le complémentaire ($\pi^c(i) = n + 1 - \pi(i)$) ou l'inverse ($\pi^{-1}(i) = j$ si et seulement si $\pi(j) = i$). Le chemin conduisant à cette preuve est composé de quatre arbres de génération dont deux seulement sont identiques et passe par la famille $S_n(2413, 41\bar{3}52)$ considérée dans ce travail (voir figure 6).

Une analyse des règles de construction de ces quatre arbres nous permet d'obtenir des formules analogues à celles obtenues ici et donnant les distributions de ces permutations suivant divers paramètres.

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Rat Races and the Hook Length Formula

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Abstract

A bijective proof of the hook length formula is presented. This is the famous expression for the number of standard Young tableaux of a given shape. A variant of the Vandermonde determinant proves that the formula can be written as an alternating sum of multinomial coefficients that enumerate certain tableaux. An interpretation of these as describing rat races suggests a bijection that proves the formula.

Key words. Young tableau, hook length, Vandermonde, bijection.

1 Introduction

The well-known *hook length formula* by Frame, Robinson and Thrall [1] gives an extremely simple expression for the number of standard Young tableaux of a given shape $\lambda \vdash n$, namely $n!/\prod h_{ij}$, where the hook-length h_{ij} is the number of squares in the hook with lower left corner in square i, j (using Latin diagrammatic conventions). No really attractive combinatorial explanation of this formula is known. The proof presented in this note uses a rewriting of the formula as an alternating sum of multinomial coefficients. For the shape $\lambda = (2, 2, 3)$ we get

$$\frac{7!}{(2\cdot 1)(3\cdot 2)(5\cdot 4\cdot 1)} = \binom{7}{2,2,3} - \binom{7}{1,3,3} + \binom{7}{0,3,4} - \binom{7}{2,1,4} + \binom{7}{1,1,5} - \binom{7}{0,2,5}$$

$$21 \quad = \quad 210 \quad - \quad 140 \quad + \quad 35 \quad - \quad 105 \quad + \quad 42 \quad - \quad 21$$

Here we interpret the multinomial coefficients as enumerating more general tableaux called *rat races* and we define an involution on rat races whose fixed points are precisely the standard Young tableaux.

In order to obtain a direct bijective proof of the hook length formula, we also construct an involution that proves the equivalence of the hook length formula and the alternating sum expression. The bijection given by Franzblau and Zeilberger [2] is more explicit but also more intricate, so we believe that our proof might be preferable. The alternating sum was used by Linial [6] in a short (nonbijective) proof of the hook length formula. One should also mention the elegant probabilistic proof by Greene, Nijenhuis and Wilf [5].

2 The hook length formula rewritten

A *Ferrers board* of shape $\lambda = (\lambda_1, \ldots, \lambda_m)$, with $0 \le \lambda_1 \le \lambda_2 \le \ldots$ and $\lambda_1 + \cdots + \lambda_m = n$, is an array of n cells arranged in m rows, left justified with λ_i cells in the i-th row. The *hook length* h_{ij} of cell (i,j) is defined as the number of cells to the right of it plus the number of cells above it plus one. A *standard Young tableau* is a filling of the diagram with the numbers $1, \ldots, n$, increasing from left to right and from bottom to top. It is convenient to have a notation for the *shifted shape* μ defined by $\mu_1 = \lambda_1, \mu_2 = \lambda_2 + 1, \ldots, \mu_m = \lambda_m + m - 1$, for which one has $0 < \mu_1 < \mu_2 < \ldots$. We use the notation $\pi = (\pi_1, \ldots, \pi_m)$ for a permutation of $\{1, \ldots, m\}$ and π_{id} for the identity permutation.

Some of the multinomial coefficients in the formula below may have negative parameters in them, and then their value is zero by definition.

Proposition 1 *The expression in the hook length formula can be written as an alternating sum of multinomial coefficients:*

$$\frac{n!}{\prod h_{ij}} = \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \binom{n}{\lambda + \pi_{\mathrm{id}} - \pi}. \tag{1}$$

Proof: The first hook length in row i is μ_i, so the product of all hook lengths in that row is like $\mu_i!$ with some factors skipped. It is easy to see that

$$\frac{n!}{\prod h_{ij}} = \frac{n!}{\prod \mu_i!} \prod_{i>j} (\mu_i - \mu_j)$$

and the familiar expression $\prod(\mu_i - \mu_j)$ is the determinant of the matrix named after Vandermonde, with entries $a_{ij} = \mu_i^{j-1}$. We can replace the powers μ_i^{j-1} by factorial powers $(\mu_i)_{(j-1)} = \mu_i(\mu_i - 1) \cdots (\mu_i - j + 2)$ and leave the value of the determinant unchanged, for to each column we have only added some multiples of other columns. For example, $m = 3$ gives

$$(\mu_2 - \mu_1)(\mu_3 - \mu_1)(\mu_3 - \mu_2) = \begin{vmatrix} 1 & \mu_1 & \mu_1^2 \\ 1 & \mu_2 & \mu_2^2 \\ 1 & \mu_3 & \mu_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \mu_1 & \mu_1(\mu_1 - 1) \\ 1 & \mu_2 & \mu_2(\mu_2 - 1) \\ 1 & \mu_3 & \mu_3(\mu_3 - 1) \end{vmatrix}$$

From the definition of the determinant we get

$$\frac{n!}{\prod \mu_i!} \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \prod_i (\mu_i)_{(\pi_i - 1)} = \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \frac{n!}{\prod(\mu_i - \pi_i + 1)!} = \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \binom{n}{\lambda + \pi_{\mathrm{id}} - \pi},$$

which is the formula that we wanted to prove. □

Dividing the j-th column of $|(\mu_i)_{(j-1)}|$ by $(j-1)!$ we get the *binomial determinant* $\left|\binom{\mu_i}{j-1}\right|$, which by the main theorem of Gessel and Viennot [4] is the number of certain noncrossing lattice paths. This interpretation is closely connected with our rat races. The authors of [4] note with regret the absence of a bijective proof of

$$\prod_{i>j} (\mu_i - \mu_j) = 1!\,2! \cdots (m-1)! \left| \binom{\mu_i}{j-1} \right|.$$

3 Rat races

A *rat race* is an event that takes place on the real line. The rats have separate starting points, one unit apart at coordinates $0, 1, \ldots, m-1$, and they also finish at separate points, namely $\mu_1, \mu_2, \ldots, \mu_m$. The distances add up to n, for $\sum(\mu_i - i + 1) = \sum \lambda_i = n$. Every time step, one of the rats moves one unit to the right. After n time steps, all rats have reached their final position.

The rat race is completely specified by recording which rat did move in each time step $1, \ldots, n$. One way to do this is to fill in the Ferrers board of the shifted shape μ with numbers $1, \ldots, n$ and with letters from the beginning of the alphabet. Each row belongs to one rat. For the rat starting from the origin, there will be no letters, only numbers. The rat starting from 1 has a row beginning with A, the rat starting from 2 has a row beginning with AB etc. The projection of a cell on the axis is a unit interval, and the number in the cell indicates at which time step that interval is crossed by the rat of that row. Therefore the numbers are left to right increasing.

A standard Young tableau of shape λ corresponds to a rat race with decreasing southeast diagonals, and a moment's thought should convince you that this means a *nontouching race*, i.e. two rats never occupy the same coordinate at the same time. So nontouching rat races are in bijective correspondence with Young tableaux.

For the benefit of the rat crowd, all rats wear yellow T-shirts that carry the row index in red digits. At the start of a nontouching race, these numbers form the permutation $\pi_{\mathrm{id}} = (1, 2, \ldots, m)$ and rat i has a distance of λ_i to run. In a general rat race, the starting order can be any permutation π, and then rat i has to run the distance $\lambda_i + i - \pi_i$.

The number of rat races corresponding to a certain startpermutation π is the number of unordered partitions of shape $\lambda + \pi_{\mathrm{id}} - \pi$, that is $\binom{n}{\lambda + \pi_{\mathrm{id}} - \pi}$.

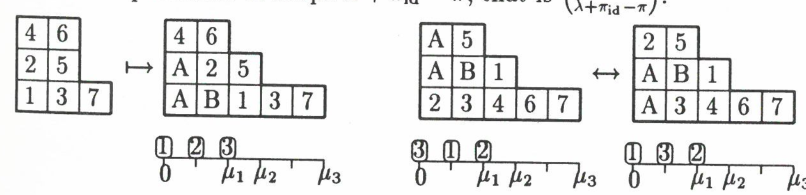

Figure 1: A standard Young tableau, a nontouching rat race and two general rat races.

Proposition 2 *The number of standard Young tableaux of a given shape λ is given by*

$$\sum_{\pi \in S_m} \operatorname{sgn}(\pi) \binom{n}{\lambda + \pi_{\mathrm{id}} - \pi}.$$

Proof: We are going to define an involution on the set of rat races that maps every race to a race with a startpermutation of opposite parity, except for nontouching races, which are left fixed. Consider a touching race with startpermutation π and let i and j be the first two rats to occupy the same position, at time t, say. Now imagine the same race but with rats i and j wearing each other's T-shirt from the start, as a token of friendship and in anticipation of the coming-together at time t. At that time, and when the crowd looks the other way, they change back into their own T-shirts with correct numbers on them. To the crowd this would look like a different race, a race with a transposition in the startpermutation relative to the previous race (cf. figure 1). We already know that nontouching rat races are in bijective correspondence with standard Young tableaux, so the proof is complete. □

4 An involution of mixed tableaux

It is not obvious how to translate the algebraic manipulations used to prove (1) into a counting argument. In order to lend itself to such a combinatorial interpretation, the formula has to be rewritten in a form not using division.

$$n! \prod_{i>j} (\mu_i - \mu_j) = \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \binom{n}{\lambda + \pi_{\mathrm{id}} - \pi} \prod_i \mu_i! \tag{2}$$

As we shall see, both sides of this equation count certain *mixed tableaux* of the following kind. The shape is μ and the entries are letters and numbers. One row contains no letters, another row one letter, namely an A, another row an A and a B etc, so for every $k < m$ there is a row containing the first k letters of the alphabet. The remaining n cells of the tableau contain the numbers $1, \ldots, n$ in any order.

A mixed tableau is a rat race if, in each row, letters precede numbers and both come in increasing order. The total number of mixed tableaux is $\prod \mu_i!$ times the number of rat races, for each row can be arbitrarily permuted. Thus,

$$\binom{n}{\lambda + \pi_{\mathrm{id}} - \pi} \prod_i \mu_i! = \#\text{ mixed tableaux of type } \pi = M_\pi,$$

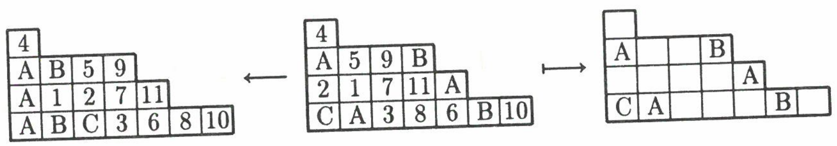

Figure 2: A rat race, a mixed tableau and the corresponding letter pattern

so the identity that needs a bijective proof is the one in the following proposition.

Proposition 3 *There is an involution on the set of mixed tableaux that proves the identity*

$$n! \prod_{i>j} (\mu_i - \mu_j) = \sum_{\pi \in S_m} \operatorname{sgn}(\pi)\, M_\pi \tag{3}$$

where M_π is the number of mixed tableaux with $\pi_i - 1$ letters in the i-th row.

Proof: As the involution will depend only on the letter pattern, the number entries are invisible in all figures from now on. We start out by describing the tableaux that are left fixed by the involution. These are all of type π_{id}, so there is no letter in the first row, one letter in the next row etc. Furthermore, the letter pattern in each row must be *legal*, a concept defined below. The definition is chosen so as to ensure that the number of legal patterns in the i-th row will be $(\mu_i - \mu_{i-1})(\mu_i - \mu_{i-2}) \cdots (\mu_i - \mu_1)$ and consequently that the number of fixed points of our involution will become $n! \prod_{i>j} (\mu_i - \mu_j)$.

Consider the i-th row with μ_i unit cells positioned on the real axis between 0 and μ_i. The cells between μ_j and μ_{j+1} form the j-th *segment*, the leftmost cell of the segment is the *niche*, the rest is the *gallery*. A horizontal line crossing a cell means that it is considered illegal to write a letter in that cell.

Start by crossing out all cells to the left of μ_1, leaving $\mu_i - \mu_1$ legal positions for A. Write a legal A, then cross out all empty cells in the first gallery. If a gallery cell is occupied by A, cross out the first niche instead. In any case, that leaves $\mu_i - \mu_2$ legal cells for B etc. In general, if A_j denotes the j-th letter, write it into any of the $\mu_i - \mu_j$ legal cells and cross out all empty cells of the j-th gallery. If a gallery cell is occupied by A_k, cross the k-th niche instead, if that niche is occupied by A_r, cross the r-th niche instead etc. Evidently, the number of *legal letter patterns* obtained in this way is $(\mu_i - \mu_{i-1})(\mu_i - \mu_{i-2}) \cdots (\mu_i - \mu_1)$.

An important observation is that for a legal pattern with $k - 1$ letters, the number of crossed cells is $\mu_k - (k-1) = \lambda_k$, independent of the row index i. For an arbitrary pattern, the first $k - 1$ letters may be legal, while letter A_k occupies one of the λ_k crossed cells. Later letters may occupy either legal or crossed cells, but the simplest case for us is when they are all in crossed cells. We are aiming at an involution on the set of all mixed tableaux, but we start by defining it for these *settled tableaux*.

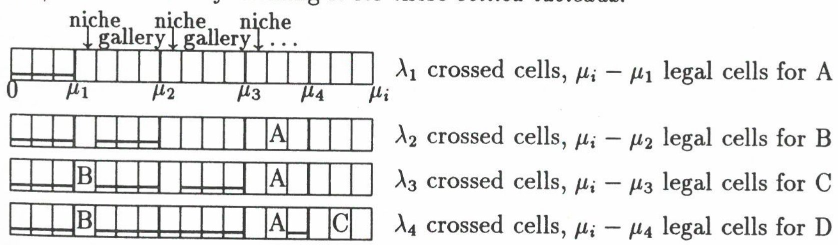

Figure 3: The construction of a legal cell pattern

Given any settled tableau such that there are two or more rows with the same number of legal letters, say $k-1$, choose the first possible such row and the next possible other row with that number of legal letters. The involution consists in switching the contents of the crossed cells of these two rows. Assuming the crossing procedure defined above, both rows have λ_1 cells that were crossed out from the beginning, both have $\lambda_2 - \lambda_1$ that were crossed out as A was written, $\lambda_3 - \lambda_2$ that were crossed in the next step etc, and every crossed cell in the first row is matched with a cell in the other row of the same crossing class, keeping the internal order of each class.

This transformation defines an involution of settled tableaux, leaving as fixed those tableaux where no two rows have the same number of legal letters. It is obvious that if the involution maps a tableau in M_π to a tableau in $M_{\pi'}$, the permutations π and π' differ by a transposition, so $\operatorname{sgn}\pi = -\operatorname{sgn}\pi'$ and their contributions to (3) cancel. If no two rows have the same number of legal letters, then the tableau must be legal, by the following argument. There is exactly one row with no legal letters, one row with one legal letter etc. On the other hand, the first row can accommodate no legal letter, the next row at most one legal letter, the row below that at most two etc. Therefore the tableau is of type π_{id} and thus legal.

It remains to be shown how to extend the involution to unsettled tableaux without destroying these properties. For a row in an unsettled tableau with $A_1, \ldots, A_{k-1}$ legal, $A_k, \ldots, A_{r-1}$ in crossed cells and A_r in a noncrossed cell, we define a *settling procedure* as follows. Switch A_k and A_r, so that A_k becomes legal and cross out the appropriate cells, i.e. empty cells in the k-th gallery and niches corresponding to letters in that gallery. If the row is still not settled, repeat by switching A_{k+1} and the first letter in a noncrossed cell etc. After some switches a settled tableau results.

If, after this settling procedure, there are two or more rows with the same number of legal letters, we can switch the contents of the crossed cells as described above. Finally, an *unsettling procedure* takes place, consisting in the same switches of letter pairs as the settling procedure, but this time in reverse order and on the other row. It is clear that

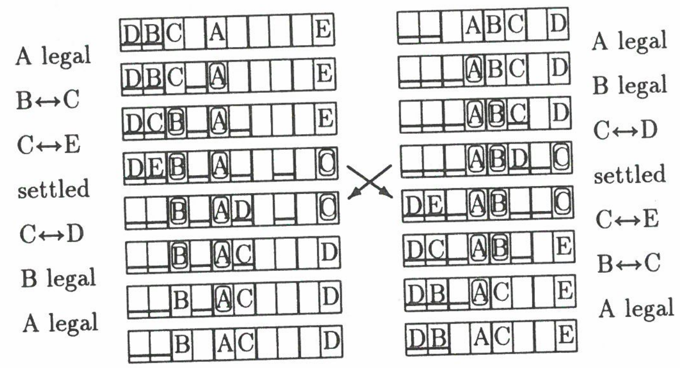

Figure 4: Settling, involuting and unsettling a pair of rows (legal letters circled)

this transformation defines an involution of mixed tableaux, leaving as fixed tableaux that become legal after settling. But the settling procedure never changes the number of letters in illegal cells, so the tableau must have been legal to begin with. □

5 The bijection

In the two previous sections we have presented two involutions on the set of mixed tableaux. Both involutions switch some entries between two selected rows. After this process, each of the rows has the number of letters that the other row used to have, so the sign of the permutation is changed. The fixed points of the first involution are the mixed nontouching rat races, the fixed points of the other one are the legal letter patterns. A bijection proving that

$$\prod_i \mu_i! \ (\#\text{ standard Young tableaux of shape } \lambda) = n! \prod_{i>j}(\mu_i - \mu_j) \qquad (4)$$

is obtained by repeated alternating application of the involutions, starting in a fixed point of one kind and ending in a fixed point of the other kind. This is a simple case of the involution principle of Garcia and Milne [3]. In our case, it is possible but not very interesting to describe the resulting bijection more explicitly.

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