the right-hand side being a rational function of the power sums of odd degree.

Proof. Theorem 3.9 shows that the function $\sum_{0\leq 2i\leq k} \Lambda_{\rho(n-1)+2i}S_{k-2i}$ may be expressed in terms of Schur functions of the type $S_{\rho(m)/1^p}$, which all belong to $\mathbb{Z}[\psi_1,\psi_3,\psi_5,\ldots]$. On the other hand, since $\mathbb{X}$ is finite of cardinality n, $\Lambda_{\rho(n-1)+2i}(\mathbb{X}) = 0$ for all $i > 0$. □

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Combinatorial S_n-Modules as Codes: A Summary

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This is a summary of a paper [8] with essentially the same title and authors that has been submitted elsewhere. The paper points out and extends connections between combinatorial t-designs, representations of the symmetric group S_n labelled by 2-part partitions, and the classical Reed-Muller codes. Motivated by the representation theory of S_n associated with partitions having more than two parts, we then introduce τ-designs and give a characteristic free construction of "Euclidean geometry" codes.

The basis of this work is the poset of flags introduced in [7]. The incidence maps of this poset are used to introduce Z-forms on certain classical QS_n-modules in, and our principal object of study $ZB\kappa_\lambda$ is described in Theorem 3 and Corollary 4. The incidence maps are also used to construct large collections of characteristic free parity checks.

Let m, n be positive integers. A *partition* λ of n having (at most) m parts is a sequence $\lambda_1, \ldots, \lambda_m$ of non-negative integers such that $\sum_{s=1}^m \lambda_s = n$. The *conjugate* partition λ' to λ is defined by $\lambda'_s = |\{\lambda_t \mid s \leq \lambda_t\}|$. The partition λ is *proper* if $\lambda_s \geq \lambda_{s+1}$ for all s. It turns out that $\lambda = \lambda''$ if and only if λ is proper. The partition λ *dominates* partition ν of n ($\lambda \trianglerighteq \nu$) if $\sum_{s=1}^t \lambda_s \geq \sum_{s=1}^t \nu_s$ for all t. The domination relation $\trianglerighteq$ defines a partially ordered set (*poset*) on the partitions of n. We say λ *covers* ν in case the interval $[\nu, \lambda]$ in this poset contains only its endpoints. This means that equality holds for all but one value of t and the difference of these sums is one for this value (and that the diagram of λ [4] is obtained from the diagram of ν by raising one node to the t-th from the $(t+1)$-th row).

There are three equivalent ways we think about the set underlying a natural permutation S_n-module. The most combinatorial mode is flags or tabloids. A *flag* $\mathbf{F}$ with (at most) m parts is a totally ordered collection $\{F_0, \ldots, F_{m-1}\}$ of m subsets of $\{1, \ldots, n\}$ such that $\{1, \ldots, n\} = F_0 \supseteq \ldots \supseteq F_{m-1}$. The type of $\mathbf{F}$ is the partition typ $\mathbf{F}$ of n with t-th term $|\Delta F_t|$ where $\Delta F_t = F_{t-1} \setminus F_t$ for $1 \leq t < m$ and $\Delta F_m = F_{m-1}$. There is a natural (partial) ordering of flags given by $\mathbf{F} \leq \mathbf{G}$ whenever $F_t \subseteq G_t$ for each t. It is easy to see that this partially ordered set is a lattice and $\mathbf{F} \leq \mathbf{G}$ implies typ$\mathbf{F} \trianglerighteq$ typ$\mathbf{G}$. If the sets ΔF_t are being emphasized rather than F_t, the term *tabloid* rather than *flag* is used.

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The more algebraic mode is to use monomials. A *monomial of degree less than* m *in each variable* is an element of $Z[x_1, \ldots, x_n]$ of the form $\prod x_s^{i_s - 1}$ where $1 \leq i_s \leq m$. Each such monomial is associated with the unique flag $\mathbf{F}$ where $F_t = \{s \mid i_s \geq t\}$. Observe the relation "is a multiple of" corresponds to $\leq$.

The most succinct mode is sequences. An n-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ with entries from $\{1, \ldots, m\}$ is associated with the monomial $\prod x_s^{i_s - 1}$ (and so also with a flag). Under this correspondence, "is a multiple of" becomes the natural product order $\leq$: $(i_1, \ldots, i_n) \leq (j_1, \ldots, j_n)$ if and only if $i_s \leq j_s$ for all $s \in \{1, \ldots, n\}$.

Let $B = B_{n,m}$ denote this lattice of n-tuples on the set $\{1, \ldots, m\}$ (alias flags,tabloids, or monomials). The symmetric group S_n acts as an automorphism group on the lattice $(B, \leq)$ by place permutation since $\mathbf{i} \leq \mathbf{j}$ implies $\mathbf{i}\pi \leq \mathbf{j}\pi$ for all $\pi \in S_n$ and all $\mathbf{i}, \mathbf{j} \in B$: For each partition ν of n into m parts, the elements B^ν of B having type ν form an S_n-orbit. By linear extension of the action of S_n on B, the free Z-module ZB with distinguished orthonormal basis B is a right ZS_n-module and S_n acts as a group of orthogonal linear transformations (with respect to the nondegenerate form $\langle , \rangle$ defined by the orthonormal basis). The decomposition $ZB = \sum \oplus ZB^\nu$ where ν ranges over all partitions of n into m parts; is an ZS_n-module decomposition. The ZS_n-module ZB^ν is, by definition, the *natural permutation* S_n*-module of type* ν [3, 17.4].

In contrast to the Möbius function on the lattice of all partitions of n ordered by dominance, the Möbius function μ on $(B, \leq)$ is easily computed.

Lemma 1 *The partition* $\lambda \trianglelefteq \nu$ *if and only if* $\mathbf{i} \leq \mathbf{j}$ *for some* $\mathbf{i} \in B^\nu$, $\mathbf{j} \in B^\lambda$. *If* $\mathbf{i}, \mathbf{j} \in B^\nu$ *with* $\mathbf{i} \leq \mathbf{j}$ *then* $\mathbf{i} = \mathbf{j}$. *Finally,* $\mu(\mathbf{i}, \mathbf{j}) = \begin{cases} (-1)^t & \mathbf{i} \textit{ is infimum of } t \textit{ distinct coatoms of } [\mathbf{0}, \mathbf{j}], \\ 0 & \textit{otherwise.} \end{cases}$

The context in which we work is generally the integral Hecke algebra of the natural permutation S_n-modules. This means that the ZS_n-endomorphisms of ZB are of central interest.

Define ϕ_ξ^ν and $\psi_\xi^\nu \in \mathrm{Hom}(ZB^\nu, ZB)$ and *projection* maps $\pi_\lambda \in \mathrm{Hom}(ZB, ZB^\lambda)$ by:

$$\psi_\xi^\nu(\mathbf{n}) = \sum_{\mathrm{typ}(\mathbf{n} \sqcup \mathbf{j}) = \xi} \mathbf{j} \ , \ \phi_\xi^\nu(\mathbf{n}) = \sum_{\mathrm{typ}(\mathbf{n} \sqcap \mathbf{j}) = \xi} \mathbf{j} \ \text{ and } \ \pi_\lambda(\mathbf{i}) = \begin{cases} \mathbf{i} & \mathrm{typ}(\mathbf{i}) = \lambda, \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{n} \in B^\nu, \mathbf{i} \in B$. In case $\nu = \xi$, we drop the subscript ξ and call $\psi^\nu := \psi_\nu^\nu$ and $\phi^\nu := \psi_\nu^\nu$ *incidence* maps. Observe that $(\ker \pi_\lambda \phi_\xi^\nu)^\perp = (\mathrm{im}\, \pi_\nu \phi_\xi^\lambda)^{\perp\perp}$; $(\ker \pi_\nu \psi_\xi^\lambda)^\perp = (\mathrm{im}\, \pi_\lambda \psi_\xi^\nu)^{\perp\perp}$. and that ϕ_ξ^ν , ψ_ξ^ν and π_λ are ZS_n-homomorphisms

A Z-submodule P of the finitely generated Z-module M is called *pure* [2, 16.15] if the quotient module M/P is torsion free. This is equivalent to the condition that $m \in P$ whenever $0 \neq t \in Z$, $m \in M$ and $tm \in P$ and appears in [10] as "P has index 1 in M". We list some of the more important properties of pure submodules of a Z-module.

Lemma 2 *Let* Q *be the field of rational numbers and* K *an arbitrary field. Then the intersection of two pure submodules is pure and the kernel of a* Z*-homorphism* ϕ *between finitely generated free* Z*-modules is pure. If* P *is a pure submodule of the free* Z*-module* M *then* P *is also free and its rank equals its dimension when coefficients are extended to* K*. If two pure submodules determine the same* Q*-subspace when coefficients are extended to* Q*, then they coincide. If* M *is* Z*-free and has an orthonormal basis, and* N *is an arbitrary submodule then* $N^\perp$ *is pure and* $N^{\perp\perp}$ *is the smallest pure submodule containing* N.

Take $\mathbf{l} \in B^\lambda$. Then the stabilizer of $\mathbf{l}$ in S_n, $(S_n)_\mathbf{l} := \{\sigma \in S_n \mid \mathbf{l}\sigma = \mathbf{l}\}$ is the *Young subgroup associated with* $\mathbf{l}$. If $\mathbf{l}'$ is a second sequence satisfying $1 = \mid \Delta \mathbf{l}_i \cap \Delta \mathbf{l}'_j \mid$ for all nodes (i, j) in the diagram of λ, then $\mathbf{l}'$ has type λ' and is called a *conjugate* of $\mathbf{l}$ [7]. For λ proper, define

$$\kappa_{\mathbf{l}'} := \sum_{\sigma \in (S_n)_{\mathbf{l}'}} \mathrm{sgn}(\sigma)\sigma \ \in \ ZS_n \text{ and } ZB\kappa_\lambda := (ZB\kappa_{\mathbf{l}'} ZS_n)^{\perp\perp}.$$

The (integral) *Specht module* $S^\lambda := \pi_\lambda(ZB\kappa_\lambda)$ is the central object of study in the module theoretic appoach to the symmetric group [3]. In this theory James defines a sequence of modules $S^{\lambda,\nu}$ when $\lambda_t \leq \nu_t$ for all $t > 1$. These (integral) *James modules* can be expressed in the above language as $(\mathbf{n}\kappa_{\mathbf{l}'} ZS_n)^{\perp\perp}$ where $\mathbf{n} \in B^\nu$ is obtained from $\mathbf{l}$ by lowering entries from (only) the first row of $\mathbf{l}$. An important result of [3] gives an algorithm for finding a submodule series of an arbitrary James module whose terms are Specht modules when coefficients are extended to a field K. This result holds also over Z.

Theorem 3 *The module* $ZB\kappa_\lambda$ *is a pure* ZS_n*-submodule of* ZB *depending only on* λ *not* $\mathbf{l}$ *or* $\mathbf{l}'$*. If* λ *is proper and* $\lambda \not\trianglerighteq \xi$ *and* $\phi \in \mathrm{Hom}_{ZS_n}(ZB^\xi, ZB^\nu)$, $\psi \in \mathrm{Hom}_{ZS_n}(ZB^\nu, ZB^\xi)$*, then* $ZB^\nu\kappa_\lambda \subseteq (\mathrm{im}\,\phi)^\perp \cap \ker\psi$*. For* λ, ν *proper:* $ZB^\nu\kappa_\lambda \neq 0$ *if and only if* $\lambda \trianglerighteq \nu$*. For* λ *proper,* $ZB^\nu\kappa_\lambda = \bigcap_{\lambda \not\trianglerighteq \xi \trianglerighteq \nu} \ker \pi_\xi\psi^\nu \ = \ \bigcap_{\lambda \not\trianglerighteq \xi} \ker \pi_\xi\psi^\nu$.

Corollary 4 *(cf. [7, 2.6]) If* $\rho \trianglerighteq \lambda$ *are proper then* $ZB^\nu\kappa_\rho \supseteq ZB^\nu\kappa_\lambda$*. The character afforded by* $QB^\nu\kappa_\lambda$ *is* $\sum_{\lambda \trianglerighteq \zeta \trianglerighteq \nu} k_{\zeta,\nu}\{\zeta\}$ *where the Kostka number* $k_{\zeta,\nu}$ *is the multiplicity of the irreducible character* $\{\zeta\}$ *in* QB^ν*. For* K *an arbitrary field, the dimension of* $KB^\nu\kappa_\lambda$ *is the degree of the above character.*

By extending coefficients to K, an arbitrary field, the module $ZB^\nu\kappa_\lambda$ and each of its submodules can be viewed as a linear code in KB^ν whose block length is the cardinality of B^ν and where (Hamming) distance is measured relative to the distinguished basis B^ν.

In order to establish lower bounds on the minimum distance of these codes, we construct parity checks with disjoint support. But care must be taken to insure that they are nontrivial in all characteristics. For this purpose, the characteristic functions of subsets of elements of the lattice introduced at the begining of this section is ideal. The simplest possible decoding (threshhold decoding) method appears in case the partitions ν and λ have just two parts.

Theorem 5 *Let* $\mathbf{i}, \mathbf{j} \in B$ *with* $\mathbf{i} \leq \mathbf{j}$ *and set* $\Phi_\nu(\mathbf{i}, \mathbf{j}) = \{\, \mathbf{n} \mid \mathbf{n} \sqcap \mathbf{j} = \mathbf{i},\ \mathrm{typ}(\mathbf{n}) = \nu \,\}$, $\Psi_\nu(\mathbf{i}, \mathbf{j}) = \{\, \mathbf{n} \mid \mathbf{n} \sqcup \mathbf{i} = \mathbf{j},\ \mathrm{typ}(\mathbf{n}) = \nu \,\}$*, and use the same symbol to denote the sum of all elements in each of these set. Then*

$$\pi_\nu \phi^{\mathrm{typ}(\mathbf{i})}(\mathbf{i}) = \sum_{\mathbf{k} \in [\mathbf{i}, \mathbf{j}]} \Phi_\nu(\mathbf{k}, \mathbf{j}) \ \textit{ and } \ \Phi_\nu(\mathbf{i}, \mathbf{j}) = \sum_{\mathbf{k} \in [\mathbf{i}, \mathbf{j}]} \mu(\mathbf{i}, \mathbf{k}) \pi_\nu \phi^{\mathrm{typ}(\mathbf{k})}(\mathbf{k}).$$

If $\lambda \triangleleft \mathrm{typ}(\mathbf{j})$ *then* $\Phi_\nu(\mathbf{i}, \mathbf{j}) \in (ZB^\nu\kappa_\lambda)^\perp$*. If* $\lambda \not\trianglerighteq \mathrm{typ}(\mathbf{i})$, $\mathrm{im}\, \pi_\nu \phi_\xi^{\mathrm{typ}(\mathbf{i})} \subseteq (ZB^\nu\kappa_\lambda)^\perp$*.* $P_{\nu,\xi}(\mathbf{j}) := \{\, \Phi_\nu(\mathbf{i}, \mathbf{j}) - \mu(\mathbf{i}, \mathbf{j}) \pi_\nu \phi^\xi(\mathbf{j}) \mid \mathbf{i} \in [\mathbf{0}, \mathbf{j}),\ \mu(\mathbf{i}, \mathbf{j}) \neq 0,\ \Phi_\nu(\mathbf{i}, \mathbf{j}) \neq 0 \,\}$ *forms a set of parity checks for* $ZB^\nu\kappa_\lambda$ *orthogonal on* $\pi_\nu \phi^\xi(\mathbf{j})$ *(in Massey's sense [9, p389]).*

Unfortunately, the claims concerning minimum weight and simple decoding algorithms made in [7, 3.4] assumed that in $P_{\nu,\lambda}(\mathbf{j})$, $\mu(\mathbf{i},\mathbf{j}) \neq 0$ for all $\mathbf{i} \in [\mathbf{0},\mathbf{j}]$. As shown in Lemma 1, this assumption is only correct for 2-part partitions. On the other hand, one of our main results provide proof of one of the claims made originally in [7, 3.4].

Theorem 6 *The minimum weight of $KB^\nu\kappa_\lambda$ equals the weight $\prod_{s\geq 1} s^{\lambda_s}$.*

A long recursive argument gives more explicit information than has been available heretofor.

Theorem 7 *The codewords of Specht module S^λ with minimal weight are the K-multiplies of the λ-polytabloids.*

We note that Theorem 7 does not extend to $ZB^\nu\kappa_\lambda$. Indeed, let $\lambda = \{4\ 2\}$ and $\nu = \{3\ 3\}$. Then $126 - 125 - 346 + 345 = (126 - 136 - 126 + 135) + (136 - 345 - 135 + 345)$ is the sum of two generators and has minimal weight by Theorem 6, but is itself not a generator.

When $m = 2$, each tabloid $\mathbf{j} \in B^\nu$ may be identified with the k-set that is its second row, or equivalently, with the k-set of positions of sequence $\mathbf{j}$ where $\mathbf{j}$ has entries 2. (The k-set associated with $\mathbf{j} \in B^\nu$ is also denoted by $\mathbf{j}$.) Note that for $m = 2$, the natural product order $\leq$ of the lattice $(B, \leq)$ then becomes set inclusion. Now any multiset of k-sets of $\{1, \ldots, n\}$ "is" an element of ZB^ν. For this reason, a wide variety of combinatorial constructions may be interpreted in our context.

In order to motivate this, consider a simple example. Let $\lambda = \nu = \{4, 3\}$. Then KB^λ can be regarded as the set of K-linear combinations of the 3-sets of the $\{1, \ldots, 7\}$. Theorem 5.5 provides seven parity checks for the Specht module $KB^\lambda\kappa_\lambda = S^\lambda$ orthogonal on 567 It is well known [4] that S^λ has dimension 14 and we showed that S^λ is a $(35, 14, 8)$-code in Theorem 7. This code is very easy to implement because each message symbol can be correctly decoded by a simple majority vote of such parity checks. This GF(2)-code is obtainable as a truncation of the classical Reed Muller code $R(3, 7)$ with parameters (128,99,8) and that any two 2-designs with parameters (7,3,1) (also known as Fano planes) have as their "difference" an element of this Z-code. It follows that a Fano plane is uniquely constructible from any 4 (but not any 3) lines. More generally we have:

Corollary 8 *A t-design S is uniquely reconstructible if fewer than 2^t blocks are lost.*

Remarkably enough this result is, in some sense, best possible. Indeed, consider the sum $S \in ZB^{4,3}$ of the 7 lines of a Fano plane above. Let σ be any transposition in the symmetric group S_7. Then S^σ is again a Fano plane on the same points but with a slightly different line set and $S - S^\sigma$ is of weight 8 in the Specht module $ZB^{4,3}\kappa_{4,3}$, so is actually a polytabloid by Theorem 7.

A combinatorial t-design is a collection of k-sets of an n-set with the property that each t-set $(t < k \leq n/2)$ is contained in a constant number λ of elements of the collection. Delsarte [9, 21.9] observed that the sum of the blocks of a combinatorial t-design is an element of the ambient space of the Bose Mesner algebra of the Johnson Scheme $J(n, k)$ that has trivial projection into the first through t-th representation (eigenspace). He then took this property to define t-designs in arbitrary Q-polynomial association schemes (those in which the representations are naturally totally ordered).

In contrast to the Johnson scheme, our incidence maps arise most naturally in the coherent configuration $\mathrm{Hom}_{ZS_n}(ZB, ZB)$. Since B^ν is an S_n-orbit, this coherent configuration has the sets B^ν as its fibres and we have been working in the fibre $\mathrm{Hom}_{ZS_n}(ZB^\nu, ZB^\nu)$ which is itself a (non commutative) association scheme based on the Young subgroups of fixed type ν. The trivial S_n-representation in ZB^ν is spanned by "the all ones vector" $\mathbf{J}^\nu := \pi_\nu\phi^{\{n\}}(\mathbf{0})$ which generates (by inspection) a pure submodule that is clearly analogous to the 0-th eigenspace for the Johnson scheme.

But the representations of $\mathrm{Hom}_{ZS_n}(ZB^\nu, ZB^\nu)$ are not totally ordered! Instead they possess domination as a natural partial order. In an effort to extend the combinatorial interpretation, we are led to define a *τ-design of type* ν is a subset $S \subset B^\nu$ such that $\pi_\tau\psi^\nu(S) = \lambda_\tau\mathbf{J}^\tau$ for some integer $\lambda > 0$ and, to avoid degeneracies, we require that $\nu_1 \geq n/2$ and $\tau \trianglerighteq \nu$. (As above, we use S to denote both the set S and the sum of all elements of S.) In order to maintain the convenient notation of the subject, further omit the first part of τ and ν writing simply "$\tau_2, \ldots, \tau_m$-design of type $\nu_2, \ldots, \nu_m$" rather than the more formal $\{\tau_1, \tau_2, \ldots, \tau_m\}$-design of type $\{\nu_1, \nu_2, \ldots, \nu_m\}$.

Higher type designs are best viewed in the"flag mode". Thus, for example, a $1, 1$-design S of type a, b, consists of a set of *blocks*, each of which is an ordered pair (X, Y) of sets, such that

$$Y \subseteq X \subseteq \{1, \ldots, n\},\ |X| = a + b,\ |Y| = b \text{ and } \pi_{\{n-2,1,1\}}\psi^{\{n-a-b,a+b,b\}}(S) = \lambda\,\mathbf{J}^{\{n-2,1,1\}}$$

for some integer λ. This means that for any ordered pair of distinct points (x, y) the number of blocks (X, Y) with $x \in X$, $y \in Y$ is always λ, independent of the choice of (x, y).

Since every t-design is automatically a $t - 1$-design [9, Thm. 2.9], it is natural to ask if a τ-design is also a λ-design whenever $\lambda \trianglerighteq \tau$. The Z-algebra $\mathcal{A}$ generated by $\{\pi_\nu\phi^\lambda, \pi_\nu\psi^\lambda\} \subseteq \mathrm{Hom}_{ZS_n}(ZB, ZB)$ is of central importance to address this question and to develop a theory of τ-designs analogous to that of t-designs. Unfortunately, the basic relations and structure constants of $\mathcal{A}$ are not as straightforward as in the Johnson scheme. For example, the intervals $[1122, 1233]$ and $[1122, 3123]$ contain a different number of tabloids of type $\{2\,1^2\}$.

We call a triple (λ, μ, ν) of partitions *balanced* if there is $a_{\lambda,\mu,\nu} \neq 0$ so that $\pi_\nu\phi^\mu\,\pi_\mu\phi^\lambda = a_{\lambda,\mu,\nu}\,\pi_\nu\phi^\lambda$, or equivalently, $\pi_\lambda\psi^\mu\,\pi_\mu\psi^\nu = a_{\lambda,\mu,\nu}\,\pi_\lambda\psi^\nu$. Thus $a_{\lambda,\mu,\nu}$ is the number of $\mathbf{m} \in B^\mu$ such that $\mathbf{n} \geq \mathbf{m} \geq \mathbf{l}$ for arbitrary chosen $\mathbf{n} \in B^\nu$ and $\mathbf{l} \in B^\lambda$ with $\mathbf{n} \geq \mathbf{l}$ and the requirement that $a_{\lambda,\mu,\nu} \neq 0$ implies that $\lambda \trianglerighteq \mu \trianglerighteq \nu$.

Given partitions $\lambda \trianglerighteq \nu$ there are many (but as shown above, not all) intermediate partitions μ for which (λ, μ, ν) is balanced. For instance, suppose λ, μ and ν are partitions of n and $\lambda_s = \mu_s = \nu_s$ for all $t - 1 \neq s \neq t$, and that $\lambda_{t-1} \geq \mu_{t-1} \geq \nu_{t-1}$. Then for $\mathbf{l} \in B^\lambda$, $\mathbf{m} \in B^\mu$ and $\mathbf{n} \in B^\nu$, the condition $\mathbf{l} \leq \mathbf{m} \leq \mathbf{n}$ implies that the associated flags $\mathbf{L} = \{L_0 \supseteq \ldots \supseteq L_{m-1}\}$, $\mathbf{M} = \{M_0 \supseteq \ldots \supseteq M_{m-1}\}$ and $\mathbf{N} = \{N_0 \supseteq \ldots \supseteq N_{m-1}\}$ coincide except that $L_t \subseteq M_t \subseteq N_t$ (or equivalently that the sequences $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ coincide at all entries different from $t - 1$ and t). Ignore the entries not in $L_{t-1} \setminus L_{t+1}$ and reduce to the situation of 2-part partitions. By, for example Wilson [10, 3.1], (λ, μ, ν) is balanced and $a_{\lambda,\mu,\nu} = \binom{\nu_t - \lambda_t}{\mu_t - \lambda_t}$.

Extend this example by defining *the natural balanced sequence $\{\lambda^{(t)}\}$ of partitions $\lambda^{(t)}$ from λ to ν* whenever $\lambda \trianglerighteq \nu$. Set $\lambda_s^{(t)} := \nu_s$ for $s < t$ and $\lambda_s^{(t)} := \lambda_s$ for $s > t$. Then $\lambda^{(t)}$ is actually a partition of n just in case $\lambda_t^{(t)} := n - \sum_{s>t}\lambda_s - \sum_{s<t}\nu_s$. Indeed, $\lambda_t^{(t)} = \lambda_t + \sum_{s<t}\lambda_s - \nu_s \geq 0$, since $\lambda \trianglerighteq \nu$. Moreover, $\lambda^{(t)} \trianglerighteq \lambda^{(t+1)}$ for all $t \geq 1$. Thus, for example, the natural balanced sequence from $\{5\,1\}$ to $\{2^2\,1^2\}$ is $\{5\,1\} \triangleright \{2\,4\} \triangleright \{2^3\} \triangleright \{2^2\,1^2\}$.

Theorem 9 *Let $\lambda \trianglerighteq \tau \trianglerighteq \nu$ and take $\mathbf{L} \leq \mathbf{T}$ for type$\mathbf{L} = \lambda$ and type$\mathbf{T} = \tau$. Let $\{\lambda^{(t)}\}, 1 \leq t \leq r$, be the natural balanced sequence of partitions from λ to τ. Then there exists a unique chain of flags $\mathbf{L} = \mathbf{L}^{(1)} \leq \ldots \leq \mathbf{L}^{(r)} = \mathbf{T}$ with $\mathbf{L}^{(t)}$ of type $\lambda^{(t)}$. Moreover, for example, $\pi_\tau \phi^\lambda = \pi_\tau \phi^{\lambda^{(r-1)}} \pi_{\lambda^{(r-1)}} \phi^{\lambda^{(r-2)}} \ldots \pi_{\lambda^{(2)}} \phi^\lambda$. Suppose (λ, τ, ν) is balanced and S is a τ-design of type ν. Then S is a λ-design of type ν and $\lambda_\lambda = \lambda_\tau a^{\sqcap}_{\lambda,\tau} / a_{\lambda,\tau,\nu}$ is given explicitly.*

Theorem 10 *Let $\mu = \{\tau_1 - 1, \tau_2 + 1, \tau_3, \ldots, \tau_m\}$. Then a τ-design S of type ν is uniquely reconstructible from $\{\pi_\xi \psi^\nu(S) \mid \xi \not\trianglerighteq \mu\}$, if fewer than $\prod_{s \geq 1} s^{\tau_s}$ blocks are lost.*

It is clear that the required list of"ξ-shadows" in Theorem 10 is much to large. At least the ζ for which there is a ξ such that (ζ, ξ, ν) is balanced are unnecessary.

Exactly what a minimal sufficient list might be is a very interesting question.

We turn finally to Geometric codes and come at last to the "algebraic mode" of presenting the natural permutation representations of S_n that was briefly mentioned earlier. Because some treatments of the classical Reed-Muller codes are based on functions, it seems necessary to post an immediate WARNING. Our polynomials are not functions, rather they are formal as in [1, 4.3] and in order to emphasize this we use the language of "generating functions".

Let $Z[x_1, \ldots, x_n]^{<m}$ denote the Z-polynomials in n variables $\{x_i\}$ of degree less than m in each variable. Recall that the monomial $p(\mathbf{n}) := \prod x_j^{n_j - 1} \in Z[x_1, \ldots, x_n]^{<m}$ is associated with the sequence $\mathbf{n} = (n_1, \ldots, n_n)$ and the associated tabloid has s-th part $\{j \mid n_j = s\}$. The type of $\mathbf{n}$ is ν where $\nu_s = \mid \{j \mid n_j = s\} \mid$.

The degree $\partial(\nu) = \sum(i-1)\nu_i$ of $p(\mathbf{n}), \mathbf{n} \in B^\nu$ depends only on ν (and equals the length of a covering chain from $\{n\}$ to ν in the domination poset of partitions of n). Unfortunately, the possible types ν for which $\partial(\nu) = k$ do not seem easy to describe in general and consequently clear connections with, for example, the polynomial codes of Kasami, Lin and Peterson [5] are awkward to establish. However, because the number of parts in a partition is bounded by m in this discussion, we are able to make some connections.

The incidence map ψ^ν takes $\mathbf{n} \in B^\nu$ to the the sum of all sequences that are termwise, less than or equal to $\mathbf{n}$. (or equivalently, all tabloids where j appears in a part no higher than n_j). These sequences are enumerated by the product
(choices for first position)(choices for second position) ... (choices for n-th position):

$$(1 + x_1 + \ldots + x_1^{n_1})(1 + x_2 + \ldots + x_2^{n_2}) \ \ldots \ (1 + x_n + \ldots + x_n^{n_n}).$$

Since this formula makes no explicit mention of ν, the Z-homomorphism

$$\psi : Z[x_1, \ldots, x_n]^{<m} \to Z[x_1, \ldots, x_n]^{<m} \text{ defined by } \psi(\prod x_j^{n_j}) = \prod \psi(x_j^{n_j}) = \prod \sum_{k=0}^{n_j} x^k$$

is a global algebraic version of the incidence map ψ. The other incidence map ϕ is given by the Z-homomorphism ϕ where $\phi(\prod x_j^{n_j}) = \prod \phi(x_j^{n_j}) = \prod x_j^{n_j} \psi(x_j^{m-1-n_j})$.

Let C_p denote the cyclic group of order p. Then $Z[x_1, \ldots, x_n]^{<p}$ may be identified with the group algebra ZC_p^n. In case p is prime, a t-*flat* is a coset of a subgroup of C_p^n having order p^t. It is helpful to extend this geometric language to the general case. For each subset $T \subseteq \{1, \ldots, n\}$ of cardinality t, define the associated *coordinate t-space* to be $\psi(\prod_{t \in T} x_t^{p-1}) = \phi(\prod_{s \notin T} x_s^{p-1})$. And say a *coordinate t-flat* is anything of the form (monomial) (coordinate t-space) $\in Z[x_1, \ldots, x_n]^{<p}$.

Theorem 11 *Let $\omega(r) = \{n - r\ 0^{p-2}\ r\}$ and set $E_r = \bigcap_{\nu \trianglerighteq \omega(r)} \ker \pi_\nu \psi = \langle \mathrm{im}\, \phi^\nu \mid \nu \trianglerighteq \omega(r) \rangle^\perp$. Then $\nu \trianglerighteq \omega(r)$ if and only if $\partial(\nu) \leq (p-1)r$. $E_r^\perp$ is generated by the characteristic funtions of the coordinate $(n-r)$-flats. Let $K = GF(p)$. Then $E_r \otimes_Z K$ is the p-ary $(n-r-1)$-th order Euclidean geometry code over K.*

Of course the Euclidean Geometry codes are well studied in their own right and they admit a multistep majority decoding scheme which yields complete decoding in case $p = 2$ [9]. It is quite instructive to compare this scheme with our poset approach. For this we content ourselves with the case $p = 2$.

Corollary 12 *Let $p = 2$ and K be an arbitrary field. Then the code $E_r \otimes_Z K$ admits a complete $(n-r)$-step majority decoding scheme.*

The"ν-puncture" of this multistep decoding algorithm survives from [7, 3.4].

The presentation of a reasonable decoding algorithm for the general $KB^\nu\kappa_\lambda$ remains open and seems to be a difficult but worthwhile problem.

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