. It is customary to let $R_{x,y}(q) \stackrel{\text{def}}{=} 0$ if $x \not\leq y$. We then have the following fundamental result that follows from (3) and Proposition 1.1 (see [2], §7.5).

Theorem 1.2 *Let $x, y \in W$ and $s \in D(y)$. Then*

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } s \in D(x), \\ q R_{xs,ys}(q) + (q-1) R_{x,ys}, & \text{if } s \notin D(x). \end{cases} \tag{4}$$

Note that the preceding theorem can be used to inductively compute the R-polynomials since $l(ys) < l(y)$. Therefore, one could take Theorem 1.2 as the definition of the R-polynomials, together with the initial conditions that $R_{x,x}(q) = 1$ and $R_{x,y}(q) = 0$, for all $x, y \in W$, $x \not\leq y$.

Even though our interest in this work is mainly in the R-polynomials, some of our results have consequences also for the Kazhdan-Lusztig polynomials, which we now define. The following result is not hard to prove (and, in fact, holds in much greater generality, see [7], Corollary 6.7 and Example 6.9) and a proof can be found, e.g., in [2], §§7.9-11, or [3], §2.2.

Theorem 1.3 *There is a unique family of polynomials $\{P_{x,y}(q)\}_{x,y \in W} \subseteq \mathbf{Z}[q]$, such that, for all $x, y \in W$:*

i) *$P_{x,y}(q) = 0$ if $x \not\leq y$;*

ii) *$P_{x,x}(q) = 1$;*

iii) *$\deg(P_{x,y}(q)) \leq \lfloor \frac{l(y)-l(x)-1}{2} \rfloor$, if $x \leq y$;*

iv)

$$q^{l(y)-l(x)} P_{x,y}\left(\frac{1}{q}\right) = \sum_{x \leq z \leq y} R_{x,z}(q)\, P_{z,y}(q) \,, \tag{5}$$

if $x \leq y$.

The polynomials $P_{x,y}(q)$ defined by the preceding theorem are called the *Kazhdan-Lusztig polynomials* of W. Note that parts iii) and iv) of Theorem 1.3 actually yield an inductive procedure to compute the polynomials $P_{x,y}(q)$ for all $x, y \in W$, taking parts i) and ii) as initial conditions. The polynomials $P_{x,y}(q)$ have been the subject of considerable study, and we refer the reader to, e.g., [2], Chapter 7, for further information about them.

Our aim in this work is to study some combinatorial properties of the R-polynomials of symmetric groups. More precisely, we point out a surprising connection between them and the enumeration and combinatorics of increasing subsequences in permutations. This leads to a simple combinatorial recurrence for computing these polynomials which in turn yields some new formulas for them. Let $\sigma \in S_n$, and $s \in S$. For $a, b, i, j \in [n]$ we let

$$\mathcal{C}_{i,j}(\sigma) \stackrel{\text{def}}{=} \{(\sigma(i_k), \dots, \sigma(i_1)) \in S_n : k \in [n],\ i = i_1 < i_2 < \dots < i_k = j,\ \sigma(i_1) < \sigma(i_2) < \dots < \sigma(i_k)\},$$

$$\mathcal{C}_{i,j}(\sigma; s) \stackrel{\text{def}}{=} \{w \in \mathcal{C}_{i,j}(\sigma) : s \in D(w\sigma)\},$$

$$\mathcal{C}(\sigma) \stackrel{\text{def}}{=} \bigcup_{1 \le i < j \le n} \mathcal{C}_{i,j}(\sigma),$$

and

$$T_\sigma[i, j; a, b] \stackrel{\text{def}}{=} \{r \in [n] : i < r < j,\ a < \sigma(r) < b\}.$$

Given $w = (\sigma(i_k), \dots, \sigma(i_1)) \in \mathcal{C}_{i,j}(\sigma)$ we let

$$n(w, \sigma) \stackrel{\text{def}}{=} \sum_{r=1}^{k-1} |T_\sigma[i_r, i_{r+1}; \sigma(i_r), \sigma(j)]|. \tag{6}$$

We also let $k(\sigma)$ be the length of the longest cycle of σ. For example, if $\sigma =$ 215496378 then $k(\sigma) = 5$, $T_\sigma[2, 6; 4, 8] = \{3\}$, $\mathcal{C}_{1,6}(\sigma) = \{(6,2), (6,5,2), (6,4,2)\}$, and if $w = (8, 6, 4, 2) \in \mathcal{C}_{1,9}(\sigma)$ then $n(w, \sigma) = 1 + 0 + 1 = 2$.

We now define a distance on S_n which will play a crucial role in all that follows. For $\sigma, \tau \in S_n$ we let

$$d(\sigma, \tau) \stackrel{\text{def}}{=} \max\{i \in [n] : \sigma^{-1}(i) \neq \tau^{-1}(i)\}, \tag{7}$$

(where $\max\{\emptyset\} \stackrel{\text{def}}{=} 0$). So, for example, $d(198265374, 298461357) = \max\{1, 2, 5, 7, 4\} =$ 7, and $d(\sigma, \tau) \neq 1$ for all $\sigma, \tau \in S_n$.

Our main result is the following simple combinatorial recurrence, defined in terms of increasing subsequences of permutations, for computing the R-polynomials of symmetric groups.

Theorem 1.4 *Let $\sigma, \tau \in S_n$ be such that $\sigma < \tau$. Then*

$$R_{\sigma,\tau}(q) = \sum_{w \in \mathcal{C}_{\tau^{-1}(d), \sigma^{-1}(d)}(\sigma)} q^{n(w,\sigma)} (q-1)^{k(w)-1} R_{w\sigma,\tau}(q), \tag{8}$$

where $d \stackrel{\text{def}}{=} d(\sigma, \tau)$.

The preceding recurrence has several advantages over the one given by (4), both of a theoretical as well as practical nature. The main one is that (8) does not "branch off" into two cases as (4) does. This allows one to use (8) repeatedly and thus explicitly solve the recurrence. While this is theoretically possible also with (4), the details are much simpler using (8). The second one is that the recurrence (8) does not change the second permutation. This is extremely useful in induction arguments, and is used the proofs of Theorems 1.9 and 1.6. The third one is that the recurrence (8) is much faster from a computational point of view. We have implemented both recursions (4) and (8) on a computer (using MAPLE) and have been able to verify this. For example, computing the R-polynomial of any pair of permutations in S_8 takes (running MAPLE V on a Sun SparcStation SLC) less than 65 seconds using (8) while it takes more than 5 minutes to compute $R_{12\dots8,8\dots21}(q)$ using (4). These MAPLE programs (which will run also on older versions of MAPLE) are available from the author upon request.

We then apply our main result to the explicit computation of Kazhdan-Lusztig and R-polynomials. More precisely, we first derive a combinatorial formula for the R-polynomials by "solving" the recurrence relation (8).

Let $\sigma, \tau \in S_n$, $\sigma \le \tau$. An *R-chain* from σ to τ is a chain $\sigma = \sigma_0 < \sigma_1 < \dots < \sigma_r = \tau$ such that:

i) $d(\sigma_i, \tau) < d(\sigma_{i-1}, \tau)$;

ii) $(\sigma_i)(\sigma_{i-1})^{-1} \in \mathcal{C}(\sigma_{i-1})$;

for all $i = 1, \dots, r$. We denote by $\mathcal{R}(\sigma, \tau)$ the set of all R-chains from σ to τ. Given any chain $C = (\sigma_0 < \sigma_1 < \dots < \sigma_r)$ in S_n we define its *R-length* to be

$$l_R(C) \stackrel{\text{def}}{=} \sum_{i=1}^{r} (k((\sigma_i)(\sigma_{i-1})^{-1}) - 1).$$

For example, $C = (1234 < 4132 < 4312 < 4321)$ is an R-chain from 1234 to 4321 and its R-length is $l_R(C) = 2 + 1 + 1 = 4$.

Theorem 1.5 *Let $\sigma, \tau \in S_n$, $\sigma \le \tau$. Then*

$$R_{\sigma,\tau}(q) = \sum_{C \in \mathcal{R}(\sigma,\tau)} q^{\frac{l(\tau) - l(\sigma) - l_R(C)}{2}} (q-1)^{l_R(C)}. \tag{9}$$

We also single out some families of pairs of elements of S_n for which the corresponding R-polynomial has a simple closed form.

Theorem 1.6 *Let $\sigma \in S_n$ and $w \in \mathcal{C}(\sigma)$. Then*

$$R_{\sigma,w\sigma}(q) = (q-1)^{k(w)-1}(q^2 - q + 1)^{n(w,\sigma)}.$$

We denote by S_3 the poset obtained by partially ordering S_3 with (strong) Bruhat order.

Theorem 1.7 *Let $\sigma, \tau \in S_n$, $\sigma \leq \tau$, be such that $[\sigma, \tau]$ does not contain an interval isomorphic to S_3. Then*

$$R_{\sigma,\tau}(q) = (q-1)^{l(\tau)-l(\sigma)}. \quad (10)$$

This last result, in turn, implies an interesting connection between the Kazhdan-Lusztig polynomials of these intervals and the h-vector of their duals.

Theorem 1.8 *Let $u, v \in S_n$, $u \leq v$, be such that $[u, v]$ does not contain an interval isomorphic to S_3. Then*

$$P_{u,v}(q) = g([u,v]^*; q). \quad (11)$$

In particular,

$$P_{u,v}(q) = 1 + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} (h_i - h_{i-1}) q^i$$

where $d \stackrel{\text{def}}{=} l(v) - l(u) - 1$, and $(h_0, h_1, \ldots, h_d)$ is the h-vector of $[u, v]^*$.

We also obtain sufficient conditions on two permutations so that their Kazhdan-Lusztig and R-polynomial factor and obtain as a consequence the result that Kazhdan-Lusztig and R-polynomials of symmetric groups are closed under products.

Let $\sigma \in S_n$, and $i, j \in [n]$, $i \leq j$. We define the *restriction* of σ to $[i, j]$ to be the unique permutation $\sigma_{[i,j]} \in S([i, j])$ such that

$$\sigma^{-1}(\sigma_{[i,j]}(i)) < \sigma^{-1}(\sigma_{[i,j]}(i+1)) < \ldots < \sigma^{-1}(\sigma_{[i,j]}(j)).$$

For example, if $\sigma = 7251634$ then $\sigma_{[3,5]} = 534$ (i.e, $\sigma_{[3,5]}(3) = 5$, $\sigma_{[3,5]}(4) = 3$, $\sigma_{[3,5]}(5) = 4$). Note that $\sigma_{[i,j]} = \mathrm{Id}([i,j])$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(i+1) < \cdots < \sigma^{-1}(j)$, and that if $\sigma([i,j]) = [i,j]$ then $\sigma_{[i,j]} = \sigma|_{[i,j]}$.

Theorem 1.9 *Let $\sigma, \tau \in S_n$, $\sigma \leq \tau$. Suppose that there exist $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $\sigma^{-1}((i_j, i_{j+1}]) = \tau^{-1}((i_j, i_{j+1}])$ for all $j = 0, \ldots, k$ (where $i_0 \stackrel{\text{def}}{=} 0$, $i_{k+1} \stackrel{\text{def}}{=} n$). Then*

$$R_{\sigma,\tau}(q) = \prod_{j=0}^{k} R_{\sigma_{(i_j,i_{j+1}]},\tau_{(i_j,i_{j+1}]}}(q), \quad (12)$$

and

$$P_{\sigma,\tau}(q) = \prod_{j=0}^{k} P_{\sigma_{(i_j,i_{j+1}]},\tau_{(i_j,i_{j+1}]}}(q). \quad (13)$$

We illustrate the preceding theorem with an example. Let $\sigma = 16573824$, and $\tau = 47583612$. Then $\sigma^{-1}([1,4]) = \{1,5,7,8\} = \tau^{-1}([1,4])$ and $\sigma^{-1}([5,8]) = \{2,3,4,6\} = \tau^{-1}([5,8])$, hence $R_{16573824,47583612}(q) = R_{\sigma_{[1,4]},\tau_{[1,4]}}(q)\, R_{\sigma_{[5,8]},\tau_{[5,8]}}(q) = R_{1324,4312}(q)\, R_{6578,7586}(q) = R_{1324,4312}(q)\, R_{2134,3142}(q) = (q(q-1)^2 + (q-1)^4)(q-1)^2 = (q-1)^4(q^2-q+1)$, and $P_{16573824,47583612}(q) = P_{\sigma_{[1,4]},\tau_{[1,4]}}(q)\, P_{\sigma_{[5,8]},\tau_{[5,8]}}(q) = P_{1324,4312}(q)\, P_{6578,7586}(q) = P_{1324,4312}(q)\, P_{2134,3142}(q) = 1 \cdot 1 = 1$.

Corollary 1.10 *The product of two R-polynomials (respectively, Kazhdan-Lusztig polynomials) is again an R-polynomial (respectively a Kazhdan-Lusztig polynomial).*

For example, $R_{1324,3412}(q)\, R_{123,321}(q) = R_{1324567,3412765}(q) = R_{5671324,7653412}(q) = R_{5136274,7346152}(q)$, $P_{1234,3412}(q)\, P_{13425,34512}(q) = P_{123457869,341278956}(q) = P_{152378649,374189526}(q) = P_{571823694,783941562}(q)$, etc... .

Note that, in particular, our results imply that if the interval between two permutations in Bruhat order is a lattice then the corresponding R-polynomial is just a power of $q-1$, and the corresponding Kazhdan-Lusztig polynomial is the g-polynomial of the dual interval.

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