Bijective Census of Rooted Planar Maps
A survey

Robert Cori

Laboratoire Bordelais de Recherche en Informatique
Unité associée C.N.R.S. 1304
351, cours de la Libération
F-33405 TALENCE
email: cori@geocub.greco-prog.fr

1 Introduction

The enumeration theory of planar maps was settled by W.T. Tutte in the sixties, mainly in four seminal articles [19, 20, 21, 22], called the census papers. He obtained very elegant formulas for the number of planar rooted maps with a fixed number of arcs, with given degrees for the vertices, and for various subfamilies of planar rooted maps; among them are bipartite maps, non separable, 3-connected ones, and triangulations of different kinds. The method he used to obtain these results consists in three steps:

- First give some recursion formulas for the number of maps of a certain family, using some combinatorial constructions like splitting faces or vertices, removal or contraction of arcs.
- Then transform into an equation for the generating power series the recursion formulas obtained. This generating power series depend often on 2 or more variables (denote it by $F(x,y)$), and the equation consists of a polynomial in $F(x,y)$ and $F(x,a)$ (for a constant a) equate to 0.
- Solve this equation using the so called "quadratic method" [7, 23], also described simply in [11].

Recently a new interest is given to this theory. Mainly because maps appear in various fields of mathematics [17] and theoretical physics. For instance topology needs effective computation of some numerical constants of surfaces [12]. Grothiendieck's celebrated "Esquisse d'un Programme" enhances the deep connection between combinatorial theory of maps, Riemann surfaces and number field theory; the word " Dessins d'enfants " is used to describe these impressive developments [2]. In theoretical physics some models of interactions use deeply graphs embedded in surfaces and need results in enumeration of these objects [5, 13, 16].

In parallel with this new interest, combinatorialists obtained new results in the enumeration theory of maps using mainly three methods:

- The systematic use of the quadratic method of Tutte for solving equations involving formal power series obtained by recursion techniques and the application of asymptotic methods [3, 4]
- The evaluation of matrix integrals as it is done by physicists [5, 13]
- The use of character theory in the symmetric group allows to obtain enumeration formulas involving Schur's functions [5, 14, 15].

The new results obtained concern various asymptotic evaluations of different kinds of maps [3, 4], enumeration of maps of arbitrary genus and careful study of the generating power series of maps by genus;[1, 15, 24], counting of maps having given degrees for faces and vertices[14], enumeration of non-rooted maps [18].

In what follows our aim is neither to introduce these new methods nor to obtain new enumerative results. We consider the bijective proof point of view and try to show that some nice combinatorial constructions are contained in these formulas. Bijective arguments are probably not the more powerful method to obtain new formulas for maps, but they shed new light on the calculations given by Tutte and others. This viewpoint to the intriguing landscape of maps is interesting by itself.

2 Planar maps

A *planar map* is a partition of the sphere Σ into three disjoint subsets V, A, F,

- V is a set of points called *vertices*
- A is a set of disjoint open arcs called *arcs* homeomorphic to $]0,1[$ whose end points are vertices
- F is a set of disjoint simply connected domains called *faces*, each face is homeomorphic to an open disc.

The definition given here implies the connectedness of the graph with vertex set V and edge set corresponding to the arcs of A. A *cell* is either a vertex, an arc or a face. Two cells are *incident* if one is contained in the boundary of the other. An arc is called a *loop* if its end points coincide it is a *link* otherwise. An *isthmus* is an arc incident twice to the same face. Note that two arcs may have the same endpoints. The *degree* of a vertex is the number of arcs incident with it. So is the degree of a face.

A map is *rooted* if an oriented arc e_0 is distinguished as the root arc, the vertex origin of the root arc is called the root vertex.

Two rooted maps (Γ, e_0) and (Γ', e'_0) are *isomorphic* if there exists a homeomorphism h of the sphere preserving orientation and mapping vertices, arcs, and faces of Γ onto vertices, arcs, and faces of Γ' such that:

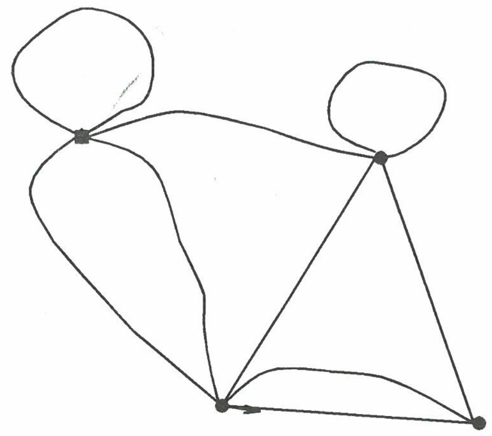

Figure 1: A planar rooted map with 4 vertices

- Two cells of Γ are incident if and only if there images by h are incident in Γ'.
- $h(e_0) = e'_0$

In *Figure 1* is given a planar map with 4 vertices 9 arcs and 7 faces. The root arc is indicated by an arrow. The degree of the root vertex is 5 and the degrees of other vertices are $3, 5, 5$.

The enumerative results presented in this paper concern only rooted planar maps.

3 Planar maps with vertices of given degrees

The oldest result in the enumeration of planar maps is that of rooted one vertex planar maps. It is well known that the number of these maps with m arcs is given by the Catalan number:

$$\frac{(2m)!}{m!(m+1)!}$$

With an arbitrary number of vertices of given degrees the problem becomes harder. W.T. Tutte [21] obtained a closed formula when all the vertices have even degree and announced a result of the same kind when only two vertices have odd degrees. We give here a bijective proof of this result[8, 9]. The first step is the following:

Proposition 3.1 *The number of rooted planar maps with k vertices of degree 2 and a root vertex of degree $2n_0$ is given by:*

$$\frac{(2n_0)!}{n_0!(n_0+1)!}\,\frac{(n_0+k-1)!}{(n_0-1)!k!}$$

Proof: To build such a map one first considers a rooted map with one vertex of degree $2n_0$ then adds k vertices of degree 2 on n_0 arcs. The operation of adding a vertex of degree 2 on an arc is the reverse of removing such a vertex, it is completely determined by the arc on which it is added. The number of different configurations is equal to the number of $k-$combinations of the set of arcs, this set is of cardinality n_0. Thus the number of rooted planar maps with k vertices of degree 2 and a root vertex of degree $2n_0$ is the product of the Catalan number by the number of k-combinations of a set of n_0 elements.

For a finite sequence $u = m_0, m_1, \ldots, m_k$ of positive integers denote by:

$$M(u) = M(m_0, m_1, m_k)$$

the number of rooted maps in which the root vertex is of degree m_0 and the other vertices are of degree $m_1, \ldots, m_k$, we have proved that:

$$M(2n_0, 2, \ldots, 2) = \frac{(2n_0)!}{n_0!(n_0+1)!}\frac{(n_0+k-1)!}{(n_0-1)!k!}$$

Let us now consider vertex-labeled planar rooted maps, that is maps in which vertices have different labels consisting of a number between 1 and k. The set of vertex labeled maps with vertices of given degrees $m_0, m_1, \ldots, m_k$ is denoted by $\mathcal{M}_l(m_0, m_1, \ldots, m_k)$ and its cardinality by: $M_l(m_0, m_1, \ldots, m_i, \ldots, m_k)$

It is easy to prove that

$$M_l(m_0, m_1, \ldots, m_k) = \prod_{j=1}^{\infty} k_j! M(m_0, m_1, \ldots, m_k)$$

where k_j is for any j the number of $m_i, i \geq 1$ equal to j.

Remark also that $m_1 m_2 \ldots m_k M_l(m_0, m_1, \ldots, m_k)$ is a symmetric function in $m_0, m_1, \ldots m_k$.

The following proposition, proved in [9] is the main tool in this section.

Proposition 3.2 *For any* $1 \leq i \leq k$ *there exists a bijection between the set*

$$\mathcal{M}_l(2n_0, 2n_1, \ldots, 2n_i, \ldots, 2n_k)$$

and the set

$$\mathcal{M}_l(2n_0 - 1, 2n_1, \ldots, 2n_{i-1}, 2n_i + 1, 2n_{i+1} \ldots, 2n_k)$$

The complete proof requires some attention, one has show that when a map has all its vertices of even degree then it is possible to move one arc incident to the root vertex to another chosen vertex.

From this Proposition it is possible to obtain a formula for the set of labeled maps with vertices of even degrees, the technique consists in the use of the previous bijection and the symmetry of the function $m_1, m_2 \ldots m_k M_l(m_0, m_1, \ldots, m_k)$. Remove iteratively arcs incident to non root vertices

such that they become incident to the root vertex. The process terminates when all the non root vertices have degree 2. Let us give some details: for any positive integers $n_0, n_1, \ldots$, let u, u' be the following $k+1$-tuples:

$$u_i = 2n_i \qquad \forall i = 0, k$$

$$u'_0 = 2n_0 + 2, u'_1 = 2n_1 - 2, u'_i = u_i = 2n_i \qquad \forall i = 2, k$$

From the above theorem one has:

$$M_l(2n_1, 2n_0, 2n_2 \ldots, 2n_k) = M_l(2n_1 - 1, 2n_0 + 1, 2n_2, \ldots, 2n_k)$$

By the same argument:

$$M_l(2n_0 + 2, 2n_1 - 2, 2n_2, \ldots 2n_k) = M_l(2n_0 + 1, 2n_1 - 1, 2n_2, \ldots 2n_k)$$

Since $m_0 m_1 \ldots m_k M_l(m_0, m_1, \ldots m_k)$ is a symmetric function on its arguments:

$$m_1 M_l(m_0, m_1, m_2, \ldots m_k) = m_0 M_l(m_1, m_0, m_2, \ldots, m_k)$$

giving:

$$M(u) = \frac{2n_0}{2n_1} M_l(2n_1, 2n_0, 2n_2, \ldots, 2n_k)$$

We then obtain a formula allowing to transfer 2 arcs from any vertex to the root one:

$$M(u) = \frac{n_0}{2n_0 + 1}\frac{(2n_1 - 1)}{n_1} M_l(u')$$

Applying this process sufficiently many times one has:

$$M_l(u) = \frac{(2n_1-1)(2n_1-3)\ldots 3}{n_1(n_1-1)\ldots 2}\frac{n_0(n_0+1)\ldots(n_0+n_1-2)}{(2n_0+1)(2n_0+3)\ldots(2n_0+2n_1-3)} M_l(2n_0 + 2n_1 - 2, 2, 2n_2, 2n_3, \ldots, 2n_k)$$

After some simplification:

$$M_l(u) = \frac{(n_0+n_1-1)!(n_0+n_1-2)!}{(2n_0+2n_1-2)!}\frac{(2n_1-1)!}{n_1!(n_1-1)!}\frac{2n_0!}{(n_0-1)!n_0!} M_l(2n_0 + 2n_1 - 2, 2, 2n_2, 2n_3, \ldots, 2n_k)$$

Transfering arcs from other vertices one obtains:

$$M_l(u) = \frac{(n-k)!(n-k-1)!}{(2n-2k)!} 2n_0 \prod_{i=0,k} \frac{(2n_i - 1)}{!n_i!(n_i-1)!} M_l(2n_0 + 2n_1 + \ldots + 2n_k - 2k, 2, 2, \ldots, 2)$$

Now applying the previous result and the relation between $M(u)$ and $M_l(u)$ gives:

Theorem 3.3 *The number of planar maps having* k *vertices,* m *arcs; the root vertex of which is of degree* $2n_0$*, and for any* i *having* k_i *non root vertices of degree* $2i$*, is given by:*

$$\frac{(m-1)!}{(m-k+1)!} 2n_0 \prod_{i=0}^{\infty} \binom{2i-1}{i}^{k_i} \frac{1}{k_i!}$$

4 Maps by number of arcs

The enumeration of rooted planar maps with m arcs was obtained by W.T. Tutte [22] using a bijective proof, let us recall his proof.

Theorem 4.1 *There exists a bijection β between rooted planar maps having m arcs and rooted planar maps with m vertices all of degree 4.*

Let $\mathcal{M}$ be a planar map, with set of vertices V set of arcs A and set of faces F. To build $\beta(\mathcal{M})$, first associate to each arc $a \in A$ four *darts* $(a,1)$, $(a,2)$, $(a,3)$, $(a,4)$, each one being associated to the incidence between the arc, the two vertices it joins and the two faces it separates. Let v', v'' be the vertices at the ends of a and let f', f'' the faces incident with a. The dart $(a,1)$ corresponds to v', f', $(a,2)$ to $v'f''$, $(a,3)$ to v'', f' and $(a,4)$ to v'', f''. The darts $(a,1), (a,3)$ are in the same face incident to a and so are $(a,2)$ and $(a,4)$. The darts $(a,1), (a,2)$ are in a neighborhood of v' and $(a,3), (a,4)$ in a neighborhood of v''.

Build a planar map $\mathcal{M}'$ having the darts of $\mathcal{M}$ as vertices in the following way:

- Each dart is of degree 3,
- Arcs connect $(a,1)$ to $(a,2)$, $(a,1)$ to $(a,3)$, $(a,2)$ to $(a,4)$, $(a,3)$ to $(a,4)$.
- Other arcs connect $(a,1)$ to (b,x), $(a,2)$ to (c,y), $(a,3)$ to (d,z) and $(a,4)$ to (e,t), where b is the other arc incident to v' and f', c is incident to v' and f'', d is incident to v'' and f' and e is incident to v'' and f''.

The map $\mathcal{M}'$ obtained in that way has $\cancel{4}$ 2 $Card(A)$ arcs, and as many faces as $\mathcal{M}$ has ~~arcs~~ faces and vertices. To each vertex v of $\mathcal{M}$, corresponds a face of $\mathcal{M}'$ of degree twice the degree of v, to each face f of $\mathcal{M}$ a face of degree twice the degree of f and to each arc of $\mathcal{M}$ a ~~face~~ vertex of $\mathcal{M}'$ of degree 4.

The last step in the bijection consists in contracting the faces of $\mathcal{M}'$ of degree 4 into points. These become the vertices of a new map $\beta(\mathcal{M})$ all having degree 4, it is also possible to choose a convention in order to determine the root of $\beta(\mathcal{M})$ from that of $\mathcal{M}$(see Figures 2 and 3).

The reverse bijection is simple to obtain composing the reverse of the elementary operations performed above.

Corollary 4.2 *The number of rooted planar maps with m arcs is:*

$$2\ 3^m \frac{(2m-1)!}{m!(m+2)!}$$

Proof Use formula of Theorem 3.3 with $k = m - 1$, and $\forall i, n_i = 2$.

Remark Another bijective proof was given using well labeled trees [10], this proof generalizes to bipartite maps [1]

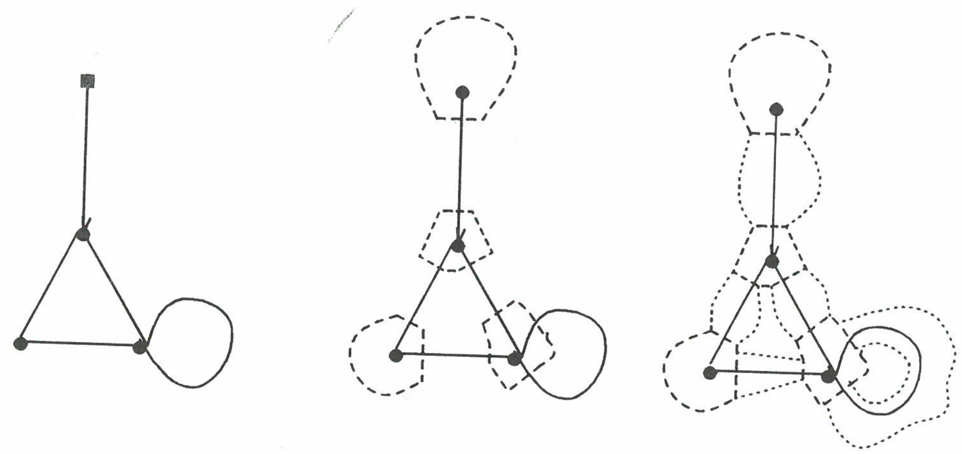

Figure 2: Bijection between planar rooted maps ...

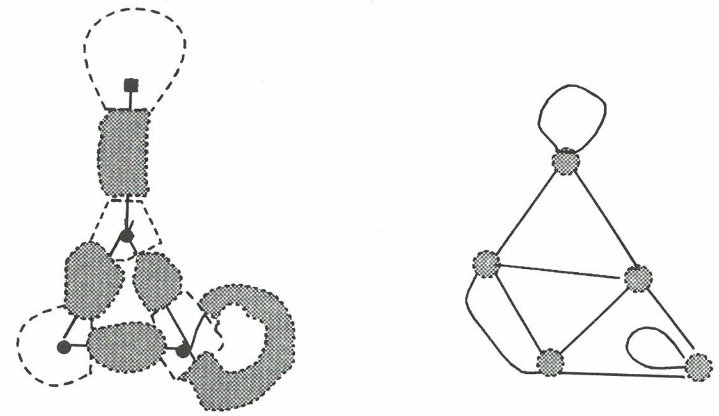

Figure 3: ... and maps with vertices of degree 4

5 Various kinds of maps

In this section we give as example the enumeration of non-separable planar maps as given by Tutte [22].

A map is *separable* if its arc set A can be partitioned into two subsets A_1 and A_2 such that there exists exactly one vertex incident to an arc of A_1 and an arc of A_2. Note that a map with more than one arc and containing a loop is separable (consider the partition consisting of the loop in one subset and the other arcs in the other subset).

In order to enumerate non separable maps one has to consider derived maps. The derived $D(\mathcal{M})$ map of $\mathcal{M}$ is constructed as follows (see Figure 4.):

- Between the two ends of each arc a of $\mathcal{M}$ put a vertex, the center $c(a)$ of a
- In any face f of $\mathcal{M}$ put a new vertex, the center $c(f)$ of f
- Join by an arc $c(f)$ to all the vertices laying in the boundary of f including the centers of the arcs incident with f. The center of an isthmus will be joined twice to the center of the face in which it lies.

Any map $\mathcal{M}'$ satisfying the following conditions is the derived map of a map $\mathcal{M}$:

1. The set of vertices can be colored in three colors V_1, V_2, V_3 such that the end points of each arc are in different classes.
2. Each element of V_2 has degree 4.
3. Each face is incident with three edges.

To reconstruct $\mathcal{M}$ from $\mathcal{M}'$ consider only the vertices of V_1 and for any vertex v_2 in V_2 draw an arc between the two vertices of V_1 which are neighbors of v_2.

A *lune* in the derived map $D(\mathcal{M})$ of a map $\mathcal{M}$ is a simple closed curve consisting of two arcs having the same endpoints v_1 and v_3 respectively in V_1 and V_3. A vertex v_1 of $\mathcal{M}$ is a cut vertex in $\mathcal{M}$ if and only if v_1 is the endpoint of a lune in $D(\mathcal{M})$. Thus a map is non separable if and only if its derived map has no lunes. The inside of a lune is the domain bounded by the lune and not containing the root. To any map $\mathcal{M}$ on associates the non separable core $\mathcal{M}_0$ by deleting all lunes in $D(\mathcal{M})$, obtaining the derived map $D(\mathcal{M}_0)$ and applying the inverse transform in order to obtain $\mathcal{M}_0$.

The maps inside the lunes are the derived maps of maps $\mathcal{M}_i, i = 1, l$. the map $\mathcal{M}$ can be reconstructed from $\mathcal{M}_0$ and the $\mathcal{M}_i, i = 1, l$. we thus get:

Proposition 5.1 *The formal power series $M(x) = \sum_{m=0}^{\infty} a_m x^m$ enumerating planar rooted maps by arcs and $M_{ns}(x) = \sum_{m=0}^{\infty} b_m x^m$ enumerating planar non separable rooted maps by arcs satisfy the following equation:*

$$M(x) = M_{ns}(x(1 + M(x))^2)$$

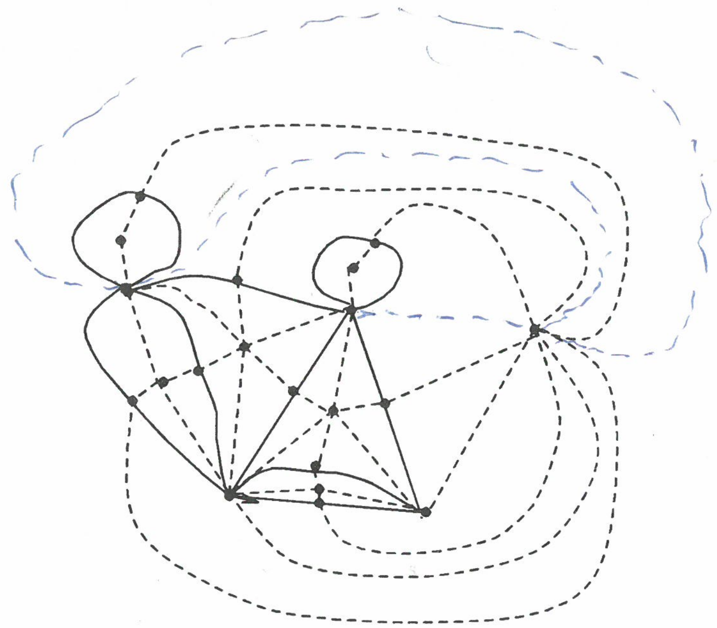

Figure 4: The derived map of the map in figure 1

Theorem 5.2 *The number of non separable maps with m edges is;*

$$b_m = 2\,\frac{(3m-3)!}{m!(2m-1)!}$$

To prove this theorem, first remark that the series $M(x)$ satisfies the parametric equations:

$$M(x) = \frac{1}{3}(3-u)(u-1)$$

$$u = 1 + 3xu^2$$

Giving by denoting $v = 1 + u$ and $F(x) = x(1 + M(x))^2$:

$$M_{ns}(F(x)) = -\frac{1}{3}(v)(v+2)$$

and $v = \frac{-27F}{(3+v)^2}$ The result is then obtained, applying Lagrange's formula.

A bijective proof of the formula on non-separable maps is still an open problem.

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