[2] A. Bertrand, Développements en base de Pisot et répartition modulo 1. *C.R.Acad. Sc.*, Paris **285**, 1977, 419–421.

[3] A. Bertrand-Mathis, Comment écrire les nombres entiers dans une base qui ne l'est pas. *Acta Math. Acad. Sci. Hungar.* **54**, 1989, 237–241.

[4] S. Eilenberg, *Automata, Languages and Machines*, vol. A, Academic Press, 1974.

[5] C.C. Elgot and J.E. Mezei, On relations defined by generalized finite automata. *IBM Journal Res. and Dev.* **9**, 1965, 47–68.

[6] Ch. Frougny, Representation of numbers and finite automata. *Math. Syst. Th.* **25**, 1992, 37–60.

[7] Ch. Frougny, Confluent linear numeration systems. *Theoret. Computer Sci.* **106**, 1992, 183–219.

[8] Ch. Frougny and J. Sakarovitch, Synchronized rational relations of finite and infinite words. *Theoret. Computer Sci.* **108**, 1993, 45–82.

[9] Ch. Frougny and B. Solomyak, Finite beta-expansions. *Ergod. Th. & Dynam. Sys.* **12**, 1992, 713–723.

[10] Ch. Frougny and B. Solomyak, On representation of integers in linear numeration systems, *in preparation.*

[11] W. Parry, On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11**, 1960, 401–416.

[12] A. Rényi, Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8**, 1957, 477–493.

[13] A.L. Rosenberg, A machine realization of the linear context-free languages, *Inform. & Control* **10**, 1967, 175–188.

Asymptotics of Orthogonal Polynomials with Applications to q-Analogs of Classical Polynomials

by
J. S. Geronimo
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332 USA

Consider the difference equation

$$a_{n+1}p_{n+1}(y) + b_n p_n(y) + a_n p_{n-1}(y) = y p_n(y) \tag{1}$$

with initial conditions

$$p_{-1}(x) = 0, \quad p_0(x) = 1. \tag{2}$$

Here $a_n > 0$ and b_n is real. As is well known the family $\{p_n(x)\}$ forms a family of polynomials orthogonal with respect to some (not necessarily unique) positive measure supported on the real line. We consider the asymptotics of these polynomials when the interval of orthogonality is, 1) a bounded interval, or 2) an infinite interval. In the first case we will assume that $\lim_{n\to\infty} a_n = 1/2$ and $\lim_{n\to\infty} b_n = 0$, and that

$$\Sigma v(n) n \{|1 - 2a(n)| + |b(n)|\} < \infty, \tag{3}$$

where $v(n)$ satisfies the equations $v(0) = 1$, $v(-n) = v(n)$, $v(n) \le v(n+1)$, $v(n+m) \le v(n)v(m)$, and $\limsup v(n)^{1/n} = R > 1$. The above equations allow us to associate to this problem a family of Banach algebras A_v where $f \in A_v$ if and only if $\|f\|_v = \sum_n v(n)|c_n|$ with $f(z) = \sum c_n z^n$, $|z| = 1$. The variable z is related to x in (1) by $x = \frac{1}{2}(z + 1/z)$. This allows a very precise description of the asymptotics of solutions of (1) as well as a precise description of the spectral measure associated with (1) and will be discussed in section (II). In section (III) these results are used to study the Askey-Wilson polynomials. These polynomials contain the q-analogs of classical polynomials when $|q| < 1$.

When the interval of orthogonality is infinite we will assume that the sequences $\{a_n\}$ and $\{b_n\}$ are regularly or slowly varying functions of n. That is we suppose there exists an increasing positive sequence $\{\lambda_n\}, n \ge 0$ such that

$$\lim_{n\to\infty} \frac{a_n}{\lambda_n} = a > 0, \quad \lim_{n\to\infty} \frac{b_n}{\lambda_n} = b \in \mathbb{R}, \tag{4}$$

with

$$\lim_{n\to\infty} n\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \alpha \ge 0. \tag{5}$$

Using a discrete analog of the Liouville-Green method for differential equations, we obtain (in section IV) strong asymptotics away from the real line for polynomials whose recurrence coefficients satisfy (4) and (5).

2. Bounded Interval of Orthogonality

We begin by assuming that (3) holds and compare this system with that associated with the Chebyshev polynomials of the second kind. For the Chebyshev polynomials we find $a(n) = 1/2$, $b(n) = 0$ for all n, and $p_n(x) = \frac{z^{n+1} - z^{-(n+1)}}{(z - z^{-1})} \equiv U_n(x)$. Here x and z are related as above. After suitable manipulation of (1) ([6], [11], [13]) the following formula can be obtained for $p_n(x)$

$$\frac{z^n p_n(x)}{\alpha(n)} = z^n U_n(x) + \sum_{i=0}^{n-1} \Big\{ (1 - 4a(i+1)^2) z^2 z^{n-i-1} U_{n-i-1}(x) - 2b(i) z z^{n-i} U_{n-i}(x) \Big\} z^i \frac{p_i(x)}{\alpha(i)}. \tag{6}$$

Here $\alpha(n) = \prod_{i=1}^{n} \frac{1}{2a(i)}$, $\alpha(0) = 1$. If we now use the discrete Gronwall inequality ([13]) we obtain the bound

$$\left\| \frac{z^n p(x,n)}{\alpha(n)} \right\|_\nu \leq (n+1)\nu(2n+2) \exp \sum_{i=1}^{n} i\nu(2i) \left\{ |1 - 4a(i)^2| + |2b(i-1)| \right\}. \tag{7}$$

Consider the function

$$\frac{\psi_n(z)}{\alpha(n)} = 1 + \sum_{i=0}^{n-1} \{ (1 - 4a(i+1)^2) z^2 - 2b(i) z \} \frac{z^i p(x,i)}{\alpha(i)}.$$

From (7) it is easy to see that if (3) holds then the sequence $\{\psi_n\}$ is a Cauchy sequence in A_v hence there is a function f_+ such that $\lim_{n\to\infty} \|\psi_n - 2zf_+\|\nu = 0$. Since the maximal ideal space for A_v is $1/R < |z| < R$, and each ψ_n is a polynomial in z, zf_+ can be extended to be analytic for $|z| < R$. The function f_+ is the discrete analog of the Jost function ([12] p. 339). We now introduce two other useful solutions of equation (1) . Let $p_n^+(z)$ and $p_n^-(z)$ be solutions of (1) satisfying the boundary conditions

$$\lim_{n\to\infty} |z^{-n} p_+(z,n) - 1| = 0, \qquad |z| \leq 1,$$

and

$$\lim_{n\to\infty} |z^n p_-(z,n) - 1| = 0, \qquad |z| \geq 1.$$

Again comparing $p_+(z,n)$ against z^n which is associated with the Chebyshev system we find the following formula ([6])

$$\frac{p_+(z,n)}{\gamma(n+1)} = z^n + \sum_{i=n+1}^{\infty} \sum_{m=i}^{\infty} \left\{ \prod_{j=i+1}^{m} (2a(j))^2 \right\} \times \left\{ (1 - 4a(m+1)^2) z - 2b(m) \right\} z^{m-2i+n+1} \frac{p_+(z,m)}{\gamma(m+1)},$$

with $\gamma(n) = \prod_{i=n}^{\infty} \frac{1}{2a(i)}$. The following bound on $p_+(z,n)$ can be obtained using the discrete Gronwall inequality,

$$\left\| \frac{z^{-n} p_+(z,n)}{\gamma(n+1)} \right\|_\nu \leq \exp \left[C \sum_{m=n+1}^{\infty} m\nu(2m) \left\{ |1 - 4a(m+1)^2| + |2b(m)| \right\} \right].$$

Using the above two equations yield

$$\left\| \frac{z^{-n} p_+(z,n)}{\gamma(n+1)} - 1 \right\|_\nu \leq D \sum_{m=n+1}^{\infty} m\nu(2m) \left\{ |1 - 4a(m+1)^2| + |2b(m)| \right\},$$

where D is a constant independent of z and n. Since $p_+(z,n)$ and $p_-(z,n)$ satisfy (1) by examining the boundary conditions they satisfy we see that $p_-(z,n) = p_+(1/z,n)$ for $\frac{1}{R} \leq |z| \leq |R|$. The above considerations lead to,

Lemma 1. [5] *If* (3) *holds then $z^{-n}p_+(z,n)$ and $z^n p_-(z,n)$ are elements of A_v, $z^{-n}p_+(z,n)$ is analytic for $|z| < R$ and $z^n p_-(z,n)$ is analytic for $|z| > 1/R$. Furthermore $p_+(z,n)$ and $p_-(z,n)$ are linearly independent solutions of* (1) *except at $z = \pm 1$*

Since $p_+(z,n)$ and $p_-(z,n)$ are linearly independent it is not difficult to show that

$$(z - 1/z) p_n(x) = f_+(1/z) p_+(z,n) - f_+(z) p_-(z,n), \quad \frac{1}{R} \leq |z| \leq R.$$

Thus for $x \subset [-1,1], x = \cos\theta$ we find the following asymptotic formula

$$\begin{aligned} \sin\theta p_n(\cos\theta) = {} & 2|f_+(e^{i\theta})| \gamma(n+1) \{ \sin((n+1)\theta - \arg e^{i\theta} f_+(e^{i\theta})) \\ & + \sum_{i=1}^{\ell} \alpha(n,i) \sin((n+i+1)\theta - \arg e^{i\theta} f_+(e^{i\theta})) \} \\ & + O\left(\sum_{m=[\ell/2]}^{\infty} (2m - \ell + 1) \{ |1 - 4a(m+n+1)^2| + 2|b(m+n)| \} \right). \end{aligned} \tag{8}$$

Here $\alpha(n,i)$ is the ith Fourier coefficient of $z^{-n}p_+(z,n)$.

If ρ is the measure with respect to which the polynomials $p_n(x)$ are orthonormal then ([6], [11]),

$$d\rho(x) = \begin{cases} \sigma(\theta)dx, & x = \cos\theta, \quad 0 \leq \theta \leq \pi, \\ \sum_{i=1}^{N} \rho_i \delta(x - x_i) dx, & x \text{ not as above}, \quad N < \infty, \end{cases}$$

with

$$\sigma(\theta)dx = \frac{\sin\theta}{2\pi |f_+(z)|^2} dx, \qquad x = \cos\theta, \quad z = e^{i\theta},$$

and

$$\rho_i = \frac{p_+(z_i, 0)}{f_+'(x_i)}, \qquad x_i = \frac{(z_i + 1/z_i)}{2}.$$

With the above relations the work of Baxter [2,3] and the Weiner-Levy Theorem may be used to obtain necessary and sufficient conditions for a certain class of measures relating (3) to the decay of the Fourier coefficients of σ. For instance ([5])

Theorem 1. *Let $\rho(x)$ be a bounded nondecreasing, absolutely continuous function on $[-1,1]$ with*

$$d\rho(x) = \sigma(\theta)dx, \qquad x = \cos\theta, \qquad 0 \le \theta \le \pi.$$

Furthermore, let $\frac{\sigma(\theta)}{\sin\theta} = \frac{\sigma(-\theta)}{\sin(-\theta)}$ and $\ln\frac{\sigma(z)}{z-1/z} \in A_\nu$ with $R > 1$. Then

$$\sum_{n=1}^{\infty} n\nu(2n)\{|1-4a(n)^2| + |b(n-1)|\} < \infty,$$

if and only if

$$\sum_{n=1}^{\infty} n\nu(n)|g(n)| < \infty.$$

Here $\frac{\sigma(z)}{z-1/z} = \sum_{m=-\infty}^{\infty} g(m)z^m$, $1/R \le |z| \le R$, with $g(m) = g(-m)$.

3. The Askey - Wilson Polynomials

The results from the previous section can be applied to the Askey-Wilson polynomials ([1],[9],[10]). In this case $a_n^2 = \frac{1}{4}A_{n-1}C_n$, $n = 1,2,\ldots$ and $b_n = \frac{1}{2}(a + 1/a - A_n - C_n)$, $n = 0,1,2,\ldots$ where

$$A_n = \frac{a^{-1}(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{(1-abcdq^{2n-1}(1-abcdq^{2n-2})}, \qquad n = 0,1,2,\ldots$$

and

$$C_n = \frac{a(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})(1-q^n)}{(1-abcdq^{2n-1}(1-abcdq^{2n-2})}, \qquad n = 0,1,2,\ldots$$

where a, b, c, d are chosen so that A_n and C_n are real and $A_{n-1}C_n > 0$ and $q < 1$. In this case $1 - 4a_n^2 = O(q^n)$ and the same is true of b_n and the results of the above section may be applied with $R = 1/q$. The function f_+ has been computed and found to be [1],

$$zf_+(z) = \frac{(az,q)_\infty(bz,q)_\infty(cz,q)_\infty(dz,q)_\infty}{(z^2q:q)_\infty}k(q),$$

where

$$k(q)^2 = \frac{1}{4}\frac{(abcd:q)_\infty}{(ab:q)_\infty(ac:q)_\infty(ad:q)_\infty(bc:q)_\infty(bd:q)_\infty(cd:q)_\infty(q:q)_\infty}$$

and

$$(a:q)_k = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & k \ge 1, \\ 1, & k = 0,. \end{cases}$$

If we substitute the above formulas into equation (8) we obtain an asymptotic formula for $p_n(\cos\theta)$ with an error term $O(\frac{q^n}{n})$. This also leads to an asymptotic formula for the zeros of $p_n(\cos\theta)$. From (8)we find $\theta = \frac{k\pi}{n+1} + \frac{1}{n+1}\arg(e^{i\theta}f_+(e^{i\theta})) + O(\frac{q^n}{n})$. Successive

iteration of this equation beginning with $\theta_0 = \frac{k\pi}{n+1}$ yields improving estimates for θ. To see this suppose for convenience that a, b, c, and d are real and less than one in magnitude. Then with $\theta_0 = \pi\frac{n-m}{n+1} = \pi(1 - \frac{m+1}{n+1})$ we find $\arg(1 - ae^{i\theta_0}q^k) = -a\pi\frac{m+1}{n+1}\frac{q^k}{1+aq^k} + O(\frac{q^k}{n^3})$. Consequently $\arg(ae^{i\theta_0}, q)_\infty = -a\pi\frac{m+1}{n+1}\sum_{k=0}^{n}\frac{q^k}{1+aq^k} + O(\max(\frac{1}{n^3}, \frac{q^n}{n}))$. An application of the Euler–Macluran formula gives

$$\sum_{k=0}^{n}\frac{aq^k}{1+aq^k} = \int_0^n \frac{aq^x}{1+aq^x}dx + c(a,q) - \frac{1}{2}\int_n^\infty [B_2 - B_2(x-[x])]\frac{d^2}{dx^2}\frac{aq^x}{1+aq^x}dx,$$

where

$$c(a,q) = \sum_{k=0}^{\infty}\frac{aq^k}{1+aq^k} - \int_0^\infty \frac{aq^x}{1+aq^x}dx.$$

B_2 is the second Bernoulli number and $B_2(x)$ the second Bernoulli polynomial.Thus we find that

$$\sum_{k=0}^{n}\frac{aq^k}{1+aq^k} = \int_0^n \frac{ae^{x\ln q}}{1+ae^{x\ln q}}dx + c(a,q) + O\left(\int_n^\infty \frac{d^2}{dx^2}\frac{aq^x}{1+aq^x}dx\right),$$

which yields

$$= \frac{\ln(1+aq^n)}{\ln q} - \frac{\ln(1+a)}{\ln q} + c(a,q) + O\left(-\frac{aq^n\ln q}{(1+aq^n)^2}\right),$$
$$= -\frac{\ln(1+a)}{\ln q} + c(a,q) + O(q^n).$$

Consequently

$$\arg(ae^{i\theta_0}, q)_\infty = \left(\frac{\ln(1+a)}{\ln q} - c(a,q)\right)\pi\frac{m+1}{n+1} + O\left(\frac{q^n}{n}, \frac{1}{n^3}\right).$$

Applying similar methods to the other factors in the formula for $zf_+(z)$ yields

$$\arg e^{i\theta_0}f_+(e^{i\theta_0}) = \pi\Big[\frac{\ln[(1+a)(1+b)(1+c)(1+d)] - 2\ln(1+q)}{\ln q} - c(a,q) - c(b,q)$$
$$- c(c,q) - c(d,q) + 2c(q,q)\Big]\frac{m+1}{n+1} + O\left(\max\left(\frac{q^n}{n}, \frac{1}{n^3}\right)\right)$$

This yields ([5])

Theorem 2. *Suppose $a, b, c,$ and d are all real and have magnitudes less than one. Then the zeros of $p_n(\cos\theta)$ are given by,*

$$\theta = \pi\frac{(n-m)}{n+1} + \pi\Big[\frac{\ln[(1+a)(1+b)(1+c)(1+d)] - 2\ln(1+q)}{\ln q} - c(a,q) - c(b,q)$$
$$- c(c,q) - c(d,q) + 2c(q,q)\Big]\frac{m+1}{(n+1)^2} + O\left(\max\left(\frac{1}{n^4}, \frac{q^n}{n}\right)\right).$$

3. Unbounded Orthogonality Intervals

In order to consider the case when the interval of orthogonality is infinite we consider the case when when the sequences $\{a_n\}$ and $\{b_n\}$ were regularly or slowly varying functions (see (4) and (5)).

In this case we can no longer compare the solutions of (1) with the Chebyshev polynomials. Instead a a good comparison system turns out to be ([8])

$$w_+(z,n) = \prod_{i=1}^{n} u(z,i)$$

where

$$u(z,i) = u_0(z,i)\left(1 - \frac{[u_0(z,i) - (a_{i+1}/a_i)u_0(z,i+1)]}{u_0(z,i) - 1/u_0(z,i)}\right)^{-1},$$

with $u_0(z,i) = \frac{z-b_i}{2a_i} + \frac{1}{2}\sqrt{\left(\frac{z-b_i}{2a_i}\right)^2 - 4}$. We now apply a discrete version of the WKB or Liouville–Green method ([4],[8]) for differential equations to show that for $y \notin [D,E] =$ convex hull $(\{0\}, [b-2a, b+2a])$

$$\lim_{n\to\infty} \frac{p(\lambda_n y, n)}{w_+(\lambda_n y, n)} = 1.$$

The above convergence is uniform on compact subsets of $\mathbb{C}\backslash[D,E]$ [8]. Now using the fact that $\{b_n\}$ and $\{a_n\}$ are regularly varying sequences we find ([8]),

Theorem 3. *If $\alpha > 0$ and*

$$\lim_{n\to\infty} n\frac{(a_{n+1}-a_n)}{\lambda_n} = a\alpha, \quad \lim_{n\to\infty} n\frac{(b_{n+1}-b_n)}{\lambda_n} = b\alpha,$$

then

$$\lim_{n\to\infty} \frac{p_n(\lambda_n y)}{\prod_{i=1}^{n} u_0(\lambda_n y, i)} = \left\{\frac{(x-b)^2 - 4a^2}{x^2}\right\}^{-1/4} \exp\left\{\frac{b}{2}\int_0^1 \frac{ds}{\sqrt{(x-bs)^2 - 4a^2s^2}}\right\},$$

uniformly on compact subsets of $\mathbb{C}\setminus[D,E]$. Here

$$[D,E] = \text{convex hull}(\{0\}, [b-2a, b+2a]),$$

and

$$u_0(x,i) = \frac{x-b_i}{2a_i} + \sqrt{\left(\frac{x-b_i}{2a_i}\right)^2 - 1}.$$

This result can also been extended to the case when $\alpha = 0$ if

$$\frac{1}{\lambda_n}\sum_{i=1}^{n} |a_{i+2} - 2a_{i+1} + a_i| = o(1) = \frac{1}{\lambda_n}\sum_{i=1}^{n} |b_{i+2} - 2b_{i+1} + b_i|,$$

and

$$\limsup\left\{\max_{1\le k\le n} \frac{a_k}{a_n}\right\} = 1 = \limsup\left\{\max_{0\le k\le n} \frac{b_k}{b_n}\right\}, \qquad b \neq 0.$$

These results can be applied to the case when $a_n = an^\alpha, n > 0$, and $b_n = bn^\alpha, n \ge 0$ to obtain ([7],[13])

$$\lim_{n\to\infty} \frac{(2\pi n)^{\alpha/2} p_n(n^\alpha y)}{u^n H(n)} = \frac{\sqrt{au(1)}}{((y-b)^2 - 4a^2)^{1/4}} \exp\left(\frac{b}{2}\int_0^1 \frac{dz}{\sqrt{(y-bz)^2 - 4a^2z^{2\alpha}}}\right),$$

where

$$H(n) = \exp\left(ny\alpha \int_0^1 \frac{1}{\sqrt{(y-bz^\alpha)^2 - 4a^2z^{2\alpha}}}\,dz\right),$$

and $u = \frac{y-b}{2a} + \sqrt{\left(\frac{y-b}{2a}\right)^2 - 1}$.

References

1. Richard Askey and J. Wilson, "Some Basic Hypergeometric Orthogonal Polynomials that Generalize Jacobi Polynomials," *Memoirs Amer. Math. Soc.*, 319, (1985).

2. G. Baxter, "A Convergence Equivalence Related to Polynomials Orthogonal on the Unit Circle," *Trans. Amer. Math. Soc.* **99** (1961), 471–487.

3. G. Baxter, "A Norm Inequality for a Finite–Section Wiener–Hopf Equation," *Ill. J. Math.* **7** (1963), 97–103.

4. P. A. Braun, "WKB method for three- term recurrence relations and quasienergies of an anharmonic oscillator",*Theoret. and Math. Phy.***37**(1978),1070-1081.

5. J. S. Geronimo, "Scattering Theory, Orthogonal Polynomials and q-Series", accepted SIAM J. Math. Anal.

6. J. S. Geronimo and K. M. Case, "Scattering Theory and Polynomials Orthogonal on the Unit Circle," *Trans. Amer. Math. Soc.* **258** (1980), 467–494.

7. J. S. Geronimo, D. T. Smith and W. Van Assche "Strong Asymptotics for Orthogonal Polynomials with Regularly and Slowly Varying Recurrence Coefficients",*J. Approx. Theory* **72**(1993), 141–158.

8. J. S. Geronimo, D. T. Smith, "WKB (Liouville-Green) analysis of second order difference equations and applications",*J. Approx. Theory* **69**(1992),269-301.

9. M. E. H. Ismail and J. A. Wilson, "Asymptotic and Generating Relations for the q–Jacobi and ${}_4\phi_3$ Polynomials," *J. Approx. Theory* **36** (1982), 43–54.

10. M. E. H. Ismail and M. Rahman, "The Associated Askey–Wilson Polynomials," *TAMS* **328** (1991), 201–237.

11. P. G. Nevai, "Orthogonal Polynomials", *Memoirs of the Amer. Math. Soc.* **18** (1979).

12. R. G. Newton, "Scattering Theory of Waves and Particles", McGraw-Hill, NY, 1966.

13. W. Van Assche, "Asymptotics for Orthogonal Polynomials and Three–Term Recurrences" in *Orthogonal Polynomials: Theory and Practice* (P. Nevai, ed.), NATO ASI Series, Kluwer Acedemic, Dordrecht (1990), 435–462.

14. W. Van Assche, J. S. Geronimo, "Asymptotics for orthogonal polynomialswith regularly varying recurrence coefficients",*Rocky Mountain J. Math.* **19** (1989), 39–49.

Asymptotics of Linear Recurrences with Rational Coefficients

*Xavier Gourdon**
`Xavier.Gourdon@inria.fr`

*Bruno Salvy**
`Bruno.Salvy@inria.fr`

Abstract

We give algorithms to compute the asymptotic expansion of solutions of linear recurrences with rational coefficients and rational initial conditions in polynomial time in the order of the recurrence.

Introduction

We investigate sequences defined by a recurrence of the form

$$a_k u_{n+k} + a_{k-1} u_{n+k-1} + \cdots + a_0 u_n = 0, \tag{1}$$

where the coefficients a_k and the initial conditions belong to $\mathbb{Q}$. This is probably the most simple type of recurrence one may encounter. Recurrences of this type are ubiquitous in many fields of applications (see [3] for numerous examples and references). Among the approximately 2300 sequences listed in Sloane's book [18], one can estimate that about 13% are of this type [13]. In the rest of this paper "linear recurrence" always means "linear recurrence with rational coefficients" and we shall refer to u_n as a "linear recurrent sequence".

Surprisingly, some problems related to linear recurrences remain open, and specially problems related to effectivity. Our aim in this paper is to describe an algorithm that computes an asymptotic expansion of a sequence obeying (1) in polynomial time in the order k of the recurrence. It is quite simple to find the asymptotic expansion of Fibonacci numbers with traditional tools, but these tools break down when the order of the recurrence gets large. The algorithm we describe works without any limitation on the value of k or those of the coefficients.

Given a recurrence such as (1), one usually computes its general term as a sum of *exponential polynomials* of the form $\sum_{k=0}^{N} p_k n^k \lambda^n$, where λ is an algebraic number. In Section 1 we shall describe an algorithm computing the coefficients p_k *without factoring any polynomial.* This general term does not solve the problem of asymptotic behaviour. To form a proper asymptotic expansion one has to order the moduli of the algebraic numbers λ occurring in the general terms. The problem which will occupy most of this paper is: How can one perform such an ordering *exactly*, i.e. we prove that the algorithms we propose work on the whole class of recurrences (1). We shall use techniques from computer algebra to free ourselves from problems of ill-conditioning related to the use of floating-point values. The result is an algorithm which, given a positive integer p and a linear recurrence (1) together with its initial conditions—or equivalently a rational function in $\mathbb{Q}(x)$ (see below)—outputs the p first exponential polynomials of the asymptotic expansion of the solution u_n of (1) as n tends to infinity.

We describe two essentially different decision procedures to compute this asymptotic expansion. The first approach, purely algebraic, completely avoids factorizations. It is made expensive by the increase of degrees due to resultant computations. Currently this is the most natural computer

*Algorithms Project, INRIA, 78153 Le Chesnay Cedex, France.