From the Fibonacci numeration system to the golden mean base and some generalizations

Christiane Frougny* and Jacques Sakarovitch†

Abstract

Every positive integer can be written as a sum of Fibonacci numbers; it can also be written as a sum of powers of the golden mean φ, with a decimal point in the middle. We show that there exists a letter-to-letter finite 2-automaton that maps the Fibonacci representation of any integer onto its φ-expansion, provided the latter is folded around the decimal point. As a corollary, one gets that the set of φ-expansions of all the integers is a context-free linear language. These results are actually proved in the more general case of quadratic Pisot units.

The aim of this paper[1] is to describe some relationships between different ways of writing numbers, with the study of a case where this relationship is achieved by means of a finite automaton.

1 *Where the problem and its solution are presented on an example: the celebrated case of Fibonacci numeration system and golden mean base.*

Let $F = \{f_n \mid n \in \mathbf{N}\}$ be the sequence of Fibonacci numbers (with $f_0 = 1$ and $f_1 = 2$) and let $A = \{0, 1\}$ be the two digit alphabet. This defines the *Fibonacci numeration system* : every integer can be written as a sum of Fibonacci numbers and thus can be represented as a sequence of 0's and 1's, *e.g.* $24 = f_6 + f_2$ and 24 is represented by 1000100. This representation is not unique; for instance $24 = f_5 + f_4 + f_2$ and is thus also represented by the word 110100. For every integer however there exists a unique *normal representation* : the one that does not contains two adjacent 1's, which is also the largest in the lexicographical ordering. The set of all normal representations of the natural integers is thus

$$R_F = 1A^* \setminus A^*11A^* ,$$

a *rational set* of words of A^*.

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*Université Paris 8 and Laboratoire Informatique Théorique et Programmation, Institut Blaise Pascal.

†Institut Blaise Pascal, Cnrs. Address of both authors : 4 place Jussieu, 75252 Paris Cedex 05.

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The Fibonacci sequence is defined by the recurrence relation $f_{n+2} = f_{n+1} + f_n$ to which is naturally associated the characteristic polynomial $P(X) = X^2 - X - 1$ the dominant root of which will be denoted by φ and is currently called *golden mean* or *golden number*. This number φ taken as a *base* also defines a *numeration system* : every real number x — and not only integer — may be developed as a sum of powers of φ and thus be represented as a sequence — possibly infinite — of 0's and 1's together with a "decimal" point. Such a sequence is then called a *φ-representation* of x; *e.g.* $5 = \varphi^3 + \varphi^{-1} + \varphi^{-4}$ and thus 5 can be represented by 1000.1001. Again such a representation is not unique, since for all z in $\mathbf{Z}$ it holds $\varphi^{z+2} = \varphi^{z+1} + \varphi^z$, but for every real number there exists a unique normal φ-representation, called its *φ-expansion* : the one that does not contain two adjacent 1's and does not terminate by the factor 10 repeated indefinitely (which is also the largest in the lexicographical ordering). It turns out that in this base φ numeration system, every integer has a finite φ-expansion (*cf* Proposition 1 below). Table 1 below gives the φ-expansion of the first 15 integers.

N	Fibonacci representations	φ-expansions	Folded φ-expansions
1	1	1.	$\genfrac{}{}{0pt}{}{1}{0}$
2	10	10.01	$\genfrac{}{}{0pt}{}{10}{10}$
3	100	100.01	$\genfrac{}{}{0pt}{}{100}{010}$
4	101	101.01	$\genfrac{}{}{0pt}{}{101}{010}$
5	1000	1000.1001	$\genfrac{}{}{0pt}{}{1000}{1001}$
6	1001	1010.0001	$\genfrac{}{}{0pt}{}{1010}{1000}$
7	1010	10000.0001	$\genfrac{}{}{0pt}{}{10000}{01000}$
8	10000	10001.0001	$\genfrac{}{}{0pt}{}{10001}{01000}$
9	10001	10010.0101	$\genfrac{}{}{0pt}{}{10010}{01010}$
10	10010	10100.0101	$\genfrac{}{}{0pt}{}{10100}{01010}$
11	10100	10101.0101	$\genfrac{}{}{0pt}{}{10101}{01010}$
12	10101	100000.101001	$\genfrac{}{}{0pt}{}{100000}{100101}$
13	100000	100010.001001	$\genfrac{}{}{0pt}{}{100010}{100100}$
14	100001	100100.001001	$\genfrac{}{}{0pt}{}{100100}{100100}$
15	100010	100101.001001	$\genfrac{}{}{0pt}{}{100101}{100100}$

Table 1: Fibonacci representations and φ-expansions

The first question to be answered is the characterization of the set R_φ of the φ-expansions of all integers. The existence of the decimal point roughly situated, as Table 1 let guess, in the middle of every expansion makes it impossible for R_φ to be rational. It will be eventually shown that R_φ is a linear context-free language (see Corollary 2 below). This is indeed the consequence of a much more precise result that will require some transformations on R_φ to be stated.

Let $f.g$ be the φ-expansion of an integer N, *i.e.* an element of R_φ; f and g belong to $\{0,1\}^*$. It is a classical result (see [13]) that the set

$$S = \{(f, \tilde{g}) \mid f.g \in R_\varphi\}$$

where $\tilde{g}$ denotes the mirror image of g, is a rational subset of $\{0,1\}^* \times \{0,1\}^*$ if and only if R_φ is a linear context-free language. Moreover, as we already noticed, the lengths of f and of g are about to be equal — the difference of these lengths is indeed bounded by 1 — and thus, it follows that S is a rational subset of $(\{0,1\} \times \{0,1\})^*$ (see [5], [4], [8]). Such a statement will be made more intelligible by means of the following convention. Every element of $J = \{0,1\} \times \{0,1\}$ will be written as a column :

$$J = \{\genfrac{}{}{0pt}{}{0}{0}, \genfrac{}{}{0pt}{}{0}{1}, \genfrac{}{}{0pt}{}{1}{0}, \genfrac{}{}{0pt}{}{1}{1}\}$$

Any element of J^* can be read as the superposition of two words of equal length, an "upper word" above a "lower word". If $f.g$ is the φ-expansion of N, its writing $\binom{f}{\tilde{g}}$ as an element of J^* will naturally be called the *folded* φ-expansion of N, *e.g.* the folded φ-expansion of 5 is $\binom{1000}{1001}$. Table 1 gives the folded φ-expansion of the first 15 integers.

Let T_φ be the set of folded φ-expansions of all integers; the characterization of R_φ announced above reads then :

PROPOSITION A .— *T_φ is a rational set of J^*.*

One could be more precise and give a regular expression describing T_φ. But the paper develops in another direction, and Proposition A appears to be the consequence of a much stronger result that relates for every integer N its Fibonacci representation and its folded φ-expansion and which is stated by the following.

THEOREM B .— *There exists a letter-to-letter finite 2-automaton $\mathcal{A}$ that maps the Fibonacci representation of any integer onto its folded φ-expansion.*

The automaton $\mathcal{A}$ is not computed directly. Its construction is rather split up into several steps. Some of the steps are derived from previous results that normalization — *i.e.* computation of the φ-expansion from any φ-representation — can be achieved by a letter-to-letter finite 2-automaton ([6]).

The main step is the construction of an automaton $\mathcal{T}$ that works for words where the letters have been grouped into blocks of length 4, and with the property that there is at most one digit 1 in every block. This automaton $\mathcal{T}$ is remarkably simple. It has 5 states, in a one-to-one correspondence with the above mentionned blocks, and consists in the complete oriented graph as indicated in Table 2 which gives the input and output label of every edge. The name given to the states of the automaton meets the notations used in the general case.

It should be noted that the output labels of the edges in $\mathcal{T}$ are not normalized by far (since digits like 2 or even negative digits like $\bar{1}$ are allowed). It is this freedom in the choice of the output labels that makes possible the construction of a 2-automaton with such a simple underlying input automaton, here and even more strikingly in the general case.

	end	0000	0001	0010	0100	1000
origin	label					
0000		0000 / $\begin{smallmatrix}0000\\0000\end{smallmatrix}$	0001 / $\begin{smallmatrix}0001\\0000\end{smallmatrix}$	0010 / $\begin{smallmatrix}0010\\0010\end{smallmatrix}$	0100 / $\begin{smallmatrix}0100\\0010\end{smallmatrix}$	1000 / $\begin{smallmatrix}1000\\1001\end{smallmatrix}$
0001		0100 / $\begin{smallmatrix}1000\\1\bar{1}00\end{smallmatrix}$	0000 / $\begin{smallmatrix}0001\\1000\end{smallmatrix}$	1000 / $\begin{smallmatrix}1010\\2000\end{smallmatrix}$	0010 / $\begin{smallmatrix}0100\\1010\end{smallmatrix}$	0001 / $\begin{smallmatrix}0010\\1010\end{smallmatrix}$
0010		1000 / $\begin{smallmatrix}1100\\0000\end{smallmatrix}$	0010 / $\begin{smallmatrix}0101\\0100\end{smallmatrix}$	0000 / $\begin{smallmatrix}0010\\0100\end{smallmatrix}$	0001 / $\begin{smallmatrix}0100\\0010\end{smallmatrix}$	0100 / $\begin{smallmatrix}1000\\1000\end{smallmatrix}$
0100		0001/ $\begin{smallmatrix}1000\\0\bar{1}00\end{smallmatrix}$	1000/ $\begin{smallmatrix}1101\\1000\end{smallmatrix}$	0100 / $\begin{smallmatrix}1010\\1000\end{smallmatrix}$	0000 / $\begin{smallmatrix}0100\\0010\end{smallmatrix}$	0010 / $\begin{smallmatrix}1000\\1001\end{smallmatrix}$
1000		0010 / $\begin{smallmatrix}1100\\0000\end{smallmatrix}$	0100 / $\begin{smallmatrix}1101\\0000\end{smallmatrix}$	0001 / $\begin{smallmatrix}1010\\0000\end{smallmatrix}$	1000 / $\begin{smallmatrix}2001\\0010\end{smallmatrix}$	0000 / $\begin{smallmatrix}1000\\0001\end{smallmatrix}$

Table 2: The labelled edges of the 2-automaton $\mathcal{T}$

The aim of the paper is to establish Theorem B — and thus Proposition A — in the more general case of quadratic Pisot units. The precise statement requires some more definitions and notations that will be given in the next section. The core of the proof will be the complete description of the 2-automaton $\mathcal{T}$ in the general case (Sections 4 and 5). This description is made possible by the identification of the underlying input automaton with a (finite) symbolic dynamical system the existence of which is "discovered" in Section 3.

2 *Where some definitions are made precise, some notations given, and some previous results recalled so as to state at last the main theorem.*

2.1 Representation of numbers

2.1.1 Representation of integers in a numeration system U

Let $U = (u_n)_{n\geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A *representation in the system U* — or a *U-representation* — of a (positive) integer N is a finite sequence of integers $(d_n)_{0\leq n\leq k(N)}$ such that

$$N = \sum_{n=0}^{k(N)} d_n u_n$$

for a convenient index $k(N) \geq 0$.

Among all possible U-representations of a given integer N one is distinguished and called the *normal U-representation* of N: the greatest for the lexicographical ordering. The normal U-representation of N is denoted by $\langle N\rangle_U$. By convention the normal representation of 0 is the empty word.

Conversely let D be a set of digits; any sequence of digits, or *word* in D^*, is given a *numerical value* by the function $\pi_U : D^* \longrightarrow \mathbf{N}$ which is defined by

$$\pi_U(w) = \sum_{n=0}^{k} d_n u_n \qquad \text{where} \qquad w = d_k \cdots d_0$$

Under the hypothesis that the ratio u_{n+1}/u_n keeps bounded when n goes to infinity, the digits of the normal U-representation of any integer N keep bounded and are all contained in a minimal alphabet A_U associated to U. For any alphabet of digits D (possibly containing negative digits) one can then define a *normalization function*

$$\nu_U : D^* \longrightarrow A_U^*$$

that maps any word w of D^* onto the normal U-representation of the integer $\pi_U(w)$.

2.1.2 Representation of real numbers in base θ

Let now θ be a real number > 1. A *representation in base θ* — or a *θ-representation* — of a real number x is an infinite sequence $(x_n)_{-\infty\leq n\leq k(x)}$ of integers such that

$$x = \sum_{n=-\infty}^{k(x)} x_n \theta^n$$

for a convenient index $k(x)$ in $\mathbf{Z}$.

As above, the greatest of all θ-representations of a given $x \geq 0$ in the lexicographical ordering is distinguished as the *normal θ-representation* of x , usually called the *θ-expansion* of x. The digits x_i of any θ-expansion are elements of the set $A_\theta = \{0, \cdots, [\theta]\}$ see [12]. (This holds indeed when θ is not an integer; when θ is an integer $A_\theta = \{0, \cdots, \theta - 1\}$ — but this later case will never occur here.)

By convention (see [12], [11]), we shall call *θ-expansion* of 1, and denote it by $d(1, \theta)$, the largest θ-representation of 1 in the lexicographical ordering *which is smaller than* "1." *i.e.* the largest sequence of integers $(t_n)_{n\geq 1}$ such that

$$d(1, \theta) = \sum_{n\geq 1} t_n \theta^{-n} .$$

The θ-expansion of x is denoted by $\langle x\rangle_\theta$. An expansion is said to be *finite* if it terminates by infinitely many 0's.

Let us introduce another definition: for every k in $\mathbf{Z}$ the *k-th initial section* of $\mathbf{Z}$ is the set of all integers smaller than or equal to k. The set of all initial sections of $\mathbf{Z}$ is denoted by $\mathbf{Z}_w$. Let D be any set of digits. The set of sequences $(x_n)_{-\infty\leq n\leq k}$ with x_i in D is thus denoted by $D^{\mathbf{Z}_w}$. It is a natural convention to consider that any finite sequence $(y_m)_{l\leq m\leq k}$ of elements in D is also an infinite sequence $(y_m)_{-\infty\leq m\leq k}$ of $D^{\mathbf{Z}_w}$ with $y_m = 0$ for all $m < l$.

Any element of $D^{\mathbf{Z}_w}$ is given a *numerical value* by the function $\pi_\theta : D^{\mathbf{Z}_w} \longrightarrow \mathbf{R}$ which is defined by

$$\pi_\theta(s) = \sum_{n=k}^{-\infty} s_n \theta^n \qquad \text{where} \qquad s = s_k \cdots s_0.s_{-1}s_{-2}\cdots$$

Now, for any alphabet of digits D (possibly containing negative digits) one can then define a *normalization function*

$$\nu_\theta : D^{\mathbf{Z}_w} \longrightarrow A_\theta^{\mathbf{Z}_w}$$

that maps any sequence s of $D^{\mathbf{Z}_w}$ onto the θ-expansion of the real $\pi_\theta(s)$.

2.2 Finite automata and 2-automata

2.2.1 Finite words

We basically follow the exposition of [5] and of [4] for the definition of finite automata over an alphabet. An *automaton over a finite alphabet* A, $\mathcal{A} = (Q, A, E, I, T)$ is a directed graph labelled by elements of A; Q is the set of *states*, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. The automaton is *finite* if Q is finite and this will always be the case in this paper. A *computation* in $\mathcal{A}$ is a finite path. It is said to be *successful* if it starts on an initial state and ends in a terminal state. The *behavior* of $\mathcal{A}$ is the subset $|\mathcal{A}|$ of A^* consisting of labels of successful computations of $\mathcal{A}$. A subset of A^* is said to be *rational* if it is the behavior of a finite automaton over A.

This definition of automata as labelled graphs extends readily to automata over $A^* \times B^*$ in which case it is called *2-automaton* : $\mathcal{A} = (Q, A^* \times B^*, E, I, T)$ is a directed graph the edges of which are labelled by elements of $A^* \times B^*$. The automaton is finite if the set of edges E is finite (and thus Q is finite). A relation is said to be *computable* by a finite 2-automaton if it is the behavior of such an automaton. In the literature 2-automata are also often called *non deterministic generalized sequential machines* or *transducers*. When the behavior $|\mathcal{A}|$ of a 2-automaton $\mathcal{A}$ is a functional relation, we say that $\mathcal{A}$ *realizes* this function, also denoted by $\mathcal{A}$.

Let $\mathcal{A}$ be a 2-automaton over $A^* \times B^*$. The (1-)automaton over A obtained by taking the projection over A^* of the label of every edge of $\mathcal{A}$ is called the *underlying input automaton* of $\mathcal{A}$. A 2-automaton is said to be *sequential* if its underlying input automaton is deterministic. A *letter-to-letter 2-automaton* is a 2-automaton with edges labelled in $A \times B$. A letter-to-letter 2-automaton can thus be viewed as *a 1-automaton with input alphabet* $A \times B$ and in particular can be determinized (see [8] for more details).

2.2.2 Infinite words

The set of infinite words — often called ω-words in the literature — on an alphabet A is denoted by $A^{\mathbf{N}}$. An infinite computation of an automaton $\mathcal{A}$ on A, $\mathcal{A} = (Q, A, E, I, T)$, is an infinite labelled path in the labelled graph $\mathcal{A}$. The computation is *successful* if it starts in an initial state and goes infinitely often through T. This definition of successfulness is usually known as the "Büchi condition of acceptance". The *infinite behavior* of $\mathcal{A}$, denoted by $||\mathcal{A}||$, is the set of labels of successful computations of $\mathcal{A}$. These definitions extend to relations on infinite words. Let $\mathcal{A} = (Q, A^* \times B^*, E, I, T)$ be a 2-automaton with edges labelled by elements of $A^* \times B^*$.

2.3 Pisot numbers

A polynomial $P(X) = a_n X^n + \cdots + a_0$ of $\mathbf{Z}[X]$ is said to be *monic* if $a_n = 1$. An *algebraic integer* is a root of a monic polynomial of $\mathbf{Z}[X]$. A *Pisot number* is an algebraic integer > 1 such that all its algebraic conjugates have modulus smaller than 1.

An algebraic integer is said to be a *unit* if the constant term a_0 of its minimal polynomial $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is equal to ± 1. Thus the minimal polynomial of a quadratic Pisot unit θ is of the form :

$$P(X) = X^2 - rX - \varepsilon$$

with either $r \geq 1$ and $\varepsilon = +1$ or $r \geq 3$ and $\varepsilon = -1$.

2.3.1 Representation of integers in base θ

When θ is not an integer, the θ-expansion of an integer is "usually" an infinite sequence over the alphabet A_θ. It turns out however that for certain numbers θ the θ-expansion of every integer is finite. More precisely,

Proposition 1 .— [9] *Let $\theta > 1$ be such that one of the following holds*

i) $d(1,\theta) = a_1 \cdots a_m$ *with* $a_1 \geq \cdots \geq a_m \geq 1$;

ii) $d(1,\theta) = t_1 \cdots t_m (t_{m+1})^\omega$ *with* $t_1 \geq \cdots \geq t_m > t_{m+1} \geq 1$.

Then θ is a Pisot number, and every integer has a finite θ-expansion.

In the sequel of this paper, we shall concentrate on quadratic Pisot numbers. We will use the following corollary of the previous proposition.

Corollary 1 .— [9] *If θ is a quadratic Pisot number, then every integer has a finite θ-expansion.*

2.3.2 Linear numeration systems associated to Pisot numbers

A very basic property of Pisot numbers (as far as θ-expansions are concerned) is given by the following.

Theorem 1 .— [2] *If θ is a Pisot number then $d(1,\theta)$, the θ-expansion of 1, is eventually periodic.*

This property makes it possible to canonically associate a linear recurrent sequence U_θ to every Pisot number θ. Two cases have to be considered, according to whether $d(1,\theta)$ is finite or infinite. We give here the construction of the sequence U_θ for the case of quadratic Pisot units we shall be studying. The general case is analogous; the formulae are just a bit more difficult to read.

Case 1. $\varepsilon = +1$, $r \geq 1$, *i.e.* θ is the dominant root of $P(X) = X^2 - rX - 1$

$$A_\theta = \{0, \cdots, r\} \qquad d(1,\theta) = r\,1$$

The linear recurrent sequence $U_\theta = (u_k)_{k\geq 0}$ associated to θ is defined by:

$$u_{k+2} = r u_{k+1} + u_k, \qquad k \geq 0 \qquad u_0 = 1, \qquad u_1 = r+1$$

Case 2. $\varepsilon = -1$, $r \geq 3$, *i.e.* θ is the dominant root of $P(X) = X^2 - rX + 1$

$$A_\theta = \{0, \cdots, r-1\} \qquad d(1,\theta) = (r-1)(r-2)^\omega$$

The linear recurrent sequence $U_\theta = (u_k)_{k\geq 0}$ associated to θ is defined by:

$$u_{k+2} = r u_{k+1} - u_k, \qquad k \geq 0 \qquad u_0 = 1, \qquad u_1 = r.$$

The sequence U_θ together with the alphabet A_θ defines the *linear numeration system associated to θ*.

2.3.3 Normalization in base θ

The fundamental property that relates representation in a Pisot base and automata theory is given by the following.

Proposition 2 .— [6] *If θ is a Pisot number, then for every alphabet D, normalization on $D^{\mathbf{Z}_w}$ in base θ is a function computable by a letter-to-letter finite 2-automaton.*

In [6] this statement is proved in the case where every element of D is non-negative. The proof extends readily to alphabets containing both positive and negative digits. As a matter of fact, the converse of this result holds as well ([1]) but this will not be used here.

2.4 Main result

2.4.1 Folded θ-representation

We now introduce the *folding* operation around the "decimal" point of a θ-representation.

Let D be an arbitrary alphabet of integers containing 0, and let $D_\rho = \{\genfrac{}{}{0pt}{}{a}{b} \mid a, b \in D\}$ be the alphabet of pairs of elements of D, conveniently written one above the other. The mirror image of a word v is denoted by $\tilde{v}$. Any element w of D_ρ^* can be written as $w = \genfrac{}{}{0pt}{}{u}{v}$, where $u, v \in D^*$ and $|u| = |v|$. The *upper* part of w will be denoted by $\overleftarrow{w}= u$ and the *lower* part of w by $\overrightarrow{w}= \tilde{v}$.

Let $s = f.g$, with $f, g \in D^*$; by completing the shorter of f and g with enough 0's (at the left for f, or at the right for g), one can assume that $|f| = |g|$. Such an s will be called a *balanced* *(θ-)representation.*

The *folding operation* ρ maps any balanced representation $s = f.g$ onto the element $\rho(s) = \genfrac{}{}{0pt}{}{f}{\tilde{g}}$ of D_ρ^*. Conversely the inverse of ρ, ρ^{-1}, called the *unfolding* operation, maps every element $w = \genfrac{}{}{0pt}{}{u}{v}$, of D_ρ^* onto the balanced representation $\rho^{-1}(w) = u.\tilde{v}$. Thus $\overleftarrow{\rho(f.g)}= f$, $\overrightarrow{\rho(f.g)}= g$, and $\rho^{-1}(w) =\overleftarrow{w} \,.\, \overrightarrow{w}$.

The numerical value function π_θ extends to folded representations : if w is a word on D_ρ^*, then, by definition $\pi_\theta(w) = \pi_\theta(\overleftarrow{w} \,.\, \overrightarrow{w})$.

Two representations (on X^* or on D^* or on D_ρ^*) will be called *equivalent* if they define the same integer, *i.e.* if they have the same numerical value.

2.4.2 The result

We are now in a position to state:

Theorem 2 .— *Let θ be a quadratic Pisot unit and let D be an arbitrary alphabet of non-negative integers. There exists a letter-to-letter 2-automaton that maps any word w on D^* onto the folded θ-expansion of the integer represented by w in the linear numeration system (U_θ, A_θ).*

As a corollary of Theorem 2 and of classical results in formal language theory (see [13]), we get:

Corollary 2 .— *Let θ be a quadratic Pisot unit. The set of folded θ-expansions of all integers is a rational language. The set of θ-expansions of all integers is a linear context-free language.*

Theorem 2 is related to the problem of normalization in the system U_θ. It is not known in full generality if normalization in a system U_θ associated to a Pisot number is a function computable by a letter-to-letter finite 2-automaton. A positive answer is given in certain cases ([7]) among which fall Case 1 above of quadratic Pisot units. We have the following.

Corollary 3 .— *If θ is a quadratic Pisot unit, then normalization in the system U_θ is a function computable by a letter-to-letter finite 2-automaton on any alphabet D.*

Proof. Let $\mathcal{A}$ be the letter-to-letter finite 2-automaton of Theorem 2, and let $L(U_\theta)$ be the set of normal U_θ-representations of the integers. $L(U_\theta)$ is a rational subset of A_θ^* ([3]). The normalization ν_{U_θ} on D is obtained by

$$\forall w \in D^*,\ \nu_{U_\theta}(w) = (\mathcal{A}^{-1} \circ \mathcal{A}(w)) \cap L(U_\theta)$$

and the result follows. ■

Subsequent work [10] proves Corollary 3 in the more general case where θ is such that its minimal polynomial is equal to the characteristic polynomial of U_θ.

3 *Where the θ-expansion of the elements of the linear recurrent sequence U_θ is computed, which leads to the reduction of the problem to a smaller set of words and at the same time puts the reader on the track of a finite 2-automaton.*

From now on θ is a quadratic Pisot unit, the dominant root of $P(X) = X^2 - rX - \varepsilon$; and U_θ is the linear recurrent sequence associated to θ as above. **Case 1** will refer to the conditions $\varepsilon = +1$, $r \geq 1$; **Case 2** to the conditions $\varepsilon = -1$, $r \geq 3$.

The result relies indeed on the very regular expression of the elements of U_θ in terms of the powers of θ as stated in the following, the proof of which is straightforward.

Proposition 3 .— **Case 1**. *For every $k \in \mathbf{N}$,*

$$\begin{aligned} u_{2k} &= \theta^{2k} + (r-1)\theta^{2k-2} + \theta^{2k-4} + \cdots + (r-1)\theta^{-2k+2} + \theta^{-2k} \\ u_{2k+1} &= \theta^{2k+1} + (r-1)\theta^{2k-1} + \theta^{2k-3} + \cdots + (r-1)\theta^{-2k-1} + \theta^{-2k-2} \end{aligned}$$

Case **2**. *For every $k \in \mathbf{N}$,*

$$u_k = \theta^k + \theta^{k-2} + \cdots + \theta^{-k}.$$

Proposition 3 can be rewritten using θ-expansions of elements of U_θ :

Proposition 4 .— **Case 1**. *For $k \in \mathbf{N}$,*

$$\begin{aligned} \langle u_{4k} \rangle_\theta &= 1(0(r-1)01)^k.(0(r-1)01)^k \\ \langle u_{4k+1} \rangle_\theta &= 10((r-1)010)^k.((r-1)010)^k(r-1)1 \\ \langle u_{4k+2} \rangle_\theta &= 10(r-1)(010(r-1))^k.(010(r-1))^k01 \\ \langle u_{4k+3} \rangle_\theta &= (10(r-1)0)(10(r-1)0)^k.(10(r-1)0)^k10(r-1)1 \end{aligned}$$

Case **2**. *For $k \in \mathbf{N}$,*

$$\langle u_{2k} \rangle_\theta = 1(01)^k.(01)^k \qquad \langle u_{2k+1} \rangle_\theta = (10)^{k+1}.(10)^{k+1}$$

Proposition 3 can be rewritten once more using folded θ-expansions.

PROPOSITION 5 .— **Case 1**. *For every $k \in \mathbf{N}$,*

$$\rho(\langle u_{4k}\rangle_\theta) = \left(\begin{smallmatrix}0&0&0&1\\0&0&0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+1}\rangle_\theta) = \left(\begin{smallmatrix}0&0&1&0\\0&0&1&r-1\end{smallmatrix}\right)\left(\begin{smallmatrix}r-1&0&1&0\\0&1&0&r-1\end{smallmatrix}\right)^k$$

$$\rho(\langle u_{4k+2}\rangle_\theta) = \left(\begin{smallmatrix}0&1&0&r-1\\0&0&1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&r-1\\r-1&0&1&0\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+3}\rangle_\theta) = \left(\begin{smallmatrix}1&0&r-1&0\\1&r-1&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&r-1&0\\0&r-1&0&1\end{smallmatrix}\right)^k$$

Case 2. *For every $k \in \mathbf{N}$,*

$$\rho(\langle u_{4k}\rangle_\theta) = \left(\begin{smallmatrix}0&0&0&1\\0&0&0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&1\\1&0&1&0\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+1}\rangle_\theta) = \left(\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right)^k$$

$$\rho(\langle u_{4k+2}\rangle_\theta) = \left(\begin{smallmatrix}0&1&0&1\\0&0&1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&1\\1&0&1&0\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+3}\rangle_\theta) = \left(\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right)^k$$

This last series of equations suggests the grouping of digits by blocks of length 4 and the following notation : for the sequel of the paper, let $X = \{z, a, b, c, d\}$ with

$$z = 0000, \qquad a = 0001, \qquad b = 0010, \qquad c = 0100, \qquad d = 1000.$$

The normal U_θ-representation of any number u_n is the word 10^n which can be written, using the new alphabet X:

$$\langle u_{4k}\rangle_{U_\theta} = az^k, \qquad \langle u_{4k+1}\rangle_{U_\theta} = bz^k, \qquad \langle u_{4k+2}\rangle_{U_\theta} = cz^k, \qquad \langle u_{4k+3}\rangle_{U_\theta} = dz^k.$$

The comparison of this series of equalities and the ones of Proposition 5 shows an obvious correspondence between every word of the form yz^k for $y \in X$ and its equivalent folded θ-expansion, obvious and computable by a letter-to-letter finite 2-automaton. The core of the paper — developed in the next two sections — consists in showing that this correspondence extends from words of the form yz^k to any words of X^*:

THEOREM 3 .— *Let f be a word of X^*. There exists a letter-to-letter 2-automaton which associates to f its folded equivalent θ-expansion.*

Before coming to this point we have to show that Theorem 3 implies Theorem 2. Let us introduce another notation. Let $f = f_n \cdots f_0$ and $g = g_n \cdots g_0$ be two words of D^* of equal length. The *digit-addition* of f and g is the word $f \oplus g = (f_n + g_n) \cdots (f_0 + g_0)$. Let m be the greatest element of D. The following clearly holds.

FACT 1 .— *Any word of D^* can be obtained by the digit-addition of at most $4m$ words of X^*.*

Since the letter-to-letter 2-automaton the existence of which is asserted in Theorem 3 has A_θ as output alphabet, it follows then that it is easy to construct a letter-to-letter finite 2-automaton that computes for every U_θ-representation on D a folded equivalent θ-representation over an alphabet of digits bounded by $4m[\theta]$. The conclusion follows then from Proposition 2 and from the following.

PROPOSITION 6 .— *Let $\mu : D^{\mathbf{Z}_w} \longrightarrow D^{\mathbf{Z}_w}$ be a relation realized by a letter-to-letter finite 2-automaton. Then the relation $\mu^\rho : D^*_\rho \longrightarrow D^*_\rho$ defined by $\mu^\rho = \rho \circ \mu \circ \rho^{-1}$ is also realized by a letter-to-letter finite 2-automaton.*

Note that Proposition 6 would not hold anymore if μ were realized by a 2-automaton not letter-to-letter. It is this step of the proof that makes necessary to specify throughout the paper that the relations we are interested in are actually realized by *letter-to-letter* 2-automata.

4 Where a (finite) dynamical system is computed to serve as underlying input automaton of the searched 2-automaton. It is a one- or two-generator finite Abelian group, whose order is equal to the discriminant of θ or to its half.

Let us come back to Proposition 5 and to the "obvious" 2-automaton it suggests for the computation of a folded equivalent θ-expansion of words of the form az^k. The letter a induces in such an automaton a transition from the initial state to a certain state, call it $\hat{a}$. The reading of a letter z ($= 0000$) when in state $\hat{a}$ causes the 2-automaton to stay in the same state $\hat{a}$ and to output the "letter" $\left(\begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}\right)$. If we thus keep reading letter z, the automaton keeps outputing letter $\left(\begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}\right)$; in such a way that one can say that this state $\hat{a}$ potentially contains the left infinite word

$${}^\omega\left(\begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}\right)$$

the unfold form of which is the bi-infinite word

$${}^\omega(0(r-1)01).(0(r-1)01)^\omega.$$

The same is true, up to a "shift", if any other letter b, c, or d of X is read from the initial state. Consider now a word of the form

$$w = abz^k = az^{k+1} \oplus z^b z^k.$$

If az^k and bz^k are associated to two bi-infinite words then abz^k should be associated to the digit-sum of these bi-infinite words. The idea behind the building of the 2-automaton announced in Theorem 3 is to maintain the identification between the states of the 2-automaton and some bi-infinite words, the reading of a letter being equivalent to an addition.

Let Y be the set of periodic bi-infinite words on $\mathbf{Z}$ of period 4. It is a commutative group isomorphic to $\mathbf{Z}^4$. The shift σ acting on the elements of Y corresponds in this isomorphism to a circular permutation on the four coordinates. If w is a 4-letter word, the corresponding element of Y is denoted by $\underline{\underline{w}} = {}^\omega w.w^\omega$. Then w is said to *represent* $\underline{\underline{w}}$.

4.1 Congruence γ_θ and set of representatives

The numerical equivalence for θ-representations has a natural correspondence on the elements of Y. It is a congruence, denoted by γ_θ, and generated by the equality $1\bar{r}\bar{\varepsilon}0 = 0000$, and the equalities obtained by shifting, that is :

$$1\bar{r}\bar{\varepsilon}0 = 0000, \qquad \bar{r}\bar{\varepsilon}01 = 0000, \qquad \bar{\varepsilon}01\bar{r} = 0000, \qquad 01\bar{r}\bar{\varepsilon} = 0000.$$

(with the convention that if k is an integer, $\bar{k}$ denotes $-k$.)

The following two propositions describe the representatives of the elements of Y modulo γ_θ. They are the exact counterpart of the result of [11] characterizing the θ-expansions of real numbers. Their proofs are a combinatorial play with the defining relations of γ_θ and are ommited here.

PROPOSITION 7 .— **(Case 1)** *Let θ be the root > 1 of $P(X) = X^2 - rX - 1$, $r \geq 1$. Every class modulo γ_θ contains a unique element represented by a word of A^4_θ such that all its conjugates are strictly less in the lexicographical ordering than the word $r0r0$.*

Proposition 8 .— (**Case 2**) *Let θ be the root > 1 of $P(X) = X^2 - rX + 1$, $r \geq 3$. Every class modulo γ_θ contains a unique element represented by a word of A_θ^4 which is different from the word $(r-2)(r-2)(r-2)(r-2)$ and such that all its conjugates are strictly less in the lexicographical ordering than the word $(r-1)(r-2)(r-2)(r-2)$.*

4.2 Computation of the dynamical system

Case 1. $\varepsilon = +1$.

Let $\hat{a}$ be the word $0(r-1)01$ and $\underline{\underline{\hat{a}}} = {}^\omega(0(r-1)01).(0(r-1)01)^\omega$ be the corresponding element of Y/γ_θ. Let G_θ be the dynamical system generated by $\underline{\underline{\hat{a}}}$ (and closed under addition and shift).

Proposition 9 .— *Let θ be the root > 1 of $P(X) = X^2 - rX - 1$, $r \geq 1$, and $\Delta = r^2 + 4$ be the discrimant of P.*
(i) If r is odd, then $G_\theta \simeq \mathbf{Z}/\Delta\mathbf{Z}$
(ii) If r is even, then
a) if $r = 4m$, then $G_\theta \simeq \mathbf{Z}/(\Delta/2)\mathbf{Z}$ *b) if $r = 4m+2$, then $G_\theta \simeq \mathbf{Z}/(\Delta/4)\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.*

Case 2. $\varepsilon = -1$.

Let $\hat{a} = 0101$ and $\underline{\underline{\hat{a}}} = {}^\omega(0101).(0101)^\omega$. Let, as in Case 1, G_θ be the dynamical system generated by $\underline{\underline{\hat{a}}}$ (and closed under addition and shift).

Proposition 10 .— *Let θ be the root > 1 of $P(X) = X^2 - rX + 1$, $r \geq 3$, and $\Delta = r^2 - 4$ be the discrimant of P.*
(i) If r is odd, then $G_\theta \simeq \mathbf{Z}/\Delta\mathbf{Z}$ *(ii) If r is even, then $G_\theta \simeq \mathbf{Z}/(\Delta/2)\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.*

In both cases, the defining relations of γ_θ yield a finite set of relations between $\underline{\underline{\hat{a}}}$ and the three words obtained by shift that allows to compute G_θ.

5 *Where the description of the searched 2-automaton is achieved.*

Let G_θ as above and let $\delta : X^* \longrightarrow G_\theta$ be the canonical surjective morphism. If f is in X^* it is convenient to note $\delta(f) = \underline{\underline{\hat{f}}}$. Then $\delta(fh) = \underline{\underline{\hat{f}}} + \underline{\underline{\hat{h}}}$. We shall denote also by δ the (right) representation of X^* over G_θ: $\forall \underline{\underline{g}} \in G_\theta$, $\forall f \in X^*$, $\delta(\underline{\underline{g}}, f) = \underline{\underline{g}} + \underline{\underline{\hat{f}}}$. The underlying (1-)automaton of the 2-automaton we are building is precisely the right representation of X^* over G_θ : the set of states is G_θ, the initial state $1_{G_\theta} = \underline{\underline{0}}$ and the transition function is defined by δ.

The description of the output function requires some notations. We call *double-digit* the elements of D_ρ for a digit alphabet D. For $x \in X$, let $\alpha(x)$ be the 4 double-digit prefix and $\beta(x)$ be the 4 double-digit periodic factor of the word $\rho(\langle \pi_{U_\theta}(xz^k)\rangle_\theta) = \alpha(x)\beta(x)^k$. Then, it holds $\beta(x) = \rho(\hat{x}.\hat{x})$ for $x \in X$.

For instance, let us take $x = a$ and $\varepsilon = +1$ (see Proposition 5, Case 1). Then

$$\pi_{U_\theta}(az^k) = u_{4k} \qquad \rho(\langle u_{4k}\rangle_\theta) = \left(\begin{smallmatrix}0&0&0&1\\0&0&0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}\right)^k \qquad \alpha(a) = \begin{smallmatrix}0&0&0&1\\0&0&0&0\end{smallmatrix} \qquad \beta(a) = \begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}.$$

The output function is then given by the following.

Lemma 1 .— *For every $\underline{\underline{g}}$ in G_θ and every x in X let $\underline{\underline{h}} = \underline{\underline{g}} + \underline{\underline{\hat{x}}}$. There exists a set B of 4 double-digit words on a certain digit alphabet and a function $\lambda : G_\theta \times X \longrightarrow B$ such that*

$$\forall \underline{\underline{g}} \in G_\theta,\ \forall x \in X,\ \forall k \in \mathbf{N} \qquad \pi_\theta(\rho(g.g)^{k+1}) + \pi_\theta(\alpha(x)\beta(x)^k) = \pi_\theta(\lambda(\underline{\underline{g}}, x)\rho(h.h)^{k+1}).$$

The essence of that statement is the fact that it is possible to find a word $\lambda(\underline{\underline{g}}, x)$ such that the equality holds for every k.

An explicit statement for Theorem 3 is given by the following.

Theorem 4 .— *Let $\mathcal{T} = (G_\theta, X, B, E, \underline{\underline{0}}, \{G_\theta\})$ be the sequential letter-to-letter 2-automaton defined by the set of edges $E = \{\underline{\underline{g}}, (x, \lambda(\underline{\underline{g}}, x)), \delta(\underline{\underline{g}}, x)\}$, (where λ is the function defined in Lemma 1). The 2-automaton $\mathcal{T}$ maps every word of X^* onto a folded equivalent θ-representation , that is*

$$\forall f \in X^*, \qquad \pi_\theta(\mathcal{T}(f)) = \pi_U(f).$$

Proof. By induction on $|f|$ we show a more general relation

$$\forall k \in \mathbf{N}, \qquad \pi_\theta(\mathcal{T}(fz^k)) = \pi_U(fz^k). \tag{6}$$

By construction of $\mathcal{T}$,

$$\forall f \in X^*,\ \forall k \in \mathbf{N}, \qquad \mathcal{T}(fz^k) = \mathcal{T}(f)\rho(\hat{f}^k.\hat{f}^k).$$

But $\mathcal{T}(xz^k) = \alpha(x)\rho(\hat{x}^k.\hat{x}^k)$, thus equation (6) is satisfied for $|f| = 1$.

Now we have

$$\begin{aligned}
\pi_U(fxz^k) &= \pi_U(fz^{k+1}) + \pi_U(xz^k) \\
&= \pi_\theta(\mathcal{T}(f)\rho(\hat{f}^{k+1}.\hat{f}^{k+1}) + \pi_\theta(\alpha(x)\rho(\hat{x}^k.\hat{x}^k)) && \text{by induction hypothesis} \\
&= \pi_\theta(\mathcal{T}(f)0^{k+1}) + \pi_\theta(\rho(\hat{f}^{k+1}.\hat{f}^{k+1})) + \pi_\theta(\alpha(x)\rho(\hat{x}^k.\hat{x}^k)) \\
&= \pi_\theta(\mathcal{T}(f)0^{k+1}) + \pi_\theta(\lambda(\underline{\underline{\hat{f}}}, x)\rho(\widehat{fx}^k.\widehat{fx}^k)) && \text{by Lemma 1 and by } \widehat{fx} = \hat{f} + \hat{x} \\
&= \pi_\theta(\mathcal{T}(fx)\mathcal{T}(\widehat{fx}^k)) \\
&= \pi_\theta(\mathcal{T}(fxz^k)).
\end{aligned}$$

■

Now, $\mathcal{T}(f)$ is not normalized but Proposition 2 and Proposition 6 yield by composition the 2-automaton announced in Theorem 3 and this concludes the proof.

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Asymptotics of Orthogonal Polynomials with Applications to q-Analogs of Classical Polynomials

by

J. S. Geronimo

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia 30332 USA

Consider the difference equation

$$a_{n+1}p_{n+1}(y) + b_n p_n(y) + a_n p_{n-1}(y) = y p_n(y) \tag{1}$$

with initial conditions

$$p_{-1}(x) = 0, \quad p_0(x) = 1. \tag{2}$$

Here $a_n > 0$ and b_n is real. As is well known the family $\{p_n(x)\}$ forms a family of polynomials orthogonal with respect to some (not necessarily unique) positive measure supported on the real line. We consider the asymptotics of these polynomials when the interval of orthogonality is, 1) a bounded interval, or 2) an infinite interval. In the first case we will assume that $\lim_{n\to\infty} a_n = 1/2$ and $\lim_{n\to\infty} b_n = 0$, and that

$$\Sigma v(n) n\{|1 - 2a(n)| + |b(n)|\} < \infty, \tag{3}$$

where $v(n)$ satisfies the equations $v(0) = 1$, $v(-n) = v(n)$, $v(n) \leq v(n+1)$, $v(n+m) \leq v(n)v(m)$, and $\limsup v(n)^{1/n} = R > 1$. The above equations allow us to associate to this problem a family of Banach algebras A_v where $f \in A_v$ if and only if $\|f\|_v = \sum_n v(n)|c_n|$ with $f(z) = \sum c_n z^n$, $|z| = 1$. The variable z is related to x in (1) by $x = \frac{1}{2}(z + 1/z)$. This allows a very precise description of the asymptotics of solutions of (1) as well as a precise description of the spectral measure associated with (1) and will be discussed in section (II). In section (III) these results are used to study the Askey-Wilson polynomials. These polynomials contain the q-analogs of classical polynomials when $|q| < 1$.

When the interval of orthogonality is infinite we will assume that the sequences $\{a_n\}$ and $\{b_n\}$ are regularly or slowly varying functions of n. That is we suppose there exists an increasing positive sequence $\{\lambda_n\}, n \geq 0$ such that

$$\lim_{n\to\infty} \frac{a_n}{\lambda_n} = a > 0, \quad \lim_{n\to\infty} \frac{b_n}{\lambda_n} = b \in \mathbb{R}, \tag{4}$$

with

$$\lim_{n\to\infty} n\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \alpha \geq 0. \tag{5}$$

Using a discrete analog of the Liouville-Green method for differential equations, we obtain (in section IV) strong asymptotics away from the real line for polynomials whose recurrence coefficients satisfy (4) and (5).