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THE CONJECTURE OF STANLEY FOR SYMMETRIC MAGIC SQUARES

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In this paper we announce a proof of Stanley's conjecture for symmetric magic squares (see [Sta76], [Sta83, p.40] and [Sta86, p.262]). The solution of the conjecture is a nice application of multivariate spline theory to combinatorics.

1. Introduction

An $m \times m$ matrix with non-negative integer entries is called *a magic r-square of order m* if every row and column sums to $r \in \mathbb{N}$ where $\mathbb{N}$ is the set of non-negative integers. Let $H_m(r)$ denote the number of all magic r-squares of order m. For instance, $H_1(r) = 1$ and $H_2(r) = r + 1$. It seems that MacMahon [Mac15, §407] first computed $H_3(r)$:

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

Guided by this evidence, Anand, Dumir and Gupta [ADG66] conjectured that $H_m(r)$ is a polynomial in r of degree $(m-1)^2$. Their conjecture was confirmed by Stanley in [Sta73].

An $m \times m$ *symmetric* matrix with non-negative integer entries is called *a symmetric magic r-square of order m* if every row (and hence every column) sums to r. Let $S_m(r)$ denote the number of all symmetric magic r-squares of order m. Carlitz [Car66] calculated $S_m(r)$ for $m \le 4$ and found that $S_m(r)$ are not polynomials in r for $m = 3$ and 4; rather, $S_m(2r)$ and $S_m(2r+1)$ are polynomials in r ($m = 3$ and 4). He conjectured that this is the case for all m. His conjecture was solved by Stanley in [Sta73]. Later, Stanley [Sta76] obtained the following result.

Theorem 1. *Let $m \ge 1$, and let $S_m(r)$ be the number of symmetric magic r-squares of order m. Then $S_m(r) = P_m(r) + (-1)^r Q_m(r)$ for all $r \in \mathbb{N}$, where $P_m(r)$ and $Q_m(r)$ are polynomials in r with $\deg P_m = \binom{m}{2}$. Moreover,*

$$\deg Q_m \le \binom{m-1}{2} - 1 \text{ if } m \text{ is odd}; \quad \deg Q_m \le \binom{m-2}{2} - 1 \text{ if } m \text{ is even}. \qquad (1.1)$$

In order to count $S_m(r)$ it is important to find the exact degree of Q_m. Stanley conjectured that equality holds for all m in (1.1). He supported his conjecture by computing $S_5(r)$ and found that the degree of Q_5 is 5. In [Sta83] and [Sta86] he raised this conjecture again and again.

Note that each function S_m is a *quasi-polynomial* (see [Sta86, p.210] for the definition). In his study of linear diophantine equations, Ehrhart introduced the so-called Ehrhart quasi-polynomials (see [Ehr77]). The relationship between symmetric magic squares and the Ehrhart quasi-polynomials of a certain type was discussed in [Sta86, pp.235–241].

We shall use *splines* (*piecewise polynomial functions*) to investigate this problem. This approach was initiated by Dahmen and Micchelli in [DM88] and is totally different from that of Stanley, who based his results on commutative algebra.

How are magic squares and symmetric magic squares related to multivariate splines? Magic squares and symmetric magic squares both are special cases of magic labelings of graphs (see [Ste66] and [Sta73]). Further, as indicated by Stanley [Sta73], the theory of magic labelings can be put into the more general context of linear diophantine equations. A study of linear diophantine equations naturally leads to truncated powers and discrete truncated powers, which are typical examples of splines.

We shall adopt the graph theoretic terminology used in [Wil85]. Thus *a graph G* is defined to be a pair (V, E), where V is a nonempty finite set of elements called *vertices*, and E is a multiset of unordered pairs of (not necessarily distinct) elements of V called *edges*. Note that this definition of graph permits the existence of loops and multiple edges.

Let $r \in \mathbb{N}$. According to Stanley [Sta73], *a magic labeling of G of index r* is an assignment $L : E \to \mathbb{N}$ of a non-negative integer label to each edge of G such that for each vertex v of G the sum of the labels of all edges incident to v is r (counting each loop at v once only). We denote by $H_G(r)$ the number of magic labelings of G of index r.

Let K_m denote the complete graph on m vertices, *i.e.*, the graph with m vertices in which every pair of distinct vertices are adjacent, and let G be the graph obtained from K_m by adding one loop to each of its vertices. If $(a_{ij})_{1\le i,j\le m}$ is a symmetric magic r-square of order m, then we assign a_{ij} $(i \neq j)$ to the edge of G joining vertex i with vertex j and assign a_{ii} to the loop around vertex i. This gives a magic labeling of G of index r. In this way we establish a one-to-one correspondence between the magic labelings of G of index r and the symmetric magic r-squares of order m, and therefore $S_m(r) = H_G(r)$.

Let G be an arbitrary graph. Suppose the vertices of G are numbered $\{1, 2, \dots, m\}$ and its edges are numbered $\{1, 2, \dots, n\}$. The *incidence matrix* M of G is the $m \times n$ matrix whose (i, j)th entry is 1 if vertex i is incident to edge j, and 0 otherwise. Suppose $L : E \to \mathbb{N}$ is a mapping assigning a label $\beta_j \in \mathbb{N}$ to edge j $(j = 1, \dots, n)$. Let β be the column n-vector whose jth coordinate is β_j, $j = 1, \dots, n$. Then L is a magic labeling of index r if and only if β satisfies the system of linear diophantine equations

$$M\beta = re,$$

where e is the column m-vector whose coordinates are all 1.

In general, a system of linear diophantine equations is of the form

$$M\beta = \alpha, \tag{1.2}$$

where M is an $m \times n$ integer matrix, α is an integer column m-vector and one seeks solutions β in $\mathbb{Z}^n$, the set of integer column n-vectors. For a given $\alpha \in \mathbb{Z}^m$, we denote by $t(\alpha|M)$ the number of *non-negative integer solutions* $\beta \in \mathbb{N}^n$ to the system (1.2) of linear diophantine equations. Thus, from the above discussion we see that $H_G(r) = t(re|M)$.

The function $t(\cdot|M) : \alpha \mapsto t(\alpha|M)$ $(\alpha \in \mathbb{Z}^m)$ was introduced by Dahmen and Micchelli in [DM83]. They pointed out that $t(\cdot|M)$ is the discrete counterpart of *the (multivariate) truncated power* introduced by Dahmen in [Dah80]. Following their lead, we call $t(\cdot|M)$ *the discrete truncated power associated with M*. It was Dahmen and Micchelli [DM88] who first revealed the close relationship between linear diophantine equations and discrete truncated powers. Thus the theory of multivariate splines developed in the past decade can be applied to linear diophantine equations.

2. Discrete Truncated Powers

Let M be an $m \times n$ integer matrix. The columns of M are integer vectors in $\mathbb{R}^m$, the m-dimensional real linear space. We use the same letter M to denote *the multiset of the column vectors of M*. Throughout this section we assume that M spans $\mathbb{R}^m$ and the convex hull of M does not contain the origin. The latter condition guarantees that $t(\alpha|M)$ is finite for every $\alpha \in \mathbb{Z}^m$. For a multiset Y of elements of $\mathbb{R}^m$, we denote by $\#Y$ the number of elements of Y, by $\mathrm{span}(Y)$ the subspace spanned by the vectors in Y, and by $\mathrm{cone}(Y)$ the cone $\{\sum_{y\in Y} a_y y : a_y \ge 0 \text{ for all } y\}$. Note that if (1.2) has a solution $\beta \in \mathbb{N}^n$, then α must lie in $\mathrm{cone}(M)$. In other words, $t(\alpha|M) = 0$ for $\alpha \notin \mathrm{cone}(M)$.

Let S denote the linear space of all mappings from $\mathbb{Z}^m$ to the complex field $\mathbb{C}$. An element of S is called *an sequence*. Given $y \in \mathbb{Z}^m$, the backward difference operator ∇_y is defined by the rule

$$\nabla_y f := f - f(\cdot - y), \quad f \in S.$$

The following difference formula was given in [DM88]:

$$\nabla_y t(\cdot|M) = t(\cdot|M\backslash y) \quad \text{for } y \in M, \tag{2.1}$$

and this can be proved by a simple combinatorial argument. It follows from (2.1) that

$$\nabla_Y t(\cdot|M) = t(\cdot|M\backslash Y) \quad \text{for } Y \subseteq M \tag{2.2}$$

where $\nabla_Y := \prod_{y\in Y} \nabla_y$.

The difference formula (2.2) motivates us to define two sets. The first is the set $\mathcal{Y}(M)$ consisting of those submultisets Y of M for which $M\backslash Y$ does *not* span $\mathbb{R}^m$. The second set $c(M)$ is the union of span$(M\backslash Y)$ where Y runs over $\mathcal{Y}(M)$. A connected component of cone$(M)\backslash c(M)$, according to [DM88], is called *an M-cone*. Let Ω be an M-cone. Then for any $Y \in \mathcal{Y}(M)$ and $\alpha \in \Omega -$ cone$(M\backslash Y)$, we have $\alpha \notin$ cone$(M\backslash Y)$, because Ω is disjoint from cone$(M\backslash Y)$, and therefore $t(\alpha|M\backslash Y) = 0$. This and (2.2) together yield

$$\nabla_Y t(\alpha|M) = 0 \quad \text{for } \alpha \in \mathbb{Z}^m \cap (\Omega - \text{cone}(M\backslash Y)).$$

Thus we are led to the following system of linear partial difference equations for $f \in S$:

$$\nabla_Y f = 0 \quad \text{for all } Y \in \mathcal{Y}(M).$$

The solutions to this system of difference equations form a linear subspace of S, which we shall denote by $\nabla(M)$. The space $\nabla(M)$ was first introduced by Dahmen and Micchelli in [DM85]. The following theorem is a modification of [DM88, Theorem 3.1], which is important to our study of discrete truncated powers. Also see [Jia93a].

Theorem 2. *For any M-cone Ω, there exists a unique element $f_\Omega \in \nabla(M)$ such that f_Ω agrees with $t(\cdot|M)$ on $\mathbb{Z}^m \cap \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω.*

Given $\theta = (\theta_1, \ldots, \theta_m) \in (\mathbb{C}\backslash\{0\})^m$, we denote by $\theta^{()}$ the sequence defined by

$$\alpha \mapsto \theta^\alpha = \theta_1^{\alpha_1} \cdots \theta_m^{\alpha_m} \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m.$$

For an integer vector $y \in \mathbb{Z}^m$, $\nabla_y \theta^{()} = 0$ if and only if $\theta^y = 1$. This motivates us to consider the multiset

$$M_\theta := \{\, y \in M : \theta^y = 1 \,\}. \tag{2.3}$$

It is easily seen that $\theta^{()} \in \nabla(M)$ if and only if M_θ spans $\mathbb{R}^m$. Let

$$A(M) := \{\, \theta \in (\mathbb{C}\backslash\{0\})^m : \text{span}(M_\theta) = \mathbb{R}^m \,\}. \tag{2.4}$$

The structure of $\nabla(M)$ was clarified by Dahmen and Micchelli in [DM85] as follows.

Theorem 3. *A sequence $f \in \nabla(M)$ if and only if it has the form*

$$f(\alpha) = \sum_{\theta\in A(M)} \theta^\alpha p_\theta(\alpha), \quad \alpha \in \mathbb{Z}^m$$

where p_θ is some polynomial in m variables of degree $\le \#M_\theta - m$ for each $\theta \in A(M)$.

If M is the incidence matrix of a graph G, then the set $A(M)$ as defined in (2.4) has nice properties as shown in the following theorem (see [Jia93b]).

Theorem 4. *Let G be a graph with m vertices and let M be its incidence matrix. If the column vectors of M spans $\mathbb{R}^m$, then*

$$\theta = (\theta_1, \ldots, \theta_m) \in A(M) \implies \theta_j = 1 \text{ or } -1 \quad \text{for all } j = 1, \ldots, m.$$

The proof of this theorem is based on the observation that there is a one-to-one correspondence between the edges of G and the columns of M. Let e_i denote the ith column of the $m \times m$ identity matrix and let $e_{ij} = e_i + e_j$, $i, j = 1, \ldots, m$. Suppose the vertices of G are labeled as $v_1, \ldots, v_m$. Then a loop around v_i corresponds to e_i, and an edge joining v_i with v_j corresponds to e_{ij}. From the definition (2.3) of M_θ we find that $e_i \in M_\theta$ implies $\theta_i = 1$, while $e_{ij} \in M_\theta$ implies $\theta_i\theta_j = 1$. Let G_θ be the subgraph of G which consists of all vertices of G and all the edges of G corresponding to the column vectors of M_θ. Then the incidence matrix of G_θ is M_θ. From the above discussion we see that if G_θ contains a loop around v_i, then $\theta_i = 1$, and if G_θ contain an edge joining v_i with v_j, then $\theta_i\theta_j = 1$, *i.e.*, $\theta_j = \theta_i^{-1}$. Furthermore, if there is a path in G_θ of length k from v_i to v_j, then

$$\theta_j = \begin{cases} \theta_i, & \text{if } k \text{ is even;} \\ \theta_i^{-1}, & \text{if } k \text{ is odd.} \end{cases} \tag{2.5}$$

It is easily seen that the column vectors of M spans $\mathbb{R}^m$ if and only if any connected component of G is not bipartite. Let K be a connected component of G_θ. Since M_θ spans $\mathbb{R}^m$, K in not bipartite; hence K contains a circuit of length k with k being an odd integer. This circuit passes through a vertex, say v_i. Then by (2.5) we have $\theta_i = \theta_i^{-1}$, since k is odd. It follows that $\theta_i = 1$ or -1. Let v_j be an arbitrary vertex in K. Since K is connected, there is a path in K from v_i to v_j. By (2.5) we have $\theta_j = \theta_i$ or $\theta_j = \theta_i^{-1}$. This shows that $\theta_j = 1$ or -1 for any vertex v_j in K. Evidently, this conclusion is valid for any vertex in G_θ.

3. Symmetric Magic Squares

Theorem 5. *Equality holds in (1.1) for all m.*

Sketch of Proof. The cases $m = 1$ and $m = 2$ are trivial. In what follows we assume that $m \geq 3$. Let G be the complete m-graph with a loop attached to each of its vertices, and let M be its incidence matrix. Then $\#M = n := \binom{m+1}{2}$ and

$$S_m(r) = H_G(r) = t(re|M), \quad r \in \mathbb{N}. \tag{3.1}$$

Obviously, e lies in cone(M), hence there exists an M-cone Ω such that $\overline{\Omega}$ contains e. By Theorem 2, one can find an element $f_\Omega \in \nabla(M)$ such that f_Ω agrees with $t(\cdot|M)$ on $\overline{\Omega}$. Since $\overline{\Omega}$ contains re for all $r \geq 0$, we have

$$t(re|M) = f_\Omega(re), \quad r \in \mathbb{N}. \tag{3.2}$$

By Theorem 3, f_Ω has a decomposition of the form:

$$f_\Omega(\alpha) = \sum_{\theta \in A(M)} \theta^\alpha p_\theta(\alpha), \quad \alpha \in \mathbb{Z}^m \tag{3.3}$$

where p_θ is a polynomial of degree $\leq \#M_\theta - m$ for each $\theta \in A(M)$. By Theorem 4, $\theta \in A(M)$ implies that all the coordinates of θ are either 1 or -1. Let $A_+(M)$ denote the set of those elements of $A(M)$ which have an even number of negative coordinates and let $A_-(M) := A(M) \backslash A_+(M)$. It follows from (3.1)–(3.3) that

$$S_m(r) = P_m(r) + (-1)^r Q_m(r) \quad \text{for all } r \in \mathbb{N},$$

where

$$P_m(r) = \sum_{\theta \in A_+(M)} p_\theta(re) \quad \text{and} \quad Q_m(r) = \sum_{\theta \in A_-(M)} p_\theta(re). \tag{3.4}$$

Evidently, $e \in A_+(M)$ and $M_e = M$. By [DM88, Proposition 5.3], the leading part of p_e agrees with $T(\cdot|M)$ on Ω, where $T(\cdot|M)$ is *the truncated power associated with M*. From this fact we can prove that $p_e(re)$ is a polynomial in r of exact degree $\#M - m$ with a positive leading coefficient. Moreover, for any $\theta \in A_+(M) \backslash \{e\}$, it can be easily proved that $M_\theta \neq M$. Hence by Theorem 3 we have

$$\deg p_\theta \leq \#M_\theta - m < \#M - m \quad \text{for } \theta \in A_+(M) \backslash \{e\}.$$

This together with (3.4) shows that the exact degree of P_m is $n - m = \binom{m}{2}$.

Now let $\theta \in A_-(M)$ and consider $t(\cdot|M_\theta)$. By an argument similar to that used before we have

$$t(re|M_\theta) = P_\theta(r) + (-1)^r Q_\theta(r), \quad r \in \mathbb{N} \tag{3.5}$$

where

$$P_\theta(r) = \sum_{\xi \in A_+(M_\theta)} q_\xi(re) \quad \text{and} \quad Q_\theta(r) = \sum_{\xi \in A_-(M_\theta)} q_\xi(re) \tag{3.6}$$

and q_ξ is a polynomial of degree $\leq \#(M_\theta \cap M_\xi) - m$ for each $\xi \in A(M_\theta)$. Evidently, $e \in A_+(M_\theta)$ and $\theta \in A_-(M_\theta)$. From (3.6) we can prove that $q_e(re)$ and $P_\theta(r)$ have the same leading term, and $q_\theta(re)$ and $Q_\theta(r)$ have the same leading term. Since θ has an odd number of negative coordinates, we can show that $t(re|M_\theta) = 0$ for odd $r \in \mathbb{N}$, and hence $P_\theta = Q_\theta$ by (3.5). Thus $q_\theta(re)$ and $q_e(re)$ have the same leading term. But $q_e(re)$ is a polynomial in r of exact degree $\#M_\theta - m$ with a positive leading coefficient, and therefore so is $q_\theta(re)$. On the other hand, $t(\cdot|M_\theta) = \nabla_{M \backslash M_\theta} t(\cdot|M)$ by (2.2). This formula relates p_θ with q_θ, so we can prove that for each $\theta \in A_-(M)$, $p_\theta(re)$ is a polynomial in r of exact degree $\#M_\theta - m$ with a positive leading coefficient. Thus by (3.4) we have

$$\deg Q_m = \max\{\#M_\theta : \theta \in A_-(M)\} - m.$$

If m is odd, then the maximum of $\#M_\theta$ for $\theta \in A_-(M)$ is achieved when $\theta = -e$. In this case, $\#M_{-e} = \binom{m}{2}$. If m is even, then the maximum of $\#M_\theta$ for $\theta \in A_-(M)$ is achieved when all but one of the coordinates of θ are -1. In this case, the maximum of $\#M_\theta$ for $\theta \in A_-(M)$ is $\binom{m-1}{2} + 1$. This shows that equality holds in (1.1) for all m.

The details of the proof of Theorem 5 will appear in [Jia93b].

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Non-crossing two-rowed arrays and summations for Schur functions

(Summary)

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Abstract. In the first part of this paper (sections 1,2) we give combinatorial proofs for determinantal formulas for sums of Schur functions "in a strip" that were originally obtained by Gessel, respectively Goulden, using algebraic methods. The combinatorial analysis involves certain families of two-rowed arrays, asymmetric variations of Sagan and Stanley's skew Knuth-correspondence, and variations of one of Burge's correspondences. In the third section we specialize the parameters in these determinants to compute norm generating functions for tableaux in a strip. In case we can get rid of the determinant we obtain multifold summations that are basic hypergeometric series for A_r and C_r respectively. In some cases these sums can be evaluated. Thus in particular, an alternative proof for refinements of the Bender-Knuth and MacMahon (ex-)Conjectures, which were first obtained in another paper by the author, is provided. Although there are some parallels with the original proof, perhaps this proof is easier accessible. Finally, in section 4, we record further applications of our methods to the enumeration of paths with respect to weighted turns.

1. Generating functions for non-crossing two-rowed arrays. We consider two-rowed arrays $P = (p \mid q)$ of the form

$$\begin{array}{cccccccccc} p_{-a} & p_{-a+1} & \dots & p_{-1} & p_1 & \dots & p_k & & & \\ & & & & q_1 & \dots & q_k & q_{-1} & \dots & q_{-b+1} & q_{-b} \end{array}, \tag{1.1}$$

where a, k, b are some nonnegative integers and where the entries p_i, q_i are positive integers such that both rows of the array are weakly increasing. (To be precise, if $k = 0$, i.e. the "middle part" of the array is empty, for a $c \le \min\{a, b\}$ we also allow the entries $p_{-1}, \dots, p_{-c}$ and $q_{-1}, \dots, q_{-c}$ to be "empty".) We say that P is *of the type* (a, b) and *of the shape* (a, k, b). If both rows of P are strictly increasing then we call P a *strict* two-rowed array. Given an array $P_1 = (p^{(1)} \mid q^{(1)})$ of the shape (a_1, k_1, b_1) and an array $P_2 = (p^{(2)} \mid q^{(2)})$ of the shape (a_2, k_2, b_2), we say that P_1 *dominates* (resp. *strictly dominates*) P_2 if the following three conditions hold:

(D1) $a_1 \le a_2$ and $p_l^{(1)} \le p_l^{(2)}$ (resp. $p_l^{(1)} < p_l^{(2)}$) for all $l = -1, -2, \dots, -\min\{a_1, a_2\}$. (By convention, these inequalities are also violated if $p_l^{(1)}$ should be an "empty" entry.)

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