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POWERS OF STAIRCASE SCHUR FUNCTIONS AND SYMMETRIC ANALOGUES OF BESSEL POLYNOMIALS

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Abstract. We present several identities involving staircase Schur functions. These identities are then interpreted in terms of a sequence of orthogonal polynomials in one variable x, with coefficients in the ring of symmetric functions. By an appropriate specialization these polynomials reduce to Bessel polynomials. This leads to a new determinantal expression for Bessel polynomials and suggests that their combinatorics might be linked to Young tableaux or shifted Young tableaux.

1 Introduction

Schur functions S_I and skew Schur functions $S_{I/J}$ are indexed by partitions I or skew partitions I/J, which are visualized graphically by a diagram of boxes. Several families of Schur functions associated with special diagrams are known to satisfy particular identities. Thus, Lascoux and Pragacz provided a determinantal expression of Schur functions and skew Schur functions in terms of ribbon functions, which generalizes the classical decomposition into hook functions given by Giambelli **[LP]**.

In this paper we shall be interested in Schur functions whose diagram is a staircase. In a recent article Goulden has provided a combinatorial proof of a formula expressing the square of such a staircase Schur function **[Go]**. We shall present a more general formula, expressing any power of a staircase Schur function.

An interpretation of these formulae in terms of polynomials is then given. Indeed, it is shown that the family of staircase Schur functions $S_{12\ldots n}(\mathbb{E}+x)$, $n \geq 0$, is a family of orthogonal polynomials in the variable x, which may be seen as a symmetric analogue of the family of Bessel polynomials. This provides a new determinantal expression for Bessel polynomials. It also shows that the powerful machinery of symmetric functions might be of some help in the study of this sequence of polynomials. Thus, using some classical identities for Schur Q-functions, Thibon and the author derived the computation of the linearization coefficients **[LT]**, which had been conjectured by Favreau **[Fa]**.

Finally, we show that our formulae are also linked to an old problem first studied by Borchardt **[Bo]** and Laguerre **[La]**, and further investigated by numerous authors including Pólya **[Po]** and Foulkes **[Fo]**. The problem is to express any symmetric polynomial of n indeterminates as a rational function of the power sums of odd degree. Set $\mathbb{X} = \{x_1, \ldots, x_n\}$. The expression given by Polya and Foulkes for the k^{th} elementary symmetric polynomial $\Lambda(\mathbb{X})$ is the coefficient of $(-x)^{n-k}$ in the ratio $S_{12\ldots n}(\mathbb{X}-x)/S_{12\ldots n-1}(\mathbb{X})$. We obtain a similar expression for the complete symmetric polynomial $S_k(\mathbb{X})$, and interpret this last formula in terms of symmetric Bessel polynomials.

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2 Schur functions and minors identities

In this section we shall define our notations and recall some basic facts about Schur functions and minors identities. General references are **[LS]**, **[Mc]** and **[Le]**.

Let $\mathbb{E}$ denote a set of indeterminates or *alphabet*. The *complete symmetric functions* $S_i(\mathbb{E})$ and the *elementary symmetric functions* $\Lambda_i(\mathbb{E})$ are defined by means of the generating series

$$\sum_i S_i(\mathbb{E})t^i = \prod_{e\in\mathbb{E}}(1-te)^{-1}\ , \tag{1}$$

$$\sum_i \Lambda_i(\mathbb{E})t^i = \prod_{e\in\mathbb{E}}(1+te)\ . \tag{2}$$

When there is no danger of confusion we shall omit the mention of the alphabet. For $I=(i_1,\dots,i_n)\in\mathbb{N}^n$, the *Schur function* S_I is defined by Jacobi–Trudi identity $S_I=\det[S_{i_l+l-k}]_{1\le k,l\le n}$. In other words, denoting by $\mathbf{S}$ the infinite Toeplitz matrix $[S_{j-i}]_{i,j\ge0}$ where $S_k=0$ for $k<0$, S_I is the minor of $\mathbf{S}$ taken on the lines $0,1,\dots,n-1$ and the columns $i_1, i_2+1,\dots,i_n+n-1$. More generally, given $J=(j_1,\dots,j_n)\in\mathbb{N}^n$ one defines the *skew Schur function* $S_{I/J}$ as the minor of $\mathbf{S}$ taken on the lines $j_1, j_2+1,\dots,j_n+n-1$ and the columns $i_1, i_2+1,\dots,i_n+n-1$, *i. e.* $S_{I/J}=\det[S_{i_l+l-j_k-k}]_{1\le k,l\le n}$. Note that this definition makes sense even if the sequences I and J are not arranged in increasing order. However, by permutation of columns and rows one can always assume that I and J are *partitions* i.e. weakly *increasing* sequences of nonnegative integers. A partition I is represented by a *diagram* of boxes like those of section 1, having i_1 boxes on its first row, i_2 boxes on its second row, etc.. $|I|=i_1+\dots+i_n$ is called the *weight* of I. The *skew diagram* associated with $S_{I/J}$ is the complement of the diagram J in the diagram I.

Schur functions may also be expressed in terms of elementary symmetric functions Λ_i. Indeed, defining by analogy the functions $\Lambda_{I/J}$ by $\Lambda_{I/J}=\det[\Lambda_{i_l+l-j_k-k}]_{1\le k,l\le n}$, one has the relation

$$\Lambda_{I/J}=S_{I^\sim/J^\sim}\ , \tag{3}$$

where $I^\sim$ denotes the *conjugate partition* of I i.e. the partition whose diagram is obtained by interchanging the rows and columns of the diagram of I.

Suppose $\mathbb{F}$ is a second alphabet and denote by $\mathbb{E}+\mathbb{F}$ the union of $\mathbb{E}$ and $\mathbb{F}$. It follows from (1), (2) that

$$S_n(\mathbb{E}+\mathbb{F}) = \sum_{0\le i\le n} S_i(\mathbb{E})S_{n-i}(\mathbb{F})\ , \tag{4}$$

$$\Lambda_n(\mathbb{E}+\mathbb{F}) = \sum_{0\le i\le n} \Lambda_i(\mathbb{E})\Lambda_{n-i}(\mathbb{F})\ . \tag{5}$$

We shall also define the difference $\mathbb{E}-\mathbb{F}$ by setting

$$\sum_i S_i(\mathbb{E}-\mathbb{F})t^i = \frac{\prod_{f\in\mathbb{F}}(1-tf)}{\prod_{e\in\mathbb{E}}(1-te)}\ ,\qquad \sum_i \Lambda_i(\mathbb{E}-\mathbb{F})t^i = \frac{\prod_{e\in\mathbb{E}}(1+te)}{\prod_{f\in\mathbb{F}}(1+tf)}\ ,$$

so that in particular

$$\Lambda_i(\mathbb{E}) = (-1)^i S_i(-\mathbb{E})\ , \tag{6}$$

which is another way of expressing the duality between complete and elementary symmetric functions. This yields a compact expression for the polynomial whose set of zeros is the alphabet $\mathbb{A}$ of cardinality n

$$\prod_{a\in\mathbb{A}}(x-a) = \sum_i x^{n-i}(-1)^i\Lambda_i(\mathbb{A}) = \sum_i S_{n-i}(x)S_i(-\mathbb{A}) = S_n(x-\mathbb{A}) = (-1)^n\Lambda_n(\mathbb{A}-x)\ .$$

Taking different alphabets $\mathbb{E}_1,\dots,\mathbb{E}_n$ in the columns of the determinant of Jacobi–Trudi formula, one gets a *multi-Schur function* $S_I(\mathbb{E}_1,\dots,\mathbb{E}_n)=\det[S_{i_l+l-k}(\mathbb{E}_l)]_{1\le k,l\le n}$. An example of multi-Schur function considered in this paper is the n^{th} orthogonal polynomial associated with the linear functional μ of moments $\mu(x^i)=S_i(\mathbb{E})$. (Note that, since the S_i are algebraically independent, one may always define a formal alphabet $\mathbb{E}$ by assigning to each S_i a given value $S_i(\mathbb{E})$ in a commutative ring.) Indeed this polynomial $p_n(x)$ has, up to a constant factor, the expression

$$p_n(x) = S_{n^n0}(\mathbb{E},\mathbb{E},\dots,\mathbb{E},x) = \begin{vmatrix} S_n(\mathbb{E}) & S_{n+1}(\mathbb{E}) & \dots & S_{2n-1}(\mathbb{E}) & x^n \\ S_{n-1}(\mathbb{E}) & S_n(\mathbb{E}) & \dots & S_{2n-2}(\mathbb{E}) & x^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ S_0(\mathbb{E}) & S_1(\mathbb{E}) & \dots & S_{n-1}(\mathbb{E}) & 1 \end{vmatrix}\ , \tag{7}$$

(see **[Sz]** p.27). By subtraction of lines in the determinant, this polynomial may also be written as an ordinary Schur function on the alphabet $\mathbb{E}-x$. Indeed, one has $p_n(x)=S_{n^n}(\mathbb{E}-x)$. This is a consequence of the following important lemma.

LEMMA 2.1 *Let m, n be two integers $m\le n$, $\mathbb{F}$ an alphabet of cardinality m, and $I=(i_1,\dots,i_n)$ a partition. Then*

$$S_I(\mathbb{E}_1,\dots,\mathbb{E}_n) = \begin{vmatrix} S_{i_1}(\mathbb{E}_1-\mathbb{F}) & \dots & S_{i_n+n-1}(\mathbb{E}_n-\mathbb{F}) \\ \vdots & & \vdots \\ S_{i_1-n+m+1}(\mathbb{E}_1-\mathbb{F}) & \dots & S_{i_n+m}(\mathbb{E}_n-\mathbb{F}) \\ S_{i_1-n+m}(\mathbb{E}_1) & \dots & S_{i_n+m-1}(\mathbb{E}_n) \\ \vdots & & \vdots \\ S_{i_1-n+1}(\mathbb{E}_1) & \dots & S_{i_n}(\mathbb{E}_n) \end{vmatrix}\ .$$

Proof. By (4), we have $S_k(\mathbb{E}_j-\mathbb{F})=S_k(\mathbb{E}_j)+S_{k-1}(\mathbb{E}_j)S_1(-\mathbb{F})+\dots+S_k(-\mathbb{F})$. But $S_k(-\mathbb{F})=(-1)^k\Lambda_k(\mathbb{F})=0$ if $k>m$. Thus, the transformation consists in adding to each of the first $n-m$ rows a linear combination of the next m rows. □

The Jacobi–Trudi formula expresses Schur functions as minors of the matrix $\mathbf{S}$ of complete symmetric functions S_i. This shows that many algebraic relations for Schur functions may be obtained by merely specializing the numerous identities satisfied by minors of a generic matrix. We have shown elsewhere that a great number of these identities are easily derived from a single one by Turnbull, that we shall now recall. We first need some appropriate notations.

Let M be a $n \times \infty$ matrix. We shall be interested in identities satisfied by maximal minors of M. Let $a,b,\ldots,c$ be n column vectors of M. The maximal minor of M taken on these n columns will be denoted by either a bracket or a one–line tableau

$$[ab\ldots c] = \boxed{a \quad b \quad \ldots \quad c}\,.$$

A product of p minors of M will be designated by a $p \times n$ tableau

$$[ab\ldots c].[de\ldots f].[gh\ldots i] = \begin{array}{|cccc|}\hline a & b & \ldots & c \\ \hline d & e & \ldots & f \\ \hline g & h & \ldots & i \\ \hline \end{array}\,.$$

To denote certain alternating sums of products of minors, we shall use tableaux with boxes enclosing some particular vectors. They are defined in the following way.

Let T be a $p \times n$ tableau and A a subset of elements of T. Given a permutation σ of the elements of A, denote by $\sigma(T)$ the tableau in which the elements of A are permuted by σ. Now, boxing in T the elements of A, we get a new tableau τ which will represent the alternating sum of all the tableaux $\sigma(T)$, taking into account the fact that a permutation of elements of the same row gives a trivial action. More precisely, $\tau = \sum_\sigma sign(\sigma).\sigma(T)$, where σ runs through the cosets of the symmetric group $\mathfrak{S}(A)$ modulo the subgroup of the permutations which leave unchanged the rows of T.

Let i_k, $k = 1,\ldots p$ denote the number of elements of A lying in the k^{th} row of τ. The number of terms in this sum will be $(i_1 + \ldots + i_p)!/i_1! \ldots i_p!$. In particular, if all the enclosed elements of τ lie in the same row, it reduces to the single tableau T.

To illustrate these notations, let us write a well–known minor identity (Plücker's relations) and then its transcription with tableaux.

Let M be the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & \ldots \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & \ldots \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & \ldots \end{pmatrix}.$$

We have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} - \begin{vmatrix} e_1 & b_1 & c_1 \\ e_2 & b_2 & c_2 \\ e_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} d_1 & a_1 & f_1 \\ d_2 & a_2 & f_2 \\ d_3 & a_3 & f_3 \end{vmatrix}$$

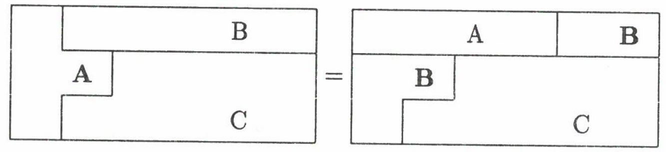
$$- \begin{vmatrix} f_1 & b_1 & c_1 \\ f_2 & b_2 & c_2 \\ f_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} d_1 & e_1 & a_1 \\ d_2 & e_2 & a_2 \\ d_3 & e_3 & a_3 \end{vmatrix} = \begin{vmatrix} a_1 & e_1 & f_1 \\ a_2 & e_2 & f_2 \\ a_3 & e_3 & f_3 \end{vmatrix} . \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

Denoting the column vectors of M by a,b,c,d,e,f, this identity written using tableaux takes the following form

$$\begin{array}{|ccc|}\hline \boxed{a} & b & c \\ \hline d & \boxed{e} & \boxed{f} \\ \hline \end{array} = \begin{array}{|ccc|}\hline a & e & f \\ \hline d & b & c \\ \hline \end{array}\,.$$

It is readily deduced from Turnbull's identity, that we shall now state.

Let τ be a tableau and denote by A the set of its enclosed elements. Let B be the set of elements of a given row, for example the first one, which do not belong to A. Finally let C denote the set of all other elements of τ. One can build another tableau υ equal to τ by merely exchanging the roles of A and B as shown by the following diagram:



More precisely, we have

THEOREM 2.1. *(Turnbull's identity,* **[Tu]**, p.209.)

Let k be the number of elements of A.

(i) *If $k > n$, $\tau = 0$.*

(ii) *If $k \leq n$, and υ is built by*

a. *exchanging each element of A which does not belong to the first row, with any element of B;*

b. *boxing the elements of B;*

c. *removing the boxes of the elements of A;*

then $\tau = \upsilon$.

For instance, taking $n = 6$, $k = 4$, $A = \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$, B $= \{\alpha,\beta,\gamma,\delta,\varepsilon\}$, one has

$$\tau = \begin{array}{|cccccc|}\hline \boxed{\mathbf{a}} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \hline \boxed{\mathbf{b}} & \boxed{\mathbf{c}} & f & g & h & i \\ \hline \boxed{\mathbf{d}} & j & k & l & m & o \\ \hline \end{array} = \begin{array}{|cccccc|}\hline \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} & \boxed{\delta} & \boxed{\varepsilon} \\ \hline \boxed{\alpha} & \boxed{\beta} & f & g & h & i \\ \hline \boxed{\gamma} & j & k & l & m & o \\ \hline \end{array} = \upsilon\,.$$

We shall now deduce from this identity some formulae for the power of a staircase Schur function.

3 Powers of a staircase Schur function and symmetric analogues of Bessel polynomials

Let $\rho(k,m)$ denote the staircase partition $(m, 2m, \ldots, km)$. The n^{th} power of $S_{\rho(k,m)}$ admits of the following expression:

THEOREM 3.1.

$$(S_{\rho(k,m)})^n = \sum_{I \subset \rho(n-1,m)} (-1)^{|\rho(n-1,m)|-|I|} \, S_{\rho(n+k-1,m)/I} \, \Delta_I \,,$$

where Δ_I is the minor taken on the lines $i_1+1, i_2+2, \dots, i_{n-1}+n-1$ of the following $(m+1)(n-1)\times(n-1)$ matrix:

$$M = \begin{pmatrix} S_{\rho(k-1,m)/1^m} & S_{\rho(k-1,m)/1^{2m+1}} & \dots & S_{\rho(k-1,m)/1^{(n-1)m+n-2}} \\ S_{\rho(k-1,m)/1^{m-1}} & S_{\rho(k-1,m)/1^{2m}} & \dots & S_{\rho(k-1,m)/1^{(n-1)m+n-3}} \\ \vdots & \vdots & & \vdots \\ S_{\rho(k-1,m)} & S_{\rho(k-1,m)/1^{m+1}} & \dots & S_{\rho(k-1,m)/1^{(n-2)(m+1)}} \\ 0 & S_{\rho(k-1,m)/1^m} & \dots & S_{\rho(k-1,m)/1^{(n-2)m+n-3}} \\ 0 & S_{\rho(k-1,m)/1^{m-1}} & \dots & S_{\rho(k-1,m)/1^{(n-2)m+n-4}} \\ \vdots & \vdots & & \vdots \\ 0 & S_{\rho(k-1,m)} & \dots & S_{\rho(k-1,m)/1^{(n-3)(m+1)}} \\ 0 & 0 & \dots & S_{\rho(k-1,m)/1^{(n-3)m+n-4}} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & S_{\rho(k-1,m)} \end{pmatrix} .$$

Note that we allow the first parts of I to be zero. Before proving the theorem, let us illustrate it by a few examples.

EXAMPLE 3.2. We take $n=3$, $m=1$ and $k=4$. The theorem states that

$$(S_{1234})^3 = \sum_{I\subset 12} (-1)^{3-|I|}\, S_{123456/I}\, \Delta_I ,$$

where I runs over the set $\{00, 01, 02, 11, 12\}$ and Δ_I is the minor taken on the lines i_1+1, i_2+2 of the 4×2 matrix:

$$M = \begin{pmatrix} S_{123/1} & S_{123/1^3} \\ S_{123} & S_{123/1^2} \\ 0 & S_{123/1} \\ 0 & S_{123} \end{pmatrix} = \begin{pmatrix} S_{1234/1112} & S_{1234/1114} \\ S_{1234/1111} & S_{1234/1113} \\ S_{1234/1110} & S_{1234/1112} \\ S_{1234/111(-1)} & S_{1234/1111} \end{pmatrix} .$$

The second expression of M shows more clearly the regularity of its columns (we recall that for any partition J, the determinants $S_{J/1110}$ and $S_{J/111(-1)}$ are null having two identical rows). Expanding this sum and the determinants Δ_I yields the following relation:

$$\begin{aligned}(S_{1234})^3 = {} & S_{123456}\, S_{123}\, S_{123/111} - S_{123456}\, S_{123/1}\, S_{123/11} + S_{123456/1}\, S_{123/1}\, S_{123/1} \\ & - S_{123456/11}\, S_{123}\, S_{123/1} - S_{123456/2}\, S_{123}\, S_{123/1} + S_{123456/12}\, S_{123}\, S_{123} .\end{aligned}$$

EXAMPLE 3.3. We take $n=2$, so that theorem 3.1 reads

$$(S_{\rho(k,m)})^2 = \sum_{0\le i\le m} (-1)^{m-i}\, S_{\rho(k+1,m)/i}\, S_{\rho(k-1,m)/1^{m-i}} ,$$

which is the formula investigated by Goulden **[Go]** for the square of a staircase Schur function.

Proof of Theorem 3.1. We shall apply Turnbull's identity 2.1. Set $N = k+(n-1)(m+1)$ and consider the $N\times\infty$ matrix

$$M = \begin{pmatrix} S_0 & S_1 & S_2 & S_3 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & S_0 & S_1 & S_2 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & S_0 & S_1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} .$$

Let $0, 1, 2, 3, \dots$ denote the column vectors of the left part of M, and $1^*, 2^*, 3^*, \dots N^*$, denote the last N column vectors of M. For brevity we shall write $m_p = pm+p-1$. The Schur function $S_{\rho(k,m)}$ may be expressed as a maximal minor of M in n different ways

$$\begin{aligned} S_{\rho(k,m)} &= [m_1\, m_2\, \dots\, m_k\, (k+1)^*\, (k+2)^* \,\dots\, N^*] \\ &= [0\, 1\, \dots\, (m_1-1)\, m_1\, m_2\, \dots\, m_{k+1}\, (k+m_1+1)^*\, (k+m_1+2)^* \,\dots\, N^*] \\ &\ \ \vdots \\ &= [0\, 1\, 2\, \dots\, (m_{n-1}-1)\, m_{n-1}\, m_n\, \dots\, m_{k+n-1}] , \end{aligned}$$

so that its n^{th} power is equal to the product of these n minors

$$(S_{\rho(k,m)})^n = \begin{array}{|lllllllll|} \hline m_1 & \dots & m_k & (k+1)^* & \dots & & & \dots & N^* \\ \hline 0 & \dots & (m_1-1) & m_1 & \dots & m_{k+1} & (k+m_1+1)^* & \dots & N^* \\ \hline \vdots & & & & \vdots & & & & \vdots \\ \hline 0 & \dots & & & \dots & (m_{n-1}-1) & m_{n-1} & \dots & m_{k+n-1} \\ \hline \end{array} .$$

Now, recalling that a tableau having two equal letters on the same row is equal to zero, we may box the letters $m_1, m_2, \dots, m_k$ in the first row, the letter m_{k+1} in the second row, …, the letter m_{k+n-1} in the n^{th} row, and apply Turnbull's identity. This yields

$$(S_{\rho(k,m)})^n$$

$$= \begin{array}{|lllllllll|} \hline \boxed{m_1} & \dots & \boxed{m_k} & (k+1)^* & \dots & & & \dots & N^* \\ \hline 0 & \dots & (m_1-1) & m_1 & \dots & \boxed{m_{k+1}} & (k+m_1+1)^* & \dots & N^* \\ \hline \vdots & & & & \vdots & & & & \vdots \\ \hline 0 & \dots & & & \dots & (m_{n-1}-1) & m_{n-1} & \dots & \boxed{m_{k+n-1}} \\ \hline \end{array}$$

$$= \begin{array}{|lllllllll|} \hline m_1 & \dots & m_k & m_{k+1} & \dots & m_{k+n-1} & \boxed{(k+n)^*} & \dots & \boxed{N^*} \\ \hline 0 & \dots & (m_1-1) & m_1 & \dots & \boxed{(k+1)^*} & (k+m_1+1)^* & \dots & N^* \\ \hline \vdots & & & & \vdots & & & & \vdots \\ \hline 0 & \dots & & & \dots & (m_{n-1}-1) & m_{n-1} & \dots & \boxed{(k+n-1)^*} \\ \hline \end{array} .$$

This last tableau represents a sum of products of Schur functions of the following type

$$\pm S_{\rho(n+k-1,m)/I}\ S_{\rho(k-1,m)/1^{j_1}} \cdots\ S_{\rho(k-1,m)/1^{j_{n-1}}}\ .$$

The coefficient of a given $S_{\rho(n+k-1,m)/I}$ in this sum is in fact a tableau having $n-1$ rows with only one letter boxed in each row, that is a determinant of order $n-1$. There is no difficulty in recognizing this determinant as the minor Δ_I defined above. □

The formula for the square of $S_{\rho(k,1)}$ may be presented as the computation of a determinant of Schur functions

$$\begin{vmatrix} S_{\rho(k-1,1)} & S_{\rho(k+1,1)} \\ S_{\rho(k-1,1)/1} & S_{\rho(k+1,1)/1} \end{vmatrix} = (S_{\rho(k,1)})^2\ . \qquad (8)$$

The same may be done for theorem 3.1, which has also a determinantal formulation. We shall write for brevity $\rho(k)$ in place of $\rho(k,1)$.

THEOREM 3.4.

$$\begin{vmatrix} S_{\rho(k-1)} & 0 & \dots & 0 & \dots & 0 & S_{\rho(k+1)} \\ S_{\rho(k-1)/1} & 0 & \dots & 0 & \dots & S_{\rho(k+2)} & S_{\rho(k+1)/1} \\ S_{\rho(k-1)/1^2} & S_{\rho(k,1)} & \dots & 0 & \dots & S_{\rho(k+2)/1} & S_{\rho(k+1)/1^2} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ S_{\rho(k-1)/1^{2n-3}} & S_{\rho(k-1)/1^{2n-5}} & \dots & S_{\rho(k+n-1)/1^{n-1}} & \dots & S_{\rho(k+2)/1^{2n-4}} & S_{\rho(k+1)/1^{2n-3}} \end{vmatrix}$$
$$= (S_{\rho(k)})^n\ S_{\rho(k+1)}\ S_{\rho(k+2)} \dots S_{\rho(k+n-2)}\ .$$

EXAMPLE 3.5. For $n=3$ and $k=4$ one has

$$\begin{vmatrix} S_{123} & 0 & 0 & S_{12345} \\ S_{123/1} & 0 & S_{123456} & S_{12345/1} \\ S_{123/1^2} & S_{123} & S_{123456/1} & S_{12345/1^2} \\ S_{123/1^3} & S_{123/1} & S_{123456/1^2} & S_{12345/1^3} \end{vmatrix} = (S_{1234})^3\ S_{12345}\ .$$

EXAMPLE 3.6. For $n=4$ and $k=6$ one has

$$\begin{vmatrix} S_{12345} & 0 & 0 & 0 & 0 & S_{1234567} \\ S_{12345/1} & 0 & 0 & 0 & S_{12345678} & S_{1234567/1} \\ S_{12345/1^2} & S_{12345} & 0 & S_{123456789} & S_{12345678/1} & S_{1234567/1^2} \\ S_{12345/1^3} & S_{12345/1} & 0 & S_{123456789/1} & S_{12345678/1^2} & S_{1234567/1^3} \\ S_{12345/1^4} & S_{12345/1^2} & S_{12345} & S_{123456789/1^2} & S_{12345678/1^3} & S_{1234567/1^4} \\ S_{12345/1^5} & S_{12345/1^3} & S_{12345/1} & S_{123456789/1^3} & S_{12345678/1^4} & S_{1234567/1^5} \end{vmatrix}$$
$$= (S_{123456})^4\ S_{1234567}\ S_{12345678}\ .$$

Proof. The proof of this last example will illustrate enough the principle of our computation. We consider the 12×30 matrix

$$M = \begin{pmatrix} S_0 & S_1 & S_2 & S_3 & \dots & S_{17} & 1 & 0 & 0 & \dots & 0 \\ 0 & S_0 & S_1 & S_2 & \dots & S_{16} & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & S_0 & S_1 & \dots & S_{15} & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & S_6 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

whose column vectors are denoted 0, 1, 2, ... 17, 1^*, 2^*, ... 12^*. The determinant D of example 3.6 is represented by the following tableau

$$D = \begin{array}{|cccccccccccc|} \hline 0 & 1 & 3 & 5 & 7 & 9 & 11 & \boxed{7^*} & 9^* & 10^* & 11^* & 12^* \\ \hline 0 & 1 & 2 & 3 & 5 & 7 & 9 & 11 & 13 & \boxed{8^*} & 11^* & 12^* \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 11 & 13 & 15 & \boxed{9^*} \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & \boxed{10^*} & 11^* & 12^* \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & \boxed{11^*} & 12^* \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & 2 & \boxed{12^*} \\ \hline \end{array}\ .$$

Now we transform this tableau by Turnbull's identity and note that all boxes may be removed because of the presence of identical letters

$$D = \begin{array}{|cccccccccccc|} \hline 0 & 1 & 3 & 5 & 7 & 9 & 11 & \boxed{13} & 9^* & 10^* & 11^* & 12^* \\ \hline 0 & 1 & 2 & 3 & 5 & 7 & 9 & 11 & 13 & \boxed{15} & 11^* & 12^* \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 11 & 13 & 15 & \boxed{17} \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & \boxed{0} & 11^* & 12^* \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & \boxed{2} & 12^* \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 7^* & 8^* & 9^* & 10^* & 11^* & 12^* \\ \hline \end{array}$$

$$= \begin{array}{|cccccccccccc|} \hline 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 9^* & 10^* & 11^* & 12^* \\ \hline 0 & 1 & 2 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 11^* & 12^* \\ \hline 0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & 11^* & 12^* \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 0 & 2 & 12^* \\ \hline 1 & 3 & 5 & 7 & 9 & 11 & 7^* & 8^* & 9^* & 10^* & 11^* & 12^* \\ \hline \end{array}$$

$$= (S_{123456})^4\ S_{1234567} S_{12345678} \qquad \square$$

We shall end this section with an interpretation of theorem 3.4 in terms of orthogonal polynomials. First we shall give an immediate generalization of formula (8).

PROPOSITION 3.7. *For $1 \le p \le k+1$ we have*

$$\begin{vmatrix} S_{\rho(k-1)} & S_{\rho(k+1)} \\ S_{\rho(k-1)/1^p} & S_{\rho(k+1)/1^p} \end{vmatrix} = S_{\rho(k)}\ S_{\rho(k)/1^{p-1}}\ ,$$

where $\rho(k)$ is short for the partition $12\dots k$.

Proof. This is an instance of Plücker relations. We consider again the matrix M of the proof of theorem 3.1, taking $N = k+2$. The determinant to be computed is then represented by the $2 \times (k+2)$ tableau

$$(-1)^p \begin{array}{|ccccccc|} \hline 0 & 1 & 3 & 5 & \dots & 2k-1 & \boxed{N^*} \\ \hline 1 & 3 & 5 & 7 & \dots & 2k+1 & \boxed{(N-p)^*} \\ \hline \end{array}\ .$$

Applying Turnbull's identity, this tableau is transformed into

$$(-1)^p \begin{array}{|ccccccc|}\hline 0 & 1 & 3 & 5 & \ldots & 2k-1 & \boxed{2k+1} \\ \hline \boxed{1} & \boxed{3} & \boxed{5} & \boxed{7} & \ldots & N^* & (N-p)^* \\ \hline \end{array}$$

$$= (-1)^p \begin{array}{|ccccccc|}\hline 0 & 1 & 3 & 5 & \ldots & 2k-1 & 2k+1 \\ \hline 1 & 3 & 5 & 7 & \ldots & N^* & (N-p)^* \\ \hline \end{array} = S_{\rho(k)}\, S_{\rho(k)/1^{p-1}} \;. \qquad \square$$

From this proposition we deduce the following result:

THEOREM 3.8. *Let $\mathbb{E}$ be an alphabet and let us denote by π_k the polynomial of degree k: $\pi_k(x) = S_{\rho(k)}(\mathbb{E} - x)$. Then π_k satisfies the three term recurrence relationship ($k \geq 1$)*

$$S_{\rho(k-1)}(\mathbb{E})\pi_{k+1}(x) + xS_{\rho(k)}(\mathbb{E})\pi_k(x) - S_{\rho(k+1)}(\mathbb{E})\pi_{k-1}(x) = 0 \;.$$

In other words $\left(\pi_k(x)\right)_{k\geq 0}$ is a family of orthogonal polynomials.

Proof. The expansion of π_k is obtained by considering it as a multi-Schur function and expanding this determinant along its last column. Indeed, by lemma 2.1,

$$\pi_k(x) = S_{12\ldots k0}(\mathbb{E}, \ldots, \mathbb{E}, x) = \sum_{0\leq i\leq k} S_{12\ldots k/1^i}(\mathbb{E})(-x)^i \;.$$

By linearity we deduce from proposition 3.7 that

$$\begin{vmatrix} S_{\rho(k-1)} & S_{\rho(k+1)} \\ \pi_{k-1}(x) & \pi_{k+1}(x) \end{vmatrix} = -xS_{\rho(k)}\; \pi_k(x) \;,$$

which is the required relationship. $\square$

Thus, theorem 3.8 states that the elimination of x^0 between $\pi_{k-1}(x)$ and $\pi_{k+1}(x)$ produces a polynomial proportional to $x\pi_k(x)$. As a consequence the elimination of x^0, x^1, x^2 between $\pi_{k-1}(x)$, $\pi_{k+1}(x)$, $x\pi_{k+2}(x)$, $x^2\pi_{k-1}(x)$ will produce a polynomial proportional to $x^3\pi_k(x)$. This elimination is expressed by the following formula

$$\begin{vmatrix} S_{\rho(k-1)} & 0 & 0 & S_{\rho(k+1)} \\ S_{\rho(k-1)/1} & 0 & S_{\rho(k+2)} & S_{\rho(k+1)/1} \\ S_{\rho(k-1)/1^2} & S_{\rho(k-1)} & S_{\rho(k+2)/1} & S_{\rho(k+1)/1^2} \\ \pi_{(k-1)}(x) & x^2\pi_{k-1}(x) & -x\pi_{k+2}(x) & \pi_{k+1}(x) \end{vmatrix} = -\left(S_{\rho(k)}\right)^2 S_{\rho(k+1)}\; x^3\pi_k(x).$$

Picking up the coefficient of x^3 on each side we find again the case $n = 3$ of theorem 3.4. And this process may clearly be carried out for larger values of n.

The computation of the moments of the sequence of orthogonal polynomials $\left(\pi_k(x)\right)_{k\geq 0}$ will result from the following identity. We shall denote by $\rho(n) + m$ the partition $(1, 2, \ldots, n-1, n+m)$.

THEOREM 3.9. *Let k, n be integers such that $1 \leq k \leq n$. There holds:*

$$\begin{vmatrix} S_{\rho(k)/1^{k-1}} & S_{\rho(k+1)/1^{k-1}} & \ldots & S_{\rho(n)/1^{k-1}} \\ S_{\rho(k)/1^k} & S_{\rho(k+1)/1^k} & \ldots & S_{\rho(n)/1^k} \\ \vdots & \vdots & & \vdots \\ S_{\rho(k)/1^{n-1}} & S_{\rho(k+1)/1^{n-1}} & \ldots & S_{\rho(n)/1^{n-1}} \end{vmatrix} = S_{\rho(k-1)}P_{\rho(k)+(n-k+1)}S_{\rho(k+1)}S_{\rho(k+2)}\ldots S_{\rho(n-1)},$$

where $P_{\rho(k)+(n-k+1)} = \sum_{0\leq 2i\leq n-k+1} \Lambda_{\rho(k)+2i}S_{n-k+1-2i}$.

EXAMPLE 3.10. For $n = 6$, $k = 3$ one has

$$\begin{vmatrix} S_{123/1^2} & S_{1234/1^2} & S_{12345/1^2} & S_{123456/1^2} \\ S_{123/1^3} & S_{1234/1^3} & S_{12345/1^3} & S_{123456/1^3} \\ S_{123/1^4} & S_{1234/1^4} & S_{12345/1^4} & S_{123456/1^4} \\ S_{123/1^5} & S_{1234/1^5} & S_{12345/1^5} & S_{123456/1^5} \end{vmatrix} = S_{12}P_{127}S_{1234}S_{12345} \;,$$

where $P_{127} = \Lambda_{123}S_4 + \Lambda_{125}S_2 + \Lambda_{127}$.

Proof. The proof is similar to the proof of theorem 3.4, the only difference being that one has to apply Turnbull's identity several times in order to obtain the required factorization.

COROLLARY 3.11. *The n^{th} moment of the sequence of orthogonal polynomials $\left(\pi_k(x)\right)_{k\geq 0}$ is the sum of all hook Schur functions of weight $n+1$: $\mu(x^n) = \sum_{0\leq i\leq n} S_{1^i(n+1-i)}(\mathbb{E})$.*

Proof. Let $q_k(x)$ denote the monic polynomial $q_k(x) = S_{\rho(k)}(\mathbb{E} - x)/(-1)^k S_{\rho(k)}(\mathbb{E})$. Theorem 3.9 may be seen as the resolution of the linear system in the unknown a_{nk}:

$$x^n = \sum_k a_{nk}q_k(x) \;. \tag{9}$$

The result is that $a_{nk} = P_{\rho(k+1)+n-k}(\mathbb{E})/S_{\rho(k+1)}$. Let μ be the functional on $\mathfrak{Sym}(\mathbb{E})[x]$ associated with the sequence of orthogonal polynomials $\left(\pi_k(x)\right)_{k\geq 0}$. In fact μ is defined up to a constant factor, so that we can add the condition $\mu(1) = S_1(\mathbb{E})$. Then, recalling that by definition $\mu(q_k(x)) = 0$, $k \geq 1$ and applying μ to the equality (9), one obtains that $\mu(x^n) = P_{n+1}(\mathbb{E}) = \sum_{0\leq i\leq n} S_{1^i(n+1-i)}(\mathbb{E})$, the last equality resulting easily from Pieri rule for the multiplication of a Schur function by an elementary symmetric function. $\square$

Remarks. (i) It has been shown in **[LLT]** that the symmetric function

$$\sum_{0\leq 2i\leq n-k+1} \Lambda_{\rho(k)+2i}S_{n-k+1-2i}$$

is nothing but the Schur P-function $P_{\rho(k)+n-k+1}$. More generally there exists for every Schur P-function a similar quadratic expansion in terms of ordinary Schur S-functions.

(ii) It is easily checked that $S_{\rho(n)/1^m} = S_{\rho(n)/m}$. It follows that the polynomials $\phi_k(x) = S_{\rho(k)}(\mathbb{E} + x)$ are equal to the polynomials $\pi_k(-x)$, and therefore also satisfy a three term recurrence relationship:

$$S_{\rho(k-1)}(\mathbb{E})\phi_{k+1}(x) - xS_{\rho(k)}(\mathbb{E})\phi_k(x) - S_{\rho(k+1)}(\mathbb{E})\phi_{k-1}(x) = 0 \;.$$

They are orthogonal for the moments $\nu(x^n) = (-1)^n \sum_{0\le i\le n} S_{1^i(n+1-i)}(\mathbb{E})$.

We must now insist on the differences between the two Schur function expressions of orthogonal polynomials hitherto encountered. As recalled in section 2, the family of orthogonal polynomials associated with a given functional μ may *always* be represented up to a constant factor by the sequence of rectangle Schur functions $S_{n^n}(\mathbb{E} - x)$, $n \ge 0$, where $\mathbb{E}$ is the alphabet formally defined by $S_k(\mathbb{E}) = \mu(x^k)$, $k \ge 0$. In contrast Schur functions of the type $S_{12\ldots n}(\mathbb{E} - x)$, $n \ge 0$ or $S_{12\ldots n}(\mathbb{E} + x)$, $n \ge 0$ represent only particular families of orthogonal polynomials. Indeed, the recurrence relationship of theorem 3.8 is not generic, for the coefficient of $\pi_n(x)$ in this relationship is of the type $\alpha_n x$ while the general form of this coefficient is $\alpha_n x + \beta_n$. The following example shows that these polynomials may be seen as symmetric analogues of Bessel polynomials.

EXAMPLE 3.12.

Let $\mathcal{E}$ be the alphabet formally defined by $S_k(\mathcal{E}) = 1/k!$, $k \ge 0$. It is a classical result that for any partition I of weight n, there holds $S_I(\mathcal{E}) = f_I/n!$, where f_I is the number of standard Young tableaux of shape I, *i.e.* the dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by I. It follows that the moments associated to the sequence $\left(\phi_k(x)\right)_{k\ge 0}$ are equal to $\nu(x^n) = (-2)^n/(n+1)!$, that is, are equal to the moments of Bessel polynomials (see for example **[Ch]**). The corresponding polynomials (suitably normalized)

$$y_n(x) = S_{12\ldots n}(\mathcal{E} + x)/S_{12\ldots n}(\mathcal{E}), \quad n \ge 0, \tag{10}$$

are therefore the *Bessel polynomials*. The first ones are

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + x, \\ y_2(x) &= 1 + 3x + 3x^2, \\ y_3(x) &= 1 + 6x + 15x^2 + 15x^3, \\ y_4(x) &= 1 + 10x + 45x^2 + 105x^3 + 105x^4. \end{aligned}$$

Formula (10) provides the following determinantal expression

$$y_n(x) = \prod_{1\le i\le n} (2(n-i)+1)^i \begin{vmatrix} 1/1! & 1/3! & 1/5! & \ldots & 1/(2n-1)! & (-x)^n \\ 1/0! & 1/2! & 1/3! & \ldots & 1/(2n-2)! & (-x)^{n-1} \\ 0 & 1/1! & 1/4! & \ldots & 1/(2n-3)! & (-x)^{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & & & & 1/n! & -x \\ 0 & & & & 1/(n-1)! & 1 \end{vmatrix} .$$

There exists a fairly extensive literature devoted to these polynomials. For a detailed account up to 1978, the interested reader is referred to **[Gr]**. We shall also mention the recent work of Dulucq and Favreau who have presented a combinatorial model for these polynomials, based upon weighted involutions **[DF]**.

Note that our point of view provides at once a q-analogue for Bessel polynomials, by merely replacing $\mathcal{E}$ by the alphabet $\mathcal{E}_q = \{1, q, q^2, \ldots\}$. More precisely, setting $v = x/(1-q)$ we define

$$y_n(x,q) = q^{\binom{n}{2}} S_{12\ldots n}(\mathcal{E}_q + v)/S_{12\ldots n}(\mathcal{E}_q).$$

The $y_n(x,q)$ are orthogonal polynomials in x, whose coefficients are polynomials in q. The first ones are

$$\begin{aligned} y_0(x,q) &= 1, \\ y_1(x,q) &= 1 + x, \\ y_2(x,q) &= q + (1+q+q^2)x + (1+q+q^2)x^2, \\ y_3(x,q) &= q^3 + q(1+q^2)(1+q+q^2)x + (1+q+q^2)(1+q+q^2+q^3+q^4)x^2 \\ &\quad + (1+q+q^2)(1+q+q^2+q^3+q^4)x^3. \end{aligned}$$

4 Expression of a symmetric polynomial in terms of the power sums of odd degree

Let $\mathbb{X}$ denote a finite set of variables $\mathbb{X} = \{x_1, \ldots, x_n\}$. It is well known that any symmetric polynomial $F(\mathbb{X})$ may be expressed as a polynomial function of the power sums $\psi_k(\mathbb{X}) = \sum_{x\in\mathbb{X}} x^k$, $k = 1,2\ldots n$. It was shown in the last century that $F(\mathbb{X})$ may also be expressed as a rational function of $\psi_k(\mathbb{X})$, $k = 1,3,5,\ldots 2n-1$ (**[Bo]**, **[La]**). Pólya **[Po]** and Foulkes **[Fo]**, among others, have proposed explicit expressions for some particular symmetric polynomials $F(\mathbb{X})$. For instance

PROPOSITION 4.1. *The k^{th} elementary symmetric function is equal to* $\Lambda_k(\mathbb{X}) = S_{\rho(n)/1^{n-k}}(\mathbb{X})/S_{\rho(n-1)}(\mathbb{X})$, *the right-hand side being a rational function of the power sums of odd degree.*

Proof. For sake of completeness we sketch the proof. It is known that $S_{\rho(n-1)}(\mathbb{X}) = \prod_{i<j}(x_i + x_j)$. Taking an additionnal variable z we have also

$$S_{\rho(n)}(\mathbb{X} + z) = \prod_{i<j}(x_i + x_j) \prod_i (x_i + z) = S_{\rho(n-1)}(\mathbb{X}) \sum_k z^{n-k} \Lambda_k(\mathbb{X}) .$$

Thus, comparing the coefficients of z^{n-k}, we get $\Lambda_k(\mathbb{X}) = S_{\rho(n)/n-k}(\mathbb{X})/S_{\rho(n-1)}(\mathbb{X}) = S_{\rho(n)/1^{n-k}}(\mathbb{X})/S_{\rho(n-1)}(\mathbb{X})$. Finally, we recall that the staircase Schur function $S_{\rho(n)}$ and all its derivatives $S_{\rho(n)/I}$ depend only on the odd power sums ψ_{2p+1}, that is, belong to the subring generated by ψ_1, ψ_3, ψ_5, ... □

Now theorem 3.9 provides at once a similar expression for the complete symmetric functions $S_k(\mathbb{X})$.

PROPOSITION 4.2. *The k^{th} complete symmetric function is equal to*

$$S_k(\mathbb{X}) = \frac{\sum_{0\le 2i\le k} \Lambda_{\rho(n-1)+2i}(\mathbb{X}) S_{k-2i}(\mathbb{X})}{S_{\rho(n-1)}(\mathbb{X})} ,$$

the right-hand side being a rational function of the power sums of odd degree.

Proof. Theorem 3.9 shows that the function $\sum_{0\leq 2i\leq k} \Lambda_{\rho(n-1)+2i} S_{k-2i}$ may be expressed in terms of Schur functions of the type $S_{\rho(m)/1^p}$, which all belong to $\mathbb{Z}[\psi_1, \psi_3, \psi_5, \ldots]$. On the other hand, since $\mathbb{X}$ is finite of cardinality n, $\Lambda_{\rho(n-1)+2i}(\mathbb{X}) = 0$ for all $i > 0$. □

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Combinatorial S_n-Modules as Codes: A Summary

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This is a summary of a paper [8] with essentially the same title and authors that has been submitted elsewhere. The paper points out and extends connections between combinatorial t-designs, representations of the symmetric group S_n labelled by 2-part partitions, and the classical Reed-Muller codes. Motivated by the representation theory of S_n associated with partitions having more than two parts, we then introduce τ-designs and give a characteristic free construction of "Euclidean geometry" codes.

The basis of this work is the poset of flags introduced in [7]. The incidence maps of this poset are used to introduce Z-forms on certain classical QS_n-modules in, and our principal object of study $ZB\kappa_\lambda$ is described in Theorem 3 and Corollary 4. The incidence maps are also used to construct large collections of characteristic free parity checks.

Let m, n be positive integers. A *partition* λ of n having (at most) m parts is a sequence $\lambda_1, \ldots, \lambda_m$ of non-negative integers such that $\sum_{s=1}^{m} \lambda_s = n$. The *conjugate* partition λ' to λ is defined by $\lambda'_s = |\{\lambda_t \mid s \leq \lambda_t\}|$. The partition λ is *proper* if $\lambda_s \geq \lambda_{s+1}$ for all s. It turns out that $\lambda = \lambda''$ if and only if λ is proper. The partition λ *dominates* partition ν of n ($\lambda \trianglerighteq \nu$) if $\sum_{s=1}^{t} \lambda_s \geq \sum_{s=1}^{t} \nu_s$ for all t. The domination relation $\trianglerighteq$ defines a partially ordered set (*poset*) on the partitions of n. We say λ *covers* ν in case the interval $[\nu, \lambda]$ in this poset contains only its endpoints. This means that equality holds for all but one value of t and the difference of these sums is one for this value (and that the diagram of λ [4] is obtained from the diagram of ν by raising one node to the t-th from the $(t+1)$-th row).

There are three equivalent ways we think about the set underlying a natural permutation S_n-module. The most combinatorial mode is flags or tabloids. A *flag* $\mathbf{F}$ with (at most) m parts is a totally ordered collection $\{F_0, \ldots, F_{m-1}\}$ of m subsets of $\{1, \ldots, n\}$ such that $\{1, \ldots, n\} = F_0 \supseteq \ldots \supseteq F_{m-1}$. The type of $\mathbf{F}$ is the partition typ$\mathbf{F}$ of n with t-th term $|\Delta F_t|$ where $\Delta F_t = F_{t-1} \setminus F_t$ for $1 \leq t < m$ and $\Delta F_m = F_{m-1}$. There is a natural (partial) ordering of flags given by $\mathbf{F} \leq \mathbf{G}$ whenever $F_t \subseteq G_t$ for each t. It is easy to see that this partially ordered set is a lattice and $\mathbf{F} \leq \mathbf{G}$ implies typ$\mathbf{F} \trianglerighteq$ typ$\mathbf{G}$. If the sets ΔF_t are being emphasized rather than F_t, the term *tabloid* rather than *flag* is used.

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