Strong convergence and the polygon property of 1-player games

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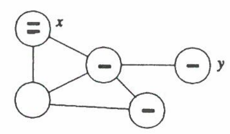
1 Introduction

The bibliography on *combinatorial games* is quite large. A. S. Fraenkel has compiled a list of 430 published papers on this subject [10]. The typical combinatorial game is a 2-player game with perfect information, no chance moves and outcome restricted to (lose, win), (tie, tie) and (draw, draw) for the two players who move alternately.

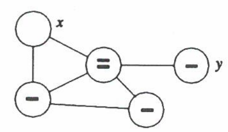
In the present paper, though, we are interested in perfect information 1-player games without chance, with a possibly infinite set of *positions*, where in each position a finite number of *moves* are possible, each leading to some position. If no move is possible, then the game has ended and we refer to such a position as *terminal*.

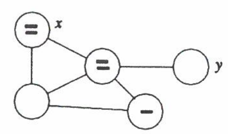
It is for this class of games that the concept of *strong convergence*, defined below, is relevant. As a canonical example we introduce the chip-firing game of Björner, Lovász and Shor (see [4]).

The game board is a graph, and a position is a distribution of chips on the nodes. For example:

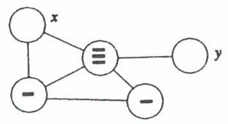


Every node with at least as many chips as outgoing edges represents a possible move, which consists of moving one chip along each outgoing edge to the adjacent nodes. In the position above there are two moves possible: either we play node x or node y. The positions obtained in this way are respectively





In each of these positions only one move is possible. Playing this one move gives, in both cases, the position



Here no move is possible, so it is a terminal position and the game ends. From this game it should be clear what we are talking about: the set of positions is in this case the set of all distributions of the chips on the nodes; the terminal positions are those where no node has enough chips; the set of moves is for every position in bijection to a subset of the nodes of the graph, hence finite. No information is hidden and no chance is involved.

Clearly, a combinatorial game in our sense can be represented, with no loss of information, by the *game graph*, whose nodes are the positions in the game and whose edges represent the moves, such that $(\overrightarrow{p,q})$ is an edge if there is a move from p to q. The example of a run in the chip-firing game above had the special property that whatever choices we made, we reached the same terminal position. This is the key issue here.

Definition. A game is said to have the *strong convergence property* if, given any starting position, either every play sequence can be continued indefinitely, or every play sequence will converge to the same terminal position in the same number of moves.

It is trivial that strong convergence implies the following property.

Definition. A game is said to have the *polygon property* if, given any position where two different moves, x and y, are legal, either there are two play sequences of the same length and beginning with x and y respectively that result in the same position, or there are two such play sequences which can be continued forever.

The 'polygon' that is referred to in the name 'polygon property' is the polygon shape in the game graph that the two play sequences, if finite, build up. Thus, we may equivalently define strong convergence as a property of directed graphs.

Indeed, the strongly convergent games can be characterized as the games that have the polygon property.

Theorem. (Polygon Property Theorem, Eriksson [5]) *A game is strongly convergent if and only if it has the polygon property.*

The idea is of course not brand new. Newman [12], 1942, shows a 'theorem of confluence', which says that if a finite directed graph is such that any two outgoing edges from a node are beginnings of a pair of directed paths with common endpoint, then every connected component has a unique sink. In a paper on ring theory by Bergman [1], 1978, this is referred to as the 'diamond lemma'. Faigle, Goecke and Schrader [9], call this same object a 'Church-Rosser system', when discussing decomposition of finite combinatorial structures. When all maximal decomposition chains are of the same length the system is said to have the 'Jordan-Dedekind property', and they show that this follows if there are always meeting paths of length two. There seems to be no precedence for the *only if* part of our theorem. The possibility of infinite paths and infinite graphs, while we demand equal length of all finite maximal paths, is natural and is, in our opinion, the correct setting.

The Polygon Property Theorem gives us a general technique for proving strong convergence; it says that it is enough to prove the existence of polygons. Again the chip-firing game provides a simple example. Since the moves are independent of each other, and since any node y that is playable before a node $x \neq y$ is played is still playable afterwards (since the number of chips on y cannot

have decreased when x was played), we have that the play sequences xy and yx constitute a polygon whenever both x and y are playable. If there are sufficiently many chips, then there will always be some node that is playable, and hence we get the case of infinite games though the polygons are of finite length.

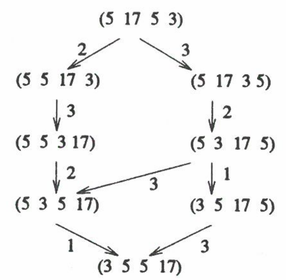
The aim of this paper is to point out the multitude of strongly convergent games (and combinatorial processes that can be regarded as strongly convergent games; see below) and how to use the Polygon Property Theorem to prove that games are strongly convergent. The games considered are the bubble-sort game, the shelling game, the chromatic game, the k-snake game and Schützenberger's jeu de taquin. We give special attention to the latter two, which were already known to be strongly convergent for algebraic reasons, though the Polygon Property Theorem gives another and simpler proof. These games are played on tableaux of squares, but we also introduce versions on tableaux of circles and of triangles. In common for all these games is that, unless the ground object (the list of numbers, the simplicial complex, the graph, or the tableau) is infinite, one will always reach a terminal position. An important example of a strongly convergent game where infinite play sequences do occur is Mozes's numbers game. We do not discuss this game here, since it has been thoroughly treated in other papers (Eriksson [6, 5, 8, 7]).

Just a couple of words on the role played by greedy algorithms in connection with strongly convergent games. *Greedy* algorithms solve combinatorial optimization problems greedily, i.e. without backtracking. Playing a strongly convergent game can be viewed as executing an algorithm that results in a certain terminal position t for any given input p, i.e. the result is well-defined despite an apparent freedom of choice in the course of the algorithm. Conversely, every algorithm of this kind can be regarded as a strongly convergent game. For instance, Kruskal's greedy algorithm for finding a minimal spanning tree in a weighted graph is in a sense a strongly convergent game: In every step, a legal move is to choose any edge of minimal weight such that it does not form any circuit with previously chosen edges. Of course, since a terminal position is a spanning tree, every play sequence to a terminal position will be of the same length, namely the number of nodes in the weighted graph minus one. The terminal positions are equivalent in the sense that they have the same weight.

2 The bubble-sort game

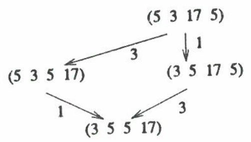
Bubble-sort is a very natural algorithm that sorts a list of n real numbers in $O(n^2)$ steps. It works by repeatedly running through the list and switching pairs of adjacent numbers in the list where a larger number comes before a smaller. The name comes from the small numbers 'bubbling up' to the front of the the list.

We shall see how this may be regarded as a combinatorial game. A position is here a list $(a_1, a_2, \ldots, a_n)$ of real numbers. A legal move is transposition of two adjacent numbers, say a_i and a_{i+1}, if they are in the wrong order, i.e. if $a_i > a_{i+1}$. Label this move by i. If the initial list is $(5, 17, 5, 3)$ then the game graph will look as follows.

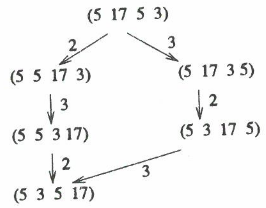


We could have guessed the terminal position in advance; it is only in the unique sorted list that no moves are legal. We could also have told beforehand that every play sequence to the terminal position would be of the same length; for every move, the number of pairs (a_i, a_j) such that a_i stands to the left of a_j while $a_i > a_j$, will decrease by one. In fact, this is a well-known quantity: the *inversion number* of the multiset permutation $(a_1, a_2, \ldots, a_n)$. Anyway, by this simple combinatorial argument we have shown that the bubble-sort game is strongly convergent (and also that the case of infinite games never occurs).

But it is also nice to see the Polygon Property Theorem work in this simple context. To prove strong convergence using this method we need to show that if two moves, $i < j$, are legal, then there exists a polygon built up from two play sequences starting with i and j respectively. Suppose first that $i < j - 1$. Then it is clear that the moves i and j can be performed independently, i.e. we have a polygon such as



In the other case, when $i = j - 1$, the moves are not independent. Instead, we always get a polygon such as



Thus we have shown the polygon property and hence strong convergence of the bubble-sort game. Note that, unfortunately, this method does not directly give us the information that the game is never infinite.

3 The shelling game

The shelling game is played on a pure d-dimensional simplicial complex Δ. A *shelling* of Δ is a linear order $F_1, F_2, \ldots, F_m$ of the facets, such that the intersection of any facet F_i, $i > 1$, with the union of its predecessors is a nonempty union of maximal proper faces, i.e. faces of dimension $d - 1$.

Let a *shelling step* mean the addition of a facet F_i, $i > 1$, to the union

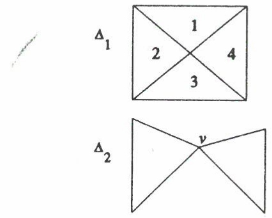
$$U_i \stackrel{\text{def}}{=} F_1 \cup F_2 \cup \ldots \cup F_{i-1}$$

such that $F_i \cap U_i$ can be expressed as $G_{i1} \cup G_{i2} \cup \ldots \cup G_{ik}$ for some k, where the G_{ij} are $(d-1)$-dimensional faces. Then $F_1, F_2, \ldots, F_m$ is a shelling of Δ if and only if adding the facets $F_2, \ldots, F_m$ in turn are proper shelling steps.

Topologically, a shelling describes the construction of Δ by starting with some facet F_1 and stepwise attaching facets to the currently constructed complex so that the intersection is a $(d-1)$-ball or a $(d-1)$-sphere.

In the *shelling game*, the first move picks any facet to begin with, and then the legal moves are the possible shelling steps. This game is like a solitaire: the player's goal is to shell the entire complex

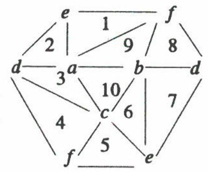
Δ. This is not always possible, but if there exists a shelling, then we say that Δ is *shellable*. Let us look at two examples.



Above we see two 2-dimensional complexes, Δ_1 and Δ_2. Δ_1 is shellable, and one shelling is given by the numbers written on the facets, which for the 2-dimensional case are triangles. The other complex, Δ_2, is not shellable. It consists of two facets, and when adding the second facet the intersection with the first will be $\{v\}$ which is a 0-dimensional face, while maximal proper faces are 1-dimensional.

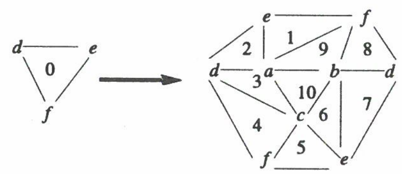
Take another look at Δ_1. It is clear that the player of the shelling game cannot fail; whatever shelling step he chooses at each time he will eventually have shelled all of Δ_1. Since the length of such a game will always be equal to the number of facets, the shelling game on Δ_1 is strongly convergent.

Can it be so that the shelling game is strongly convergent on all shellable complexes? No. The concept we are discussing is known by name of *extendable shellability*. A complex is extendably shellable if every partial shelling can be completed to a shelling. Clearly this is the same as saying that the shelling game will never end before the entire complex is shelled, which by the above implies strong convergence of the game. And it is known that there exist shellable complexes that are not extendably shellable. Below is an example, due to Anders Björner.



The complex above has ten triangles, fifteen edges and six vertices (points with equal marking are identified). This is a triangulation of the real projective plane, and it is not shellable, for homology reasons. For example, no shelling step is possible after shelling 10,3,6,9. Check for yourself!

Now, glue in an additional triangle with vertices d, e, f:



Then 0,1,2,3,4,5,6,7,8,9,10 is a shelling of this new complex, so it is shellable. However, there is still no shelling step possible after shelling 10,3,6,9, so it is not extendably shellable.

Simon [15] has conjectured that the full d-dimensional complex Δ on V (where every $(d+1)$-subset of V is a face) is always extendably shellable. Björner and Eriksson [3] has shown this to be true for $d = 2$, and indeed that all matroid complexes (see Björner [2] for a definition) of dimension 2 are extendably shellable. We conjecture that this holds for all dimensions.

Since extendable shellability of Δ is equivalent to strong convergence of the shelling game on Δ, the Polygon Property Theorem provides a technique to prove extendable shellability. It says that it suffices to show that whenever two different shelling steps, F and F', are possible, there exist two sequences of shelling steps, one beginning with F, the other with F', that ends with the same subcomplex of Δ shelled.

4 The chromatic game

Let $P(G)$ be the chromatic polynomial (in some indeterminate λ) of a simple graph G. $P(G)$ can be computed recursively by

$$P(G) = P(G \setminus e) - P(G/e)$$

where $P(G \setminus e)$ and $P(G/e)$ results from respectively deletion and contraction of an arbitrary edge e. We can remove any multiple edges that arise after a contraction so that the graphs are always simple. The basecase of the recursion is the edgeless graph; n isolated nodes have chromatic polynomial λ^n.

For combinatorial reasons this yields a unique polynomial, independent of the choices of edges, because if the chromatic polynomial is evaluated with

$$\lambda = \text{the number of colors available},$$

the result is the number of possible colorings of the nodes such that no two neighbors get the same color. Hence it is tempting to formulate the procedure as a strongly convergent game. But what should constitute a position in this game? The correct objects are *multisets* of *labeled graphs*.

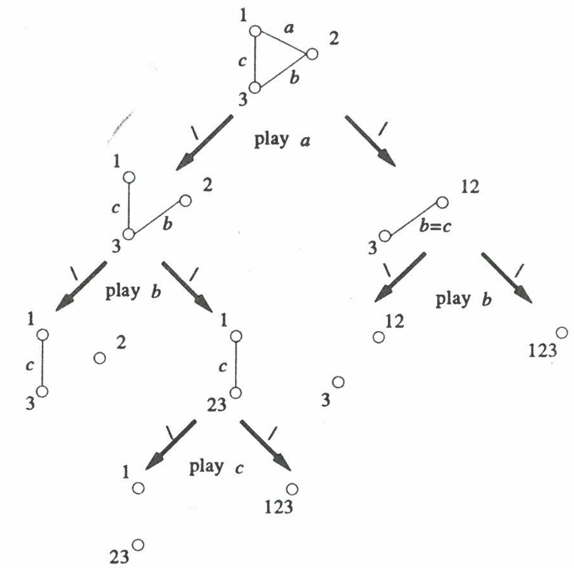
Definition. The *chromatic game* is a game version of the computation of the chromatic polynomial. A position in the game is a multiset of graphs where the nodes are marked by disjoint subsets of integers. A move consists of choosing a graph G_i from the current multiset, choosing an edge $e \in G_i$ and replacing G_i by $G_i \setminus e$ and G_i/e, where the node resulting from contraction of e is marked with the union of the marks of the two nodes of e. The game is over when all graphs in the multiset are edgeless.

Theorem. *The chromatic game is strongly convergent.*

Proof We shall verify the polygon property of the game. Suppose we have a position where two different moves are legal, i.e. we have a multiset M of labeled graphs and two edges a and b that are each in some graph in the multiset.

There are three cases to consider.

- *Case 1:* a and b are in different graphs in M. Then it is obvious that a and b can be played independently, giving a polygon of length 2.
- *Case 2:* a and b are in the same graph G, but a and b are not two edges in a triangle in G. Let Q be the multiset
$$(M \setminus \{G\}) \cup \{G \setminus a \setminus b, G \setminus a/b, G/a \setminus b, G/a/b\}$$
which is the position reached after first playing a and then playing b in both resulting graphs. Since, in this case, the contractions and deletions commute ($G \setminus a \setminus b = G \setminus b \setminus a$ etc.), Q is also the position reached by first playing b and then playing a in both resulting graphs. Thus we have a polygon of length 3.
- *Case 3:* a and b are two edges in a triangle of G with third edge c. Let Q be the multiset
$$(M \setminus \{G\}) \cup \{G \setminus a \setminus b, G \setminus a/b/c, G \setminus a/b \setminus c, G/a \setminus b, G/a/b\}$$
which is the position reached after first playing a, then playing b in both resulting graphs, and finally playing c on $G \setminus a/b$. See the picture below.



Since

$$G/a \setminus b = G \setminus b/a \setminus c \qquad \text{and} \qquad G \setminus a/b \setminus c = G/b \setminus a$$

and a and b commutes in the other three sequences, we get position Q also by first playing b, then playing a in both resulting graphs, and finally playing c in $G \setminus b/a$. In this case we accordingly have a polygon of length 4.

Hence the chromatic game has the polygon property and thus it is strongly convergent by the Polygon Property Theorem. □

Suppose the initial position is the set $\{G\}$, where G is a graph on N nodes marked $1, 2, \ldots, N$. Play the chromatic game until the terminal position M is reached. A graph $G' \in M$ with k nodes must have endured a total of $N - k$ contractions, thus it corresponds to a term $(-1)^{N-k}\lambda^k$ in the chromatic polynomial computation. Hence the chromatic polynomial

$$P(G) = a_N\lambda^N - a_{N-1}\lambda^{N-1} + \cdots + (-1)^N a_0$$

(alternating signs) of the graph can be read off easily from the terminal position: a_k is the number of graphs with k nodes.

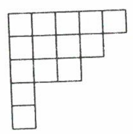
It is known that the chromatic polynomial may be equal for two graphs which are not isomorphic. However, the terminal position of the chromatic game, played from a single graph G, does uniquely determine G, due to the extra information given by the marks. In other words, two graphs which are not isomorphic cannot give the same terminal position in the chromatic game. Let us prove this.

If G and H are not isomorphic then either they are both edgeless, in which case they already constitute different terminal positions, or, for any proper marking of the nodes of G and H, there must be some pair of marks, say i and j, such that the nodes marked i and j are neighbors in one graph, say G, while they are not neighbors in H. Then we will get some node marked ij in some graph in the terminal position of the game played from G, for example by contracting (i, j) and deleting all other edges. However, it is impossible to obtain a node marked ij when playing from H, since if i and j are not neighbors, they can only merged via contractions involving also additional nodes, which will contribute with their node marks. Hence the terminal positions are different.

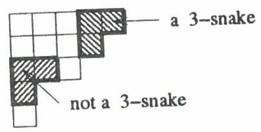
The combinatorial interpretation of the terminal position is as an inclusion/exclusion formula for the number of proper colorings. The graph with N nodes denotes the λ^N different colorings possible without restrictions. Every graph with $N-1$ nodes, i.e. with some node doubly marked, e.g. bc, corresponds to subtraction of the λ^{N-1} colorings where b and c are colored alike, despite there being an edge beteween b and c. Continue in the usual manner of inclusion/exclusion.

5 The k-snake game

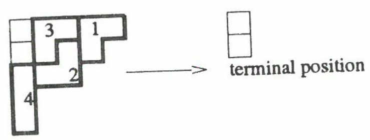
A *tableau* is the graphical representation of an integer partition ($n = m_1 + m_2 + m_3 + \cdots$, where $m_1 \geq m_2 \geq m_3 \geq \cdots$) that one gets by letting each part m_i be represented as a row of m_i squares. Thus,



is a tableau with 14 squares representing 5+4+3+1+1. A *k-snake* is a contiguous strip with k squares that live in the rightmost boundary of the tableau, such that if it is removed, then what is remaining is still a tableau.



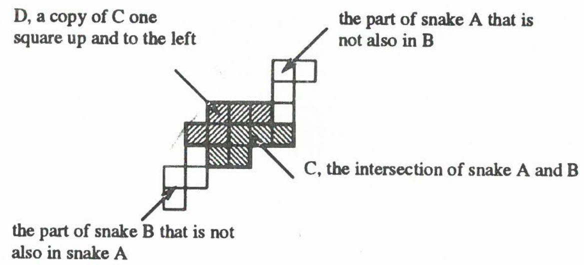
The *k-snake game* for some fixed integer k is played on a tableau by repeatedly removing k-snakes. If there are several k-snakes at any time, one has to choose one of them to be removed. The game ends when there are no k-snakes left. An example with $k = 3$ is shown below.

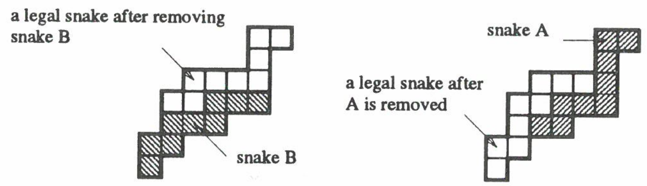


Originally, this game comes from representation theory, where it is known to have a unique terminal position, called the k-core, and hence it is strongly convergent. We refer to the book by James and Kerber [11].

Here, we show a rather elegant way to verify the polygon property. Suppose that in some tableau there are two different k-snakes, A and B. If they have no squares in common they can be removed independently of each other, in which case we have a polygon of length 2.

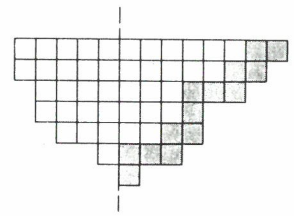
If the snakes do have some common squares, we must have something like in the picture below. The lower snake, B, must continue with one square directly to the left of the part C common with A, because otherwise it would have a square below the common part and then it would be be illegal to remove A. Analogously, A must continue with one square directly above the common part C. Hence, the tableau that remains when the union of A and B is removed contains a copy of C, say D, on the boundary, placed one square up and to the left of C. If A is played first, it is now legal to play the snake that consists of D and the remaining part of B. If, on the other hand, B is played first, it is then legal to play the snake that consists of D and the remaining part of A. All these are of course k-snakes.





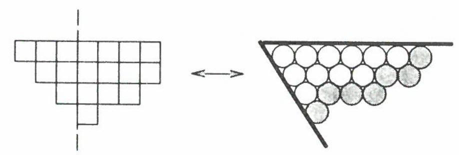
Hence, also in this case we get a polygon of length two, so the game is strongly convergent.

There is a generalization of this that can be obtained immediately from the proof. Instead of playing on an ordinary tableau, play on the shape obtained by gluing together a (left-to-right) reversed tableau with an ordinary tableau. The boundary is still only the rightmost part, as indicated in the figure below.



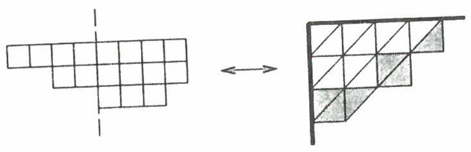
The k-snake game on such a shape is strongly convergent, because the proof of the polygon property above is still perfectly valid.

Two particularly interesting special cases should be pointed out. If the left tableau has r rows with row lengths $1, 2, \ldots, r$, then the game is equivalent to the k-snake game on the corresponding pile of circles. See the figure below.



The boundary is the rightmost part, as indicated with grey. A k-snake is legal if removing it leaves no holes in the east/west and north-east/south-west directions. The equivalence between the games should be obvious if the circles of every row are shifted one half-step to the right relative the row above.

If the lengths of consecutive rows of the left tableau differ by 2, then the game is equivalent to the k-snake game on the corresponding triangle diagram.

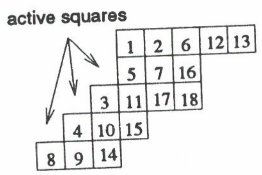


Again, the rule is that no holes may arise in the east/west and north-east/south-west directions, and the boundary is defined in accordance with this.

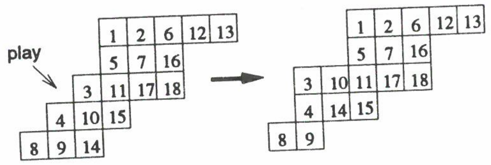
Since the k-snake games on glued tableaux are strongly convergent, due to the equivalence above, the k-snake games on circles and triangles are also strongly convergent.

6 Jeu de Taquin

Schützenberger [14] introduced a game played on skew partial tableaux which he called *jeu de taquin*. A *partial tableau* is a tableau, such as the in the k-snake game, with a number in each square such that rows and columns are increasing. A *skew* partial tableau is what you get when you remove a smaller partial tableau from the upper left corner of a larger one.



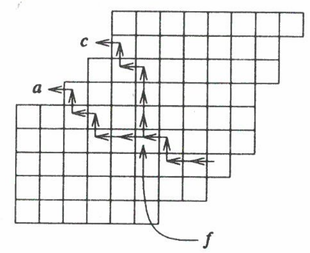
Every such skew partial tableau is a position in the game. An *active* square is a void square that has a neighbor in the skew tableau both directly to the right and directly below. A move consists of choosing an active square and then moving there the neighbor square with the least number. Then we get a hole in the tableau, which we fill in the same fashion with the neighbor square with the least number. If there is only one neighbor to the right and below, then this is the one. If there is no neighbor, then we are at the lower right boundary and the process stops. Clearly this results in a new skew partial tableau. A terminal position is when there are no more active squares, i.e. when no void squares exist in the upper left corner, which is exactly when we have an ordinary partial tableau. One move is shown below.



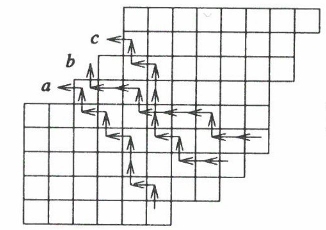
As was the case for the k-snake game, the unique terminal position has algebraic meaning, as the partial tableau obtained by taking the *row word*, i.e. the word obtained by reading the rows left-to-right from the bottom row to the top row, as input to the Robinson-Schensted algorithm. The interested reader should consult the book by Sagan [13]. It should be pointed out that if one starts with proving strong convergence like we do here, then it is quite simple to derive the theory of the Robinson-Schensted algorithm and Knuth equivalence of permutations as corollaries, thus giving a slightly new approach to the subject.

We shall as usual prove that the terminal position is unique by showing strong convergence using the polygon property. The polygons will in this game be of length either 2 or 3.

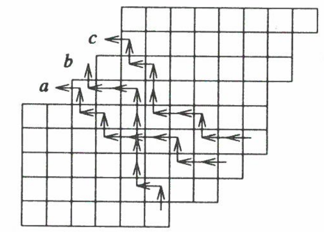
Let a and c be two possible moves in a position, i.e. two different active squares, such that a is to the left of (and thus below) c. If no square is moved by both moves, then a and c can be played independently, yielding a polygon of length 2. Otherwise there is some square f closest to a and c such that f is moved both when a and c are played.



After playing both a and c there must be some active square b to the right of a and to the left of c. We claim that the pair of play sequences acb and cab constitute a polygon.



The movement of squares when playing *cab*

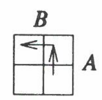


The movement of squares when playing *acb*

We claim that the paths of moving squares will always have a pattern as above, so that every square is moved equally in the two play sequences. Let an *arrow path* be a contiguous path of left-arrows and up-arrows in a tableau of squares, as in the figures. An arrow path A' is said to *run to the left* of another arrow path A if, in every row visited by both paths, there is one square visited by A that is to the right of every square visited by A' on this row. By substituting 'column' for 'row' and 'below' for 'to the right', we get an analogous definition for when A' runs *above* A.

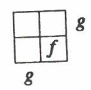
Lemma. *Let a be an active square and A its corresponding arrow path. Let b be an active square after a is played and let B be its corresponding arrow path. If b is above a, then B runs above A. Analogously, if b is to the left of a, then B runs to the left of A.*

PROOF The first statement follows from observing that no square can be at the same time at the end of an up-arrow in A and at the beginning of a left-arrow in B.



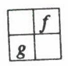
This is impossible since the integer that was initially in the lower right square would be placed above (and hence be less than) the number in the square it was initially to the right of (and hence greater than). The second statement follows analogously. □

Now return to the situation where a below b below c are squares such that a and c are active and b is active after both a and c are played. Let f be the closest square that is moved both if a is played and if c is played. Then it must be so that playing a would slide f leftwards while playing c would slide f upwards. Hence the situation in the vicinity of f is



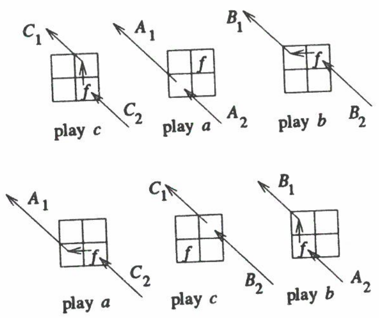
where the g's denote squares with numbers greater than f. (Void squares on the outside of the tableau may be considered have infinite content.)

Now we study the play sequence cab. Then the arrow path A of a will run to the left of the arrow path C of c. The arrow path B of b will run above A. But B will also run to the left of C until it is forced to do otherwise by A, that is, until it reaches the square to the left of (the new place of) f. There, after playing ca, it will look like

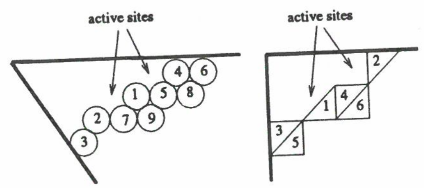


where, as above, g denotes a number greater than f. Following the rules, B will slide f to the left. From this point on, B will run above C.

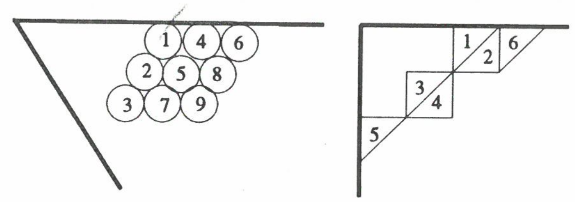
We want to compare this with the play sequence acb. The figure below shows what happens in the vicinity of f during the two play sequences. Clearly f goes one square up and to the left. The diagonal arrows represent the other parts of the arrow paths, and a moment of thought should make it clear that they will coincide as indicated by the labels. Hence. the two play sequences give the same result, and hence we have the desired polygon of length 3.



What about playing jeu de taquin on circles or triangles, in analogy with the k-snake game? Yes, it can be done, and the game will still be strongly convergent.



Active sites are void spaces with numbered neighbors both to the right (east) and to the lower-left (south-west). Holes are filled with the least of the numbers to the right and the lower-left. For the start positions above, the terminal positions will be



As one can see, these games terminate before the upper left corner is filled. In fact, for the circle game, the terminal position will always have rows of void circles of length differing by 1, while in the triangle game the rows will differ by 2. The whole secret is that the transfomation between circle diagrams and glued tableaux with left tableau of type $1 + 2 + \cdots + r$, as well as the transformation between triangle diagrams and glued tableaux with left tableau of type $2 + 4 + \cdots + 2r$, commute with the moves of jeu de taquin. In other words, transforming the diagram to a tableau and playing the game will give the same result as first playing jeu de taquin on the diagram and then pushing everything to the left.

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ON THE NUMBER OF COLUMN CONVEX POLYOMINOES WITH GIVEN PERIMETER AND NUMBER OF COLUMNS

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Abstract. In Part I for the perimeter and number of columns generating function for column convex polyominoes a remarkably simple algebraic expression is obtained and also, for the first time, explicit formulas for the numbers, stated in the title, are found and are given as certain triple sums. A correspondence between the polyominoes and the encoding class of Motzkin words, which is not a bijective one, is used and it lead us to a system of four quadratic equations. By a series of manipulations and a "magic" substitution the system is reduced to a biquadratic equation. The method of proof can be widely generalized yielding interesting results not only for Motzkin paths but also for Dyck paths with arbitrary steps. In Part II an extension of Temperley's methodology, as an alternative to difference methods, is developed, including a complete solution, which includes area, perimeters, contacts, and sources, of the Temperley's Model Q on square lattice as well as unidirectionally convex-polyomino problem on the hexagonal lattice.

PART I – Language technique

1. Introduction

The perimeter generating function for column-convex polyominoes was first found by Delest [3], and then, in a different way, by Brak, Enting and Guttman [1], who relied on an earlier paper of Temperley [11]. In both papers [3] and [1] a degree four algebraic equation satisfied by that generating function is also given. Following the approach of Brak et al., Lin [6] has obtained, the more general, column-convex polyominoes perimeter and number of columns generating function.

Let $F(a,b)$ (resp. $G(a,b)$) be the power series such that the coefficient of a^cb^v in F (notation $\langle F, a^cb^v\rangle$) is the number of column-convex polyominoes with vertical perimeter $\leq 2v$ (resp. $= 2v$) and with exactly c columns (i.e. horizontal perimeter equal to $2c$). Our main results imply the following algebraic equation and the consequent theorem:

$$F = \frac{ab(1-F)^4}{(1-b)^2(1-2F)[(1-3F)^2 - a(1-F)^2]}, \quad G(a,b) = (1-b)F(a,b) \tag{1.1}$$

THEOREM A *i) The number of column-convex polyominoes having vertical perimeter $\leq 2v$ and exactly c columns (i.e. horizontal perimeter $= 2c$) is given by*

$$\langle F, a^cb^v\rangle = \frac{1}{c}\sum_{i,j,k\geq 0}(-1)^k(k+1)\binom{v+i}{2i+1}\binom{c}{i+1}\binom{2c+j-1}{j}\binom{2(c+i)-k}{i-j-k} \tag{1.2}$$

ii) The number of column-convex polyominoes having perimeter $2p$ and exactly c columns is given by

$$\begin{aligned}\langle G, a^cb^{p-c}\rangle &= \langle F, a^cb^{p-c}\rangle - \langle F, a^cb^{p-c-1}\rangle \\ &= \sum_{i,j,k\geq 0}\frac{2}{c}\left(k + \frac{c+j+k}{2c+i+j}\right)\binom{p-c+i-1}{2i}\binom{c}{i+1}\binom{2c+j-1}{j}\binom{2(c+i-k)-1}{i-j-2k}\end{aligned} \tag{1.3}$$