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ON THE NUMBER OF COLUMN CONVEX POLYOMINOES WITH GIVEN PERIMETER AND NUMBER OF COLUMNS

Svjetlan FERETIĆ
Šetalište Joakima Rakovca 17, 51 000 Rijeka, Croatia

Dragutin SVRTAN
Department of Mathematics, University of Zagreb, Bijenička c. 30
41 000 Zagreb, Croatia

Abstract. In Part I for the perimeter and number of columns generating function for column convex polyominoes a remarkably simple algebraic expression is obtained and also, for the first time, explicit formulas for the numbers, stated in the title, are found and are given as certain triple sums. A correspondence between the polyominoes and the encoding class of Motzkin words, which is not a bijective one, is used and it lead us to a system of four quadratic equations. By a series of manipulations and a "magic" substitution the system is reduced to a biquadratic equation. The method of proof can be widely generalized yielding interesting results not only for Motzkin paths but also for Dyck paths with arbitrary steps. In Part II an extension of Temperley's methodology, as an alternative to difference methods, is developed, including a complete solution, which includes area, perimeters, contacts, and sources, of the Temperley's Model Q on square lattice as well as unidirectionally convex-polyomino problem on the hexagonal lattice.

PART I – Language technique

1. Introduction

The perimeter generating function for column-convex polyominoes was first found by Delest [3], and then, in a different way, by Brak, Enting and Guttman [1], who relied on an earlier paper of Temperley [11]. In both papers [3] and [1] a degree four algebraic equation satisfied by that generating function is also given. Following the approach of Brak et al., Lin [6] has obtained, the more general, column-convex polyominoes perimeter and number of columns generating function.

Let $F(a,b)$ (resp. $G(a,b)$) be the power series such that the coefficient of $a^c b^v$ in F (notation $\langle F, a^c b^v\rangle$) is the number of column-convex polyominoes with vertical perimeter $\leq 2v$ (resp. $= 2v$) and with exactly c columns (i.e. horizontal perimeter equal to $2c$). Our main results imply the following algebraic equation and the consequent theorem:

$$F = \frac{ab(1-F)^4}{(1-b)^2(1-2F)[(1-3F)^2 - a(1-F)^2]}, \quad G(a,b) = (1-b)F(a,b) \tag{1.1}$$

THEOREM A *i) The number of column-convex polyominoes having vertical perimeter $\leq 2v$ and exactly c columns (i.e. horizontal perimeter $= 2c$) is given by*

$$\langle F, a^c b^v\rangle = \frac{1}{c}\sum_{i,j,k\geq 0}(-1)^k(k+1)\binom{v+i}{2i+1}\binom{c}{i+1}\binom{2c+j-1}{j}\binom{2(c+i)-k}{i-j-k} \tag{1.2}$$

ii) The number of column-convex polyominoes having perimeter $2p$ and exactly c columns is given by

$$\begin{aligned}\langle G, a^c b^{p-c}\rangle &= \langle F, a^c b^{p-c}\rangle - \langle F, a^c b^{p-c-1}\rangle \\ &= \sum_{i,j,k\geq 0}\frac{2}{c}\left(k + \frac{c+j+k}{2c+i+j}\right)\binom{p-c+i-1}{2i}\binom{c}{i+1}\binom{2c+j-1}{j}\binom{2(c+i-k)-1}{i-j-2k}\end{aligned} \tag{1.3}$$

A "magic" substitution $L := (1-3F)/(1-F) \quad (\Rightarrow F = (1-L)/(3-L))$, turns the equation (1.1) to a biquadratic one: $L^4 - (1+a)L^2 + a + 4ab/(1-b)^2 = 0$ By using the generalized binomial series $B_t(z)^r = \sum_{k\geq 0} \binom{tk+r}{k} \frac{r}{tk+r} z^k$ (cf. [4], (5.58)) for $t=-1$, $r=\frac{1}{2}$, the solution of (1.1) reads as follows: $F = 1 - \frac{2}{3-L}$, $\quad L(a,b) = (1+a)^{1/2} B_{-1}\left(\frac{-a(1+b)^2}{(1+a)^2(1-b)^2}\right)^{1/2}$. In more explicit manner we have the following:

THEOREM B *For column convex polyominoes, the generating functions are given explicitly:*

i) for ($\frac{1}{2}$ horizontal perimeter = # columns, $\frac{1}{2}$ vertical perimeter) by:

$$G(a,b) = (1-b)\left[1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{1+a+\sqrt{(1-a)^2-4d}}}\right], \qquad \text{where } d := \frac{4ab}{(1-b)^2} \tag{1.4}$$

ii) for ($\frac{1}{2}$ perimeter) by:

$$G(a,a) = (1-a)\left[1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{(1+a)e}}\right], \qquad \text{where } e = 1 + \frac{\sqrt{1-6a+a^2}}{1-a} \tag{1.5}$$

Remark 1. This result may be compared with an extremaly complicated formula, obtained for the first time for $G(a,a)$ by [3] and later in [1], or to a result of [6] which can be obtained by clearing the denominator of $G(a,b)$ and written more compactly as

$$G(a,b) = (1-b)\left[1 - \frac{(1-b)^2(L+3)(L^2-ab+8)}{4[(1-b)^2(9-ab)+ab^2]}\right].$$

The methodology that we a going to use in Part I was developped by Schützenberger [9], [10]. Hovewer in the present work the correspondence between the polyominoes and the encoding class of Motzkin words is not a bijective one.

2. Preliminaries

In this work "a polyomino" always means " a column-convex polyomino". The polyominoes whose first column consists of exactly r cells will be called r-source polyomino (or simply r-polyomino). Similarly (r,s)-polyomino will mean that first and last column consist of r and s cells respectively.

Let P be a polyomino with c columns. The minimal and the maximal ordinate of the i^{th} column will be denoted by y_i and Y_i respectively. If for some i, $1 \leq i < c$

$$(y_i < y_{i+1} \quad \text{and} \quad Y_i < Y_{i+1}) \qquad \text{or} \qquad (y_i > y_{i+1} \quad \text{and} \quad Y_i > Y_{i+1})$$

then we shall say that $i \to i+1$ is an in-out passage. In the case of the four polyominoes shown in Figure 1 the in-out passages are $1 \to 2$ and $2 \to 3$.

Let w be a word over the alphabet $\{x, y, \overline{y}\}$ and let $|w|_x = n$. For $i = 2, \ldots, n$, the subword running from the $(i-1)^{\text{st}}$ up to the i^{th} letter x in w (the two x's excluded) will be called the i^{th} **nest** of w. The first nest is the left factor of w ending just before the first x and the $(n+1)^{\text{th}}$ nest is the right factor beginning immediately after the last x in w. We shall be mainly concerned with Motzkin words ($|w|_y = |w|_{\overline{y}}$, $w = uv \Rightarrow |u|_y \geq |u|_{\overline{y}}$) with $y, \overline{y}, x$ interpreted as northeast, southeast and east steps.

3. A coding for column-convex polyominoes

Let P be a polyomino with c columns. We shall encode P by the word

$$f(P) = a_1 x a_2 x \cdots x a_{2c-1} x a_{2c} \in \{x, y, \overline{y}\}^*$$

where the a_i's are defined as follows:

- ▹ $a_1 = y^{Y_1 - y_1 - 1}$
- ▹ $a_{2c} = \overline{y}^{Y_c - y_c - 1}$
- ▹ for $i = 1, \ldots, c-1$:

if $i \to i+1$ is an in-out passage such that $y_i > y_{i+1}$ and $Y_i > Y_{i+1}$
then $a_{2i} = \overline{y}^{Y_i - Y_{i+1}}$ and $a_{2i+1} = y^{y_i - y_{i+1}}$
else
if $y_i > y_{i+1}$, then $a_{2i} = y^{y_i - y_{i+1}}$
if $y_i \leq y_{i+1}$, then $a_{2i} = \overline{y}^{y_{i+1} - y_i}$
if $Y_i < Y_{i+1}$, then $a_{2i+1} = y^{Y_{i+1} - Y_i}$
if $Y_i \geq Y_{i+1}$, then $a_{2i+1} = \overline{y}^{Y_i - Y_{i+1}}$
endelse.

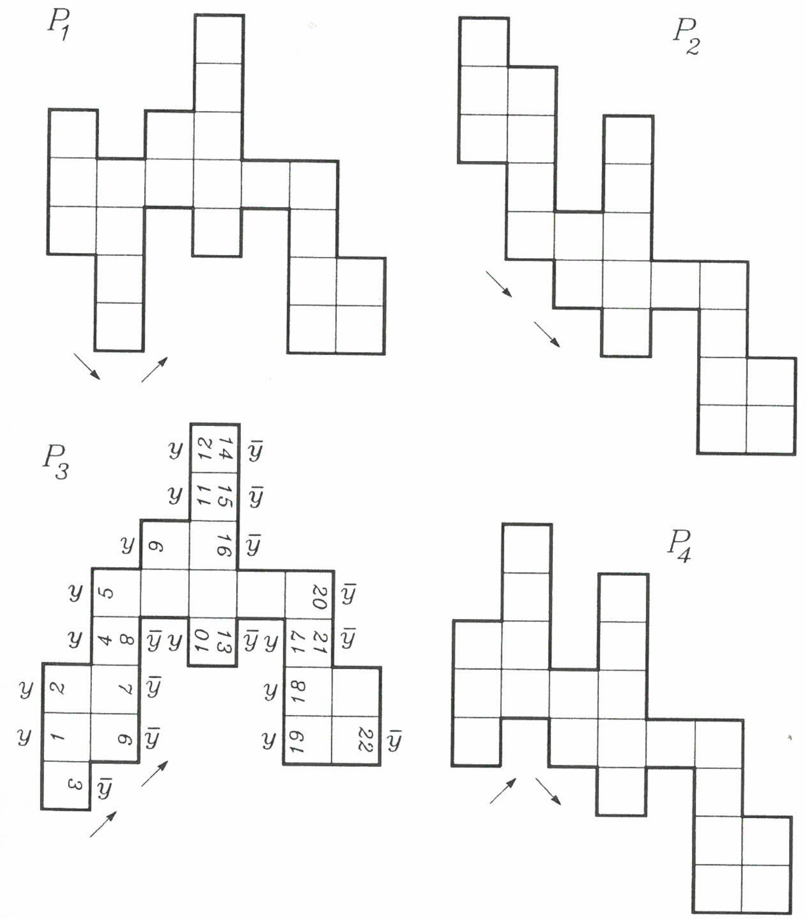

Figure 1. These four polyominoes are (and no other) encoded by the word:

Col.1		Col.1 → Col.2		2 → 3		3 → 4		4 → 5		5 → 6		6 → 7		Col.7
yy	x	$\overline{y}xyy$	x	$\overline{y}\,\overline{y}\,\overline{y}xy$	x	$yxyy$	x	$\overline{y}x\overline{y}\,\overline{y}\,\overline{y}$	x	$yyyx$	x	$x\overline{y}\,\overline{y}$	x	$\overline{y}$

In brief, the in-out passages are encoded by the rule **"first in then out"**, whereas all the other passages are encoded by the rule **"first the lower side and then the upper side"**. See Figure 1 for a few examples.

THEOREM 3.1 *Let P be a polyomino with perimeter $2p$ and c columns. The word $f(P)$ has the following properties*

a) It is a Motzkin word

b) It contains $2c-1$ x's and its length is $2p-3$

c) It contains neither of the syllables $y\overline{y}$ and $\overline{y}y$, as well as no y's in an even nest followed by $\overline{y}$'s in the next (odd) nest.[1]

Conversely, let w be a word possessing properties a) b) and c) and let the event "$\overline{y}$'s in an even nest of w followed by y's in the next nest" take place m times. Than w encodes exactly 2^m polyominoes. Each one out of those polyominoes has m in-out passages.

Remark 3.1. In [3] it was constructed an injective coding for the perimeter of column convex polyominoes, similar in spirit to our, with more letters, so it makes algebra a bit harder.

4. A non-ambiguous grammar

Let J_1 (resp. J_2) be the language over the alphabet $\{x,y,\overline{y}\}$ consisting of all those Motzkin words satisfying the following conditions

i) w contains neither of the syllables $y\overline{y}$ and $\overline{y}y$.

ii) $|w|_x$ is an odd number (resp. even number or zero)

The languages J_1 and J_2 satisfy the following system of equations

$$\begin{aligned} J_1 &= xJ_2 + yJ_1\overline{y}(1+xJ_1) + y(J_2-1)\overline{y}xJ_2 \\ J_2 &= 1 + xJ_1 + yJ_1\overline{y}xJ_2 + y(J_2-1)\overline{y}(1+xJ_1) \end{aligned} \tag{4.1}$$

which follow simply from the following (obvious) equation $J = 1+y(J-1)\overline{y}(1+xJ)$. for the language $J = J_1 + J_2$ of the all Motzkin words with no factors $y\overline{y}$, $\overline{y}y$.

5. Attribute grammars

Let w be a word over the alphabet $\{x,y,\overline{y}\}$ and let $|w|_x = n$. Write $w \in J = J_1 \cup J_2$ in the form

$$w = a_1xa_2x\cdots a_nxa_{n+1}$$

with $a_i \in \{y\}^*$ or $a_i \in \{\overline{y}\}^*$. In our terminology a_i is then called the i^{th} nest of w.

It will be customary to write w also in the form $w = a_1x_1a_2x_2a_3x_1a_4x_2\cdots x_{n\bmod 2}a_{n+1}$ $(x_1 = x_2 = x)$ in which the index 1 (resp. 2) only indicate that the order of appearance of x in w from left is odd (resp. even).

Now we introduce two attribute grammars $J^1 = J^1_1 + J^1_2$, $J^2 = J^2_1 + J^2_2$, by extending the grammar $J = J_1 + J_2$ with atributes describing the types of two consecutive nests around x_1's (resp. x_2's). More precisely, for J^1 we use the additional (commuting) letters u, v, c, e marking maximal factors in $w \in J$ belonging to $\{y\}^+x_1\{\overline{y}\}^+$, $\{\overline{y}\}^+ x_1\{y\}^+$, $\{\overline{y}\}^*x_1\{\overline{y}\}^* \setminus \{x_1\}$, $\{y\}^*x_1\{y\}^* \setminus \{x_1\}$ respectively. We call such events **left odd expansion, right odd expansion, odd contraction, odd expansion** respectively.

[1] Of course, when some nest contains zero y's (resp. $\overline{y}$'s) then we consider that it does not contain y's (resp. $\overline{y}$'s).

The definition of J^2 is defined with the same marking letters but with x_1 replaced by x_2 (i.e. around "even" x's).

The following simple facts will be useful in writing a grammar for J^1 and J^2.

PROPOSITION 5.1 *i)* $(xJ_1)^1 = ex(J^2_1 - xJ^1_2) + x^2J^1_2$, $(xJ_1)^2 = xJ^1_1$

ii) $(xJ_2)^1 = ex(J^2_2 - 1 - xJ^1_1) + x(1 + xJ^1_1)$, $(xJ_2)^2 = xJ^1_2$

iii) $(yJ_1\overline{y})^1 = (u-1)yx\overline{y} + yJ^1_1\overline{y} + (c-1)y(J^1_2-1)x\overline{y} + (e-1)yx(J^2_2-1)\overline{y} + (c-1)(e-1)yxJ^2_1x\overline{y}$, $(yJ_1\overline{y})^2 = yJ^2_1\overline{y}$

iv) $(y(J_2-1)\overline{y})^1 = y(J^1_2-1)\overline{y} + (e-1)yxJ^2_1\overline{y}$, $(y(J_2-1)\overline{y})^2 = y(J^2_2-1)\overline{y} + (c-1)yJ^2_1x\overline{y}$

v) $(\overline{y}xJ_2)^1 = \overline{y}x[v(J^2_2 - 1 - xJ^1_1) + c(1+xJ^1_1)]$, $(\overline{y}xJ_2)^2 = \overline{y}xJ^1_2$

vi) $(\overline{y}xJ_1)^1 = \overline{y}x[v(J^2_1 - xJ^1_2) + cxJ^1_2]$, $(\overline{y}xJ_1)^2 = \overline{y}xJ^1_1$

By using this Proposition, one obtains from (4.1) four language equations (with attributes) for J^1_1, J^1_2, J^2_1 and J^2_2 which we will not write down. By letting $x, y, \overline{y}$ commute we obtain four quadratic equations for the corresponding generating functions which we denote by $\mathbf{J}^1_1$, $\mathbf{J}^1_2$, $\mathbf{J}^2_1$, $\mathbf{J}^2_2$. By applying the morphism $x \mapsto x$, $y \to \overline{y}$, $\overline{y} \to y$ and reversion on words we get that

$$\begin{aligned} \mathbf{J}^1_2(x,y,\overline{y},u,v,c,e) &= \mathbf{J}^2_2(x,y,\overline{y},u,v,e,c) \\ \mathbf{J}^1_1(x,y,\overline{y},u,v,c,e) &= \mathbf{J}^1_1(x,y,\overline{y},u,v,e,c) \\ \mathbf{J}^2_1(x,y,\overline{y},u,v,c,e) &= \mathbf{J}^2_1(x,y,\overline{y},u,v,e,c) \end{aligned} \tag{5.1}$$

so we only need to find three generating functions $\mathbf{J}^2_2$, $\mathbf{J}^2_1$, $\mathbf{J}^1_1$.

Define the normalized generating functions F_1, F_2, F_3 by

$$F_1 := \alpha\mathbf{J}^2_1, \quad F_2 := \alpha\mathbf{J}^2_2, \quad F_3 := \alpha\mathbf{J}^1_1, \qquad \text{where } \alpha := \frac{y\overline{y}x}{1-y\overline{y}} \tag{5.2}$$

6. Functional equations for F_i's

The generating functions F_1, F_2 and F_3 defined in (5.2) satisfy the following system of functional equations, obtained from the equations for $\mathbf{J}^2_2$, $\mathbf{J}^2_1$, $\mathbf{J}^1_1$, when the attributes are assigned to (4.1)

$$F_1(1-vF_1) = (F_2 + x(1+(2c-v-1)F_1))\widetilde{F}_2 \tag{6.1}$$

$$F_2(1-vF_1) = (1+(2c-v-1)F_1)(\alpha + xF_3) + F_2F_3 \tag{6.2}$$

$$F_2(1-vF_1-\alpha\Delta x+2c'x^2) = \alpha(1+(2c-v-1)F_1) + F_2F_3 + (2c-1)F_1 + (2e-v-1)F^2_2 \tag{6.3}$$

where $\widetilde{F}_2(x,y,\overline{y},u,v,c,e) := F_2(x,y,\overline{y},u,v,e,c)$, $\Delta := u+v-2c-2e+2$, $c' := c-1$.

The rough idea how to handle such huge system is as follows: First by applying $^\sim$ to (6.1) one gets $(1+(2c-v-1)F_1)\widetilde{F}_2 = (1+(2e-v-1)F_1)F_2$ what is a linear relation between $\widetilde{F}_2$ and F_2 in terms of F_1. By subtracting (6.3) from (6.2) we get a relation quadratic in F_2 and linear in F_3. Then (6.1) multiplied by F_3 minus (6.2) multiplied by $\widetilde{F}_2$ gives us linear relation between F_1F_3, $F_2\widetilde{F}_2$ and F_2 (or $\widetilde{F}_2$). This is sufficient to isolate F_1, thus obtaining a polynomial equation for F_1. We only state the result for F_1 in this generality, in the following theorem:

THEOREM 6.1 *a) The generating function $F_1(x,y,\overline{y},u,v,c,e)$ satisfies the following equation of sixth degree:*

$$F_1 = \frac{\alpha\beta[1+(2e'-v')F_1]^2\{(1-v'F_1)^2+\varepsilon F^2_1+\delta F_1(1-vF_1)\}\{(1-v'F_1)^2+\varepsilon F^2_1+y\overline{y}\delta F_1(1-vF_1)\}}{(1-vF_1)\{[(1-v'F_1)(1-v''F_1)-\varepsilon F^2_1]^2-x^2(1-v'F_1)(1-v'F_1+\varepsilon F_1)(1+(2e'-v')F_1)^2\}} \tag{6.4}$$

where $\alpha := \frac{y\overline{y}x}{1-y\overline{y}}$, $\beta := \frac{x}{1-y\overline{y}}$, $c' := c-1$, $e' := e-1$, $v' := v-1$, $v'' := v+1$, $\varepsilon := (2c-1)(2e-1)-1$, $\delta := u+v-2$.

b) If $c = e$, then F_1 satisfies the forth degree equation:

$$F_1 = \frac{\alpha\beta\{(1-v'F_1)^2 + 4e(e-1)F_1^2 + \delta F_1(1-vF_1)\}\{(1-v'F_1)^2 + 4e(e-1)F_1^2 + y\overline{y}\delta F_1(1-vF_1)\}}{(1-vF_1)\{[1-(v'+2e)F_1]^2 - x^2(1-v'F_1)(1-v'F_1+4e(e-1)F_1)\}} \tag{6.5}$$

In the special case where $c = e = 1$ we state the result for all three generating functions $F_1(x,y,\overline{y},u,v)$, $F_2(x,y,\overline{y},u,v)$ and $F_3(x,y,\overline{y},u,v)$

THEOREM 6.2 *For $F_i(x,y,\overline{y},u,v)$, $i = 1,2,3$ we have*

a) $$F_1 = \frac{\alpha\beta[(1-v'F_1)^2 + \delta F_1(1-vF_1)][(1-v'F_1)^2 + y\overline{y}\delta F_1(1-vF_1)]}{(1-vF_1)[(1-v''F_1)^2 - x^2(1-v'F_1)^2]} \tag{6.6}$$

b) $$F_2 = \beta\frac{F_1(1-v'F_1) + y^2[(1-vF_1)(1-v'F_1) - \delta(1-vF_1)F_1]}{1-v''F_1} \tag{6.7}$$

c) $$F_3 = \frac{F_1 - v'F_2^2 - \delta\beta F_2}{1-v'F_1} \tag{6.8}$$

where $\alpha = \frac{y\overline{y}x}{1-y\overline{y}}$, $\beta = \frac{x}{1-y\overline{y}}$, $v' = v-1$, $v'' = v+1$, $\delta = u+v-2$.

Finally we remark here that the system can be solved explicitly by using more general "magic substitution" $L = [1-(v+1)F_1]/[1-(v-1)F_1]$, which reduces (6.6) to a biquadratic equation, but we will not write this solution here.

The following modification of attribute grammars J^1 and J^2 is also interesting. By keeping the attributes u and v as before and introducing a new attribute, denoted by d which marks every maximal factor of $w \in J$ belonging to $y^+x_1 \cup \{x_1\} \cup x_1\overline{y}^+$ **(odd small descent)** for J^1, and to $y^+x_2 \cup \{x_2\} \cup x_2\overline{y}^+$ **(even small descent)** for J^2, and using other notations from Section 5, we can get the following result

THEOREM 6.3 *The generating function $F_1(x,y,\overline{y},u,v,d)$ is given by* $F_1 = \frac{K}{1+vK}$ *where* $(1+K^2)\Lambda = 4K$ *and where Λ satisfies the following quadratic equation*

$$(1-\Lambda)(x^2+\Lambda) = \beta^2(1+\eta+\frac{1}{2}z\Lambda)^2,$$

where $\beta := \frac{x}{1-y\overline{y}}$, $\eta := y\overline{y}$, $z := \Delta\eta + d - 1$, $\Delta := u+v+d-3$.

The importance of this result is that perimeters generating function for hexagonal lattice case can be obtained from the square lattice case. Namely, the generating function G_{UC} for unidirectionally convex polyominoes on the honeycomb lattice (see [7]) we obtain the following expression $G_{UC} = \frac{z^2\Phi_1}{1-\Phi_1}$, where $\Phi_1 = xy^2F_1(x,y,z,0,z^{-2}+z^2,z^2)$, which is simpler than that obtained by Lin and Wu in [7].

Remark. The coefficients of F_1, Λ, K can be written as certain 6-fold, 5-fold and 4-fold summations respectively. In case $d = 1$ they are reduced to 5-fold, 4-fold and 3-fold summations.

7. Applications to polyominoes

According to our group coding rule "first in then out", in the coding word of a column convex polyomino (which belong to J_1^2) the attribute u is ruled out, and for each attribute v we have two polyominoes with the same relative position of two consecutive columns when the sizes of columns are given. So if we set $u = 0$, $v = 2$ into the generating function $\mathbf{J}_1^2$ we obtain that

$\langle \mathbf{J}_1^2(x,y,\overline{y},0,2), x^i(y\overline{y})^j\rangle$ = # polyominoes with $\frac{i+1}{2}$ columns (i.e. horizontal perimeter $i+1$) and vertical perimeter $2j+2$. Thus the number of polyominoes with vertical perimeter $2v$ and c columns is $\langle \mathbf{J}_1^2(x,y,\overline{y},0,2), x^{2c-1}(y\overline{y})^{v-1}\rangle$. The coefficients of $\mathbf{J}_2^2$ and $\mathbf{J}_1^1$ also have combinatorial interpretations. $\langle \mathbf{J}_2^2, x^{2c-2}(y\overline{y})^{v-1}\rangle$ is the number of 1-(source)-polyominoes having vertical perimeter $2v$ and c columns.

For $c \geq 2$, $\langle \mathbf{J}_1^1, x^{2c-3}(y\overline{y})^{v-1}\rangle$ is the number of 1,1-polyominoes with vertical perimeter $2v$ and c columns.

Now we shall specialize Theorem 6.2 for $u = 0$, $v = 2$.

THEOREM 7.1 *Let $F_i = F_i(x,y,\overline{y},0,2)$, $i = 1,2,3$. Then*

$$\begin{aligned} &\text{a)} \quad F_1 = \frac{\alpha\beta(1-F_1)^4}{(1-2F_1)[(1-3F_1)^2 - x^2(1-F_1)^2]} \\ &\text{b)} \quad F_2 = \beta\frac{F_1(1-F_1) + y^2(1-2F_1)(1-F_1)}{(1-3F_1)} \\ &\text{c)} \quad F_3 = \frac{F_1 - F_2^2}{1-F_1}, \qquad \text{where } \alpha = \frac{y\overline{y}x}{1-y\overline{y}},\ \beta = \frac{x}{1-y\overline{y}} \end{aligned} \tag{7.1}$$

Now for the coefficient evaluation see Theorem A in the Introduction for case of all vertically convex polyominoes. Here we state the result for 1-polyominoes

THEOREM 7.2 *For $c \geq 2$ and $v \geq 1$, the number of 1-(source)-polyominoes having vertical perimeter $2v$ and c columns is*

$$\sum_{i,j,k} \frac{(-1)^{i+j+k}2^{k+1}3^{j-1}}{(i+1)(v+i)} \left[\frac{(3i-v+2)(c-1)(2c-1)}{(2c+j-2)(2c+j-1)} + 2v - 1\right] \cdot \\ \cdot \binom{v+i}{2i+1}\binom{c-2}{i}\binom{i+k}{i}\binom{2c+j-1}{j}\binom{2(c+i)}{i-j-k}$$

Using Theorem A and Theorem 7.2 together with the results in Part II, Theorem 2.1 we can write formulas for r-polyominoes, $r \geq 1$.

PART II – (An alternative to Temperley's methodology)

Introduction. Here we illustrate by two examples an approach to Temperley type (infinite) systems of equations without reducing them first to difference equations. The method requires some computations in the algebraic closure of formal power series (a bit of Galois theory).

1. Unidirectionally-convex polyominoes on the honeycomb lattice

In this section we consider unidirectionally-convex polyominoes on the honeycomb lattice by more direct approach than that undertaken by Lin and Wu in [7]. We also start with the Temperley type equations (cf. (6) in [7]) and solve this infinite system directly by the generating function approach without any reference to difference equations. Lin and Wu considered first the fourth difference of the original system, which turned to be a difference equation of the forth order, and stated the solution in an extremely complicated form. In an earlier version of this paper we obtained simpler result then theirs by introducing an additional class of polyominoes denoted by g_0, whose first column consists of one degenerate hexagon what led us to simpler initial conditions g_0, g_1 than those (g_1 and g_2) in formula (19) of [7]. Then we used only second difference (c.f. (8) in [7]) which is still an infinite system, but somewhat simpler than (6) in [7]. In the approach presented bellow we dont even use g_0.

Imagine a plane hexagonal (or honeycomb) lattice with x-axis as one of its three axes of symmetry and an "column" convex polyomino in it where by column convex we mean that the vertical lines trough the centers of the hexagons intersect the polyomino in a convex set, and of course the consecutive columns have at least one edge in common (see Fig 2)

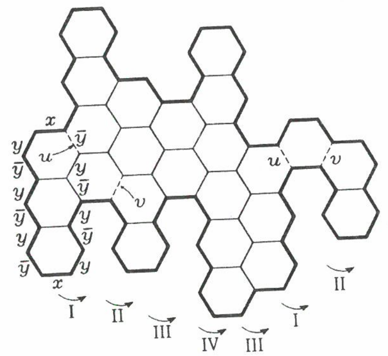

Figure 2. Unidirectionally-convex polyomino

Observe that there are four types of transitions from the i^{th} column to the $(i+1)^{\text{th}}$ column: Type I: upper contact (= in-out passage), Type II: lower contact (= out-in passage), Type III: contraction (= in-in passage) and Type IV: expansion (= out-out passage) marked by u, v, c, e respectively.

The edges of hexagons are labelled by $x, y, \bar{y}$ so that all edges with the same label are parallel to the same direction. Following Temperley we write

$$G(t) = G(x; y, \bar{y}; q; u, v, c, e; t) = \sum_{r\geq 1} g_r(x; y, \bar{y}; q; u, v, c, e)t^r \tag{1.1}$$

where g_r is the generating function for column convex polyominoes whose left-most column contains r hexagons, where $x, y, \bar{y}$ marks edges contributing to the perimeter, q marks area (i.e. number of hexagons), u marks upper contacts, v marks lower contacts, c marks contractions, e marks expansions.

By extending an argument used by Temperley for the square case and by generalizing Lin and Wu equations (6) in [7] we obtain the following system for g_r's, $r \geq 1$:

$$g_r = x^2(y\bar{y})^{2r}q^r\left\{1+\sum_{s=1}^{\infty} g_s \sum_{n=1}^{\min\{r,s\}} \alpha\beta^{n-1} + c\sum_{s=1}^{r}(r-s)g_s\beta^s + e\sum_{s=r+1}^{\infty}(s-r)g_s\beta^r\right\} \tag{1.2}$$

with $\quad \alpha = u\bar{y}^{-2} + vy^{-2}, \quad \beta = (y\bar{y})^{-2}$.

The left hand column, by itself, contributes in (1.2) a term $x^2q^r(y\bar{y})^r$, because its perimeter contains two horizontal lines, $2r$ labelled by y and $2r$ lines labelled by $\bar{y}$. If however, the next column to the right contains s hexagons and has contact of Type I, then out of n hexagons, $1 \leq n \leq \min\{r, s\}$, having contact with the first column one hexagon has partial contact along $\bar{y}$ line (thus contributing $u\bar{y}^{-2}$) and the remaining $n-1$ hexagons below it have full contact and contribute $(y\bar{y})^{-2(n-1)}$. The situation with Type II contact is symmetrical with contribution $vy^{-2}(y\bar{y})^{-2(n-1)}$. The type III contact is possible in $r-s$ ways if $r > s$, and all s hexagons contribute $c\cdot(y\bar{y})^{-2s}$. The type IV contact is possible in $s-r$ ways if $r < s$ and only r out of s hexagons contribute by the factor $e(y\bar{y})^{-2r}$. In this way the system (1.2) is explained.

By letting $\eta := (y\bar{y})^2 (= \beta^{-1})$ and writing the first sum $\sum_{s=1}^{\infty}$ in (1.2) as $\sum_{s=1}^{r} + \sum_{s=r+1}^{\infty}$, by using that $\sum_{n=1}^{r}\beta^{n-1} = (1-\beta^r)/1-\beta$, $\sum_{s=r+1}^{\infty} g_s = G - (g_1 + \cdots + g_r)$, $\sum_{s=r+1}^{\infty}(s-r)g_s = H - rG + \sum_{s=1}^{r}(r-s)g_s$, where $G = \sum_{r\geq 1} g_r$, $H := \sum_{s\geq 1} sg_s$, we can rewrite (1.2) as follows:

$$g_r = x^2q^r\left\{\eta^r + \left[\frac{\alpha\eta}{1-\eta} - \frac{\alpha\eta^{r+1}}{1-\eta} - re\right]G + \sum_{s=1}^{r}\left[-\frac{\alpha\eta}{1-\eta}(1-\eta^{r-s}) + (r-s)(c\eta^{r-s}+e)\right]g_s + eH\right\} \tag{1.2'}$$

By multiplying both sides of (1.2′) by t^r and summing over $r \geq 1$, and using the formula $\sum_{k\geq 1} kx^k = x/(1-x)^2$ we get **the functional equation for unidirectionally convex polyominoes on the honeycomb lattice:**

$$G(t) = a(t) + b_1(t)G + b_2(t)H + c(t)G(qt) \tag{1.3}$$

where

$$a(t) := \frac{x^2\eta qt}{1-\eta qt}, \quad b_1(t) := \frac{x^2qt}{1-qt}\left(\frac{uy^2+v\bar{y}^2}{1-\eta qt} - \frac{e}{1-qt}\right), \quad b_2(t) = \frac{x^2qet}{1-qt}$$

$$c(t) := x^2qt\left(\frac{e}{(1-qt)^2} - \frac{uy^2+v\bar{y}^2}{(1-qt)(1-\eta qt)} + \frac{c\eta}{(1-\eta qt)^2}\right)$$

Toward solving the equation (1.3) we introduce more notations:

$$A(t) := \begin{pmatrix} a(t) \\ a'(t) \end{pmatrix}, \qquad B(t) := \begin{pmatrix} b_1(t) & b_2(t) \\ b_1'(t) & b_2'(t) \end{pmatrix}, \qquad C(t) := \begin{pmatrix} c(t) & 0 \\ c'(t) & c(t)q \end{pmatrix} \tag{1.4}$$

(where $'$ denotes the derivative $\frac{d}{dt}$), the equation (1.3) can be written as the following matrix equation:

$$\begin{pmatrix} G(t) \\ G'(t) \end{pmatrix} = A(t) + B(t)\begin{pmatrix} G \\ H \end{pmatrix} + C(t)\begin{pmatrix} G(qt) \\ G'(qt) \end{pmatrix} \tag{1.5}$$

If we further denote

$$\mathcal{G}(t) := \begin{pmatrix} G(t) \\ G'(t) \end{pmatrix} \qquad \left(\Rightarrow \quad \mathcal{G}(1) = \begin{pmatrix} G \\ H \end{pmatrix}\right)$$

then the solution, obtained simply by iteration, can be writen as

$$\mathcal{G}(t) = \mathcal{A}(t) + \mathcal{B}(t)\begin{pmatrix} G \\ H \end{pmatrix} \tag{1.6}$$

where

$$\mathcal{A}(t) = \sum_{m\geq 0} A(q^mt)C^{[m]}(t), \quad \mathcal{B}(t) = \sum_{m\geq 0} B(q^mt)C^{[m]}(t) \tag{1.7}$$

and where $C^{[m]}(t)$ denotes the q-analog of the m-th pover of the matrix $C(t)$ which is defined as:

$$C^{[m]}(t) := C(t)C(qt)C(q^2t)\cdots C(q^{m-1}t) \tag{1.8}$$

By plugging $t = 1$ into (1.6) we obtain the "initial" values

$$\begin{pmatrix} G \\ H \end{pmatrix} = \mathcal{G}(1) = [I - \mathcal{B}(1)]^{-1}\mathcal{A}(1), \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.9}$$

Finally the general solution of the system (1.2) (or (1.3)) reads as follows:

MAIN THEOREM (area-perimeters-contacts-sources) *With the notations above the generating function $G(t) = \sum_{r\geq 1} g_r(x, y, \bar{y}; q, ; u, v, c, e)t^r$ for column convex polyominoes on the hexagonal lattice is given by the following formula:*

$$\begin{pmatrix} G(t) \\ G'(t) \end{pmatrix} (= \mathcal{G}(t)) = \mathcal{A}(t) + \mathcal{B}(t)[I - \mathcal{B}(1)]^{-1}\mathcal{A}(1) \tag{1.10}$$

Now we shall consider some special cases in which more explicit formulas are available.

Case 1. $q = 1$ (i.e. by neglecting area parameter). From (1.3) we see that $G(t)$ is rational with respect to t, say

$$G(t) = \frac{N(t)}{D(t)} \tag{1.11}$$

where

$$N(t) = (1-\eta t)x^2t\{\eta(1-t)^2 + [(1-t)(uy^2+v\overline{y}^2) - e(1-\eta t)]G + e(1-t)(1-\eta t)H\} \tag{1.12}$$

and

$$D(t) = 1 - a_1t + a_2t^2 - a_3t^3 + a_4t^4 \tag{1.13}$$

has coefficients given by

$$\begin{aligned} a_1 &= 2 + 2\eta + x^2(e + c\eta - uy^2 - v\overline{y}^2) \\ a_2 &= 1 + 4\eta + \eta^2 + x^2(2c\eta + 2e\eta - (1+\eta)(uy^2 + v\overline{y}^2)) \\ a_3 &= \eta(2 + 2\eta + x^2(c + e\eta - uy^2 - v\overline{y}^2)) \\ a_4 &= \eta^2 \qquad (\text{with} \quad \eta = y^2\overline{y}^2) \end{aligned} \tag{1.14}$$

Thus one obtains the following difference equation

$$g_r - a_1g_{r-1} + a_2g_{r-2} - a_3g_{r-3} + a_4g_{r-4} = 0 \tag{1.15}$$

for g_r's, which can be solved by using the explicit formulas for the solution of the quartic $D(t) = 0$ by radicals (c.f. [5], p.258 Ex.2) and by considering as initial values g_0, g_1, g_2, g_3 (rather then $g_1, \ldots, g_4$) where g_0 is also given by (1.2). To save the space, we will not write these formulas.

Note that, from (1.3) with

$$s = \eta t$$

$$D(t) = 0 \Leftrightarrow (s-1)^2(s-\eta)^2 - x^2(e+c\eta-uy^2-v\overline{y}^2)(s-1)(s-\eta)s - x^2(1-\eta)([e(1-s)+c(s-\eta)]s^2 = 0 \tag{1.16}$$

Case 2. Extension of results of Lin and Wu.

Now we specialize $q = 1$, $c = e =: d$. Then the equation (1.16) tourns to a quadratic one with respect to

$$w = \frac{(s-1)(s-\eta)}{s}$$

with the roots:

$$w_{1/2} = \left(x^2(d(1+\eta) - uy^2 - v\overline{y}^2) \pm x\sqrt{4(1-\eta)^2d + x^2(d(1+\eta) - uy^2 - v\overline{y}^2)^2}\right)/2 \tag{1.17}$$

Then, by choosing appropriate solution of each of the equations

$$s^2 - (1+\eta)s + \eta = w_1s \qquad \text{and} \qquad s^2 - (1+\eta)s + \eta = w_2s$$

$$s_1 = (1+\eta-w_1-\sqrt{(1+\eta-w_1)^2-4\eta})/2, \quad s_3 = (1+\eta-w_2-\sqrt{(1+\eta-w_2)^2-4\eta})/2 \tag{1.18}$$

or the corresponding t's:

$$t_1 = (1+\eta-w_1-\sqrt{(1+\eta-w_1)^2-4\eta})/2\eta, \quad t_3 = (1+\eta-w_2-\sqrt{(1+\eta-w_2)^2-4\eta})/2\eta \tag{1.18'}$$

and substituting into (1.3), with $q = 1$, we obtain two equations for G, H

$$\eta(1-t_i)^2 + [(1-t_i)(uy^2+v\overline{y}^2) - d(1-\eta t_i)]G + d(1-t_i)(1-\eta t_i)H = 0, \quad i = 1, 2$$

whose solutions are:

$$G = \frac{\eta(\eta-1)(1-t_1)(1-t_3)}{d(1-\eta t_1)(1-\eta t_3) - \eta(uy^2+v\overline{y}^2)(1-t_1)(1-t_3)}$$

$$H = \frac{[\eta(uy^2+v\overline{y}^2) - d](1-t_1)(1-t_3) + d(\eta-1)(1-t_1t_3)}{d[d(1-\eta t_1)(1-\eta t_3) - \eta(uy^2+v\overline{y}^2)(1-t_1)(1-t_3)]}$$

By substituting this in (1.11), (1.12) with $q = 1$ one obtains the generating function $G(t) = G(x, y, \overline{y}, u, q = 1, u, v, c = d, e = d; t)$.

If we put $u = v = d = 1$, we obtain a result equivalent to the extremely complicated formula of Lin and Wu. Our result is seems to be more appropriate for coefficient evaluation.

Finally we note that Case $x = 1$ in (1.10) gives the result for **diagonally convex polyominoes on the square lattice**, what will be treated in a future paper.

2. Solution of the Temperley's Model Q and extension

In this section we adapt the technique developed in the previous section to solve the Temperley's Model Q, described in [11] p.7, concerning vertically convex polyominoes. By a polyomino P we understand here a finite union of unit squares in the plane such that the vertices of the squares have integer coordinates, and P is connected and has no finite cut set. Two polyominoes will be considered equivalent if there is a translation that transforms one into the other. A polyomino P is vertically convex if each "column" of P is an unbroken line of squares, that is, if L is any line segment parallel to the y-axis with its two endpoints in P then $L \subset P$.

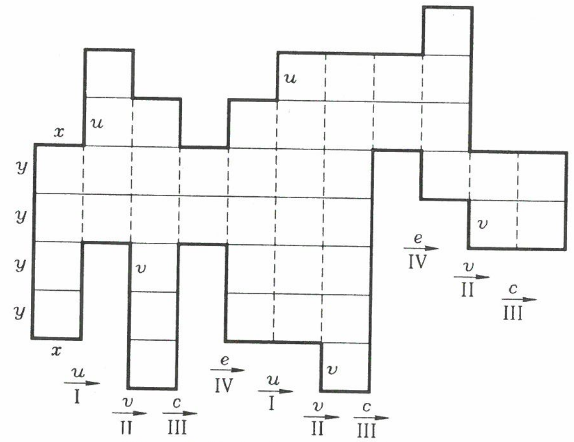

Figure 3. Vertically convex polyomino

Observe that there are four types of transitions from the i^{th}-column to the $(i+1)^{\text{th}}$-column of the $(i+1)^{\text{th}}$ column has a cell higher and no cells lower than the i^{th} column (Type I), has a cell lower and no cells higher (Type II) has no cells higher and no cells lower (Type III), has a cell higher and has a cell lower (Type IV). Let us mark such events by letters u **(upper extension)**, v **(lower extension)**, c **(contraction in weak sense)**, e **(expansion)**. As well we mark by x and y horizontal (unit) contributions to the perimeter, and by q parameter of area.

Remark. Note that the meaning of the symbols u, v, c, e here is slightly different than in Part I, section 5, but the method would work in the case corresponding to that too.

Now we introduce the (perimeters-area-extensions-sources)-generating function

$$G(t) = G(x, y; q; u, v, c, e; t) = \sum_{r \geq 1} g_r(x, y; q; u, v, c, e)t^r \tag{2.1}$$

where g_r counts column convex polyominoes whose first column contains exactly r squares (i.e. r sources). By imagining the polyomino built up, column by column from left to right and taking into account of various possible positions between adjacent columns classified into four types defined above, we find the following set of equations for g_r's.

$$g_r = x^2y^{2r}q^r \left\{ 1 + \sum_{s=1}^{\infty} g_s \sum_{n=1}^{\min\{s-1,r\}} \alpha\beta^n + c\sum_{s=1}^{r} g_s\beta^s(r-s+1) + e\sum_{s=r+1}^{\infty} g_s\beta^r(s-r-1) \right\}, \quad r \geq 1 \tag{2.2}$$

where $\alpha = u + v$, $\beta = y^{-2}$. Here n corresponds to the number of pairs of edges being glued between two adjacent columns. Note that the last sum actually starts with $s = r + 2$, corresponding to the fact the type IV transition (expansion) requires at last two more cells in the next column.

By summing in both casses $1 \leq s \leq r$, and $r < s$ over n we obtain

$$g_r = x^2y^{2r}q^r \left\{ 1 + \sum_{s=1}^{r} \left[\alpha\frac{\beta - \beta^s}{1-\beta} + (r-s+1)c\beta^s\right] g_s + \sum_{s=r+1}^{\infty} \left[\alpha\frac{\beta - \beta^{r+1}}{1-\beta} + (s-r-1)e\beta^r\right] g_s \right\} \tag{2.3}$$

Then letting $\eta := y^2 (= \beta^{-1})$, and using $\sum_{s=r+1}^{\infty} g_s = G - (g_1 + \ldots + g_r)$, $\sum_{s=r+1}^{\infty}(s-r+1)g_s = H - (r+1)G + \sum_{s=1}^{r}(r-s+1)g_s$, we obtain

$$g_r = x^2q^r \left\{ \eta^r + \left(\frac{\alpha(\eta^r - 1)}{\eta - 1} - (r+1)e\right) G + \sum_{s=1}^{r}\left[\frac{\alpha(1-\eta^{r-s+1})}{\eta - 1} + (r-s+1)(c\eta^{r-s} + e)\right] g_s + eH \right\} \tag{2.4}$$

By multiplying both sides of (2.4) by t^r and summing over $r \geq 1$, and using $\sum_{k\geq 1}(k+1)x^k = \frac{1}{(1-x)^2} - 1$, we get the **the functional equation for vertically convex polyominoes on square lattice:**

$$\begin{aligned} G(t) - x^2 \left[\frac{c}{(1-\eta qt)^2} - \frac{u+v}{(1-qt)(1-\eta qt)} + \frac{e}{(1-qt)^2}\right] G(qt) &= \\ = x^2qt \left\{ \frac{\eta}{1-\eta qt} + \frac{e}{1-qt}H + \frac{1}{1-qt}\left[\frac{u+v}{1-\eta qt} - \frac{2-qt}{1-qt}e\right] G \right\} \end{aligned} \tag{2.5}$$

In a similar vein as in the previous section we obtain the following

MAIN THEOREM (area-perimeters-extensions-sources)

The generating function $G(t)$, defined in (2.1), for column convex polyominoes on the square lattice is given by

$$\begin{pmatrix} G(t) \\ G'(t) \end{pmatrix} = \mathcal{A}(t) + \mathcal{B}(t)[I - \mathcal{B}(1)]^{-1}\mathcal{A}(1) \tag{2.6}$$

where $\mathcal{A}$, $\mathcal{B}$ are defined in the same way as in the previous section starting with

$$\begin{aligned} a(t) &= \frac{x^2\eta qt}{1-\eta qt}, \quad b_1(t) = \frac{x^2qt}{1-qt}\left[\frac{u+v}{1-\eta qt} - \frac{2-qt}{1-qt}e\right], \quad b_2(t) = \frac{x^2eqt}{1-qt} \\ c(t) &= x^2\left[\frac{e}{(1-qt)^2} - \frac{u-v}{(1-qt)(1-\eta qt)} + \frac{c}{(1-\eta qt)^2}\right] \end{aligned} \tag{2.7}$$

Remark 1. For perimeter and area generating function another formulas are given in [2], [13].

Remark 2. Note that if, in a polyomino, expansions are forbidden ($e = 0$), then (2.5) reduces to the functional equation

$$G(t) - \frac{x^2}{1-\eta qt}\left(\frac{c}{1-\eta qt} - \frac{u+v}{1-qt}\right) G(qt) = \frac{x^2\eta qt}{1-\eta qt} + \frac{x^2q(u+v)t}{(1-qt)(1-\eta qt)}G \tag{2.5'}$$

having only one initial condition. Its solution, obtained by iteration, we can visualize better, then the general solution (2.6) (given by 2×2 matrices).

Further specialization ($v = 0, e = 0$) correspond to **vertically convex directed animals**, and the case ($c = 0, v = 0, e = 0$) correspond to a **subclass of parallelogram polyominoes** with no repeated horizontal steps in the upper part of the boundary. In the former case u has no significance (it gives # columns $-$ 1 and x^2 marks columns) so we may put $u = 1$ and obtain the following

COROLLARY 2.1 *The generating function for parallelogram polyominoes with "strictly climbing" upper boundary is given by*

$$G(x, y, q, t) = \alpha(t) + \beta(t)\frac{\alpha(1)}{1 - \beta(1)} \tag{2.8}$$

where

$$\alpha(t) = \sum_{m\geq 0} \frac{(-1)^m(x^2q)^{m+1}y^2t}{(qt)_m(y^2qt)_{m+1}}, \quad \beta(t) = \sum_{m>0} \frac{(-1)^{m-1}(x^2q)^m t}{(qt)_m(y^2qt)_m}$$

and $(a)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$.

In particular $G = G(x, y, q, 1) = \frac{\alpha(1)}{1-\beta(1)}$.

Remark 3. If we let $q = 1$ (i.e. disregarding area). Then from (2.5) we obtain

$$G(t) = N(t)/D(t) \tag{2.9}$$

where $N(t) = (1-\eta t)x^2t\{\eta(1-t)^2 + e(1-t)(1-\eta t)H + [(u+v)(1-t) - e(2-t)(1-\eta t)]G\}$
$D(t) = (1-t)^2(1-\eta t)^2 - x^2[c(1-t)^2 - (u+v)(1-t)(1-\eta t) + e(1-\eta t)^2]$.

Let us now consider only the special case $u = v = c = e = 1$ corresponding to a problem considered by Lin in [6] concerning perimeters-generating function for column convex polyominoes on the rectangular lattice. In this case

$$\begin{aligned} D(t) &= (1-t)^2(1-\eta t)^2 - (1-\eta)^2x^2t^2 \\ N(t) &= (1-\eta t)x^2t\{\eta(1-t)^2 + (1-t)(1-\eta t)H + t[2\eta - 1 - \eta t]G\} \end{aligned} \tag{2.10}$$

By using the fact that $g_1 = x^2\eta + x^2H$, for $N(t)$ we can write somewhat nicer expression

$$N(t) = (1-\eta t)t\{x^2\eta(\eta-1)(1-t) + (1-t)(1-\eta t)g_1 + x^2t[\eta(1-t) + \eta - 1]G\} \tag{2.11}$$

which we will exploit in computations.

By splitting $D(t)$ into two quadratic factors and by choosing one root from each factor, say t_1 and t_3 respectively, for which $G(t_1)$ and $G(t_3)$ formally exist, then we will obtain two equations for our yet unknown quantities H and G (or g_1 and G).

The choice is the following

$$t_1 = [a_+ - (a_+^2 - 4\eta)^{1/2}]/2\eta, \quad t_3 = [a_- - (a_-^2 - 4\eta)^{1/2}]/2\eta \quad \text{with } a_\pm = 1 + \eta \pm x(1-\eta) \tag{2.12}$$

We found simpler working with g_1 and G. So our equations read as follows:

$$\left.\begin{aligned} N(t_1) &= 0 \\ N(t_2) &= 0 \end{aligned}\right\} \Leftrightarrow \begin{aligned} (1-t_1)(1-\eta t_1)g_1 + x^2t_1[\eta(1-t_1) + \eta - 1]G &= x^2\eta(1-\eta)(1-t_1)t_1 \\ (1-t_3)(1-\eta t_3)g_1 + x^2t_3[\eta(1-t_3) + \eta - 1]G &= x^2\eta(1-\eta)(1-t_3)t_3 \end{aligned} \tag{2.13}$$

By using that $(1-t_1)(1-\eta t_1) = x(1-\eta)t_1$ and $(1-t_3)(1-\eta t_3) = -x(1-\eta)t_3$, we get

$$\begin{aligned} (1-\eta)g_1 + x[\eta(1-t_1) + \eta - 1]G &= x\eta(1-\eta)(1-t_1) \\ -(1-\eta)g_1 + x[\eta(1-t_3) + \eta - 1]G &= x\eta(1-\eta)(1-t_3) \end{aligned} \tag{2.13'}$$

Thus

$$G = (1-\eta)\left[1 - \frac{1}{1 + \frac{\eta(t_1+t_3-2)}{2(1-\eta)}}\right], \quad g_1 = \frac{(1-\eta)x\eta(t_3 - t_1)}{1 + \frac{\eta(t_1+t_3-2)}{2(1-\eta)}}, \quad H = x^{-2}g_1 - \eta \tag{2.14}$$

THEOREM 2.1 *The (perimeters, sources)-generating function for column convex polyominoes on the square lattice is given explicitly by*

$$G(t) = \sum_{r\geq 0} g_r(x,y)t^r = (1-\eta t)t[A(t) + B(t)g_1 + C(t)G]/D(t) \tag{2.15}$$

where $A(t) = x^2\eta(\eta-1)(1-t)t$, $B(t) = (1-t)(1-\eta t)$, $C(t) = x^2t[\eta(1-t)+\eta-1]$ *and* $D(t)$ *given by* (2.10), g_1 *and* G *by* (2.14), t_1, t_3 *by* (2.12) *and* $\eta = y^2$.

Remark 1. Note that, by "squaring" aplied to the relations (2.14) one obtains algebraic equations satisfied by G and g_1. Simpler equations are obtained if we consider the quantities $F := \frac{G}{1-\eta}$, $\mathbf{f}_1 = \frac{g_1}{1-\eta}$. It is interesting to note that in this process our "magic" substitution $L := \frac{1-3F}{1-F}$, mentioned in the Introduction, Part I, appear naturally and the computations agree with those obtained by the language method. Let us only mention the equation satisfied by $\mathbf{f}_1$:

PROPOSITION 2.1 *The algebraic equation satisfied by the generating function* $\mathbf{f}_1 = \frac{g_1}{1-\eta}$ *reads as follows:*

$$\left[(3\mathbf{f}_1 + x^2)^2 - x(1+x^2)\mathbf{f}_1 + x\frac{1+\eta}{1-\eta}(x^2+\mathbf{f}_1^2)\right]^2 = (3\mathbf{f}_1+x^2)^2(x+\mathbf{f}_1)^2\left(1+x^2+2x\frac{1+\eta}{1-\eta}\right).$$

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The Yang-Baxter equation, symmetric functions, and Schubert polynomials

SERGEY FOMIN

Department of Mathematics, Massachusetts Institute of Technology

Theory of Algorithms Laboratory SPIIRAN, Russia

ANATOL N. KIRILLOV

Steklov Mathematical Institute, St. Petersburg, Russia

1. Introduction

The Yang-Baxter operators $h_i(x)$ satisfy the following relations (cf. [B, DWA]):

$$h_i(x)h_j(y) = h_j(y)h_i(x) \quad \text{if} \quad |i-j| \geq 2\,;$$

$$h_i(x)h_{i+1}(x+y)h_i(y) = h_{i+1}(y)h_i(x+y)h_{i+1}(x)\,.$$

The role the representations of the Yang-Baxter algebra play in the theory of quantum groups [Dr], the theory of exactly solvable models in statistical mechanics [B], low-dimensional topology [DWA, RT, J], the theory of special functions, and other branches of mathematics (see., e.g., the survey [C]) is well-known.

We study the connections between the Yang-Baxter algebra and the theory of symmetric functions and Schubert polynomials. Let us add to the above conditions an equation

$$h_i(x)h_i(y) = h_i(x+y)$$

thus getting the so-called colored braid relations (see [KB, FS] for examples of their representations). It turns out that, once these relations hold, one can introduce a whole class of symmetric functions (and even "double", or "super-" symmetric functions) and respective analogues of the [double] Schubert polynomials [L2, M2] as well. These analogues are proved to have many of the properties of their prototypes; e.g., we generalize the Cauchy identities and the principal specialization formula.

The simplest solution of the above equations involves the nilCoxeter algebra of the symmetric group [FS]. Exploring this special case, we construct super-analogues of Stanley's symmetric functions G_w (see [S]), provide another combinatorial interpretation of Schubert polynomials $\mathfrak{S}_w$, and reprove the basic facts concerning G_w's and $\mathfrak{S}_w$'s. Recently, the construction of this paper has been used [BB] to produce a Pieri rule for Schubert polynomials and yet another algorithm that generates the monomials of $\mathfrak{S}_w$.

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