THE COMBINATORICS OF q–CHARLIER POLYNOMIALS

ANNE DE MÉDICIS
DENNIS STANTON
DENNIS WHITE

ABSTRACT. We describe various aspects of the Al-Salam-Carlitz q-Charlier polynomials. These include combinatorial descriptions of the moments, the orthogonality relation, and the linearization coefficients.

1. Introduction.

The Charlier polynomials $C_n^a(x)$ are well-known analytically [4], and have been studied combinatorially by various authors [8], [12], [15], [16], [19]. The moments for the measure of these orthogonal polynomials are

$$\mu_n = \sum_{k=1}^{n} S(n,k)a^k, \tag{1.1}$$

where $S(n,k)$ are the Stirling numbers of the second kind. The purpose of this paper is to study combinatorially an appropriate q-analogue of $C_n^a(x)$, whose moments are a q-Stirling version of (1.1). It turns out that our q-analogues are not what have classically been called q-Charlier; in fact they are rescaled versions of the Al-Salam-Carlitz polynomials [4]. Zeng [23] has also studied these polynomials from the associated continued fractions. While studying these polynomials, we use statistics on set partitions which are q-Stirling distributed.

In the $q=1$ case, the linearization coefficients are given as a polynomial in a, whose coefficients are quotients of factorials (see (4.4)). This has a simple combinatorial explanation. However, in the q-case the coefficients are not the analogous quotients of q-factorials (see Theorem 3). They are alternating sums of quotients of q-factorials, and thus a combinatorial explanation is much more difficult. We provide such an explanation in this paper.

The basic combinatorial interpretation of the polynomials is given in Theorem 1. Several facts about the polynomials can be proven combinatorially. The statistic for the moments is given in Theorem 2. The linearization problem is discussed in §4 and the linearization coefficient for a product of three q-Charlier polynomials is given in Theorem 3. Some comparisons to the classical q-Charlier are given in §5.

We use the standard notation for q-binomial coefficients and shifted factorials found in [11]. We will also need

$$[n]_q = \frac{1-q^n}{1-q},$$

and

$$[n]!_q = [n]_q[n-1]_q \cdots [1]_q.$$

2. The q-Charlier polynomials.

We define the q-Charlier polynomials by the three term recurrence relation

$$C_{n+1}(x,a;q) = (x - aq^n - [n]_q)C_n(x,a;q) - a[n]_q q^{n-1} C_{n-1}(x,a;q), \tag{2.1}$$

where $C_{-1}(x,a;q) = 0$ and $C_0(x,a;q) = 1$.

It is not hard to show that these polynomials are rescaled versions of the Al-Salam Carlitz polynomials [4]

$$C_n(x,a;q) = a^n U_n\left(\frac{x}{a} - \frac{1}{a(1-q)}, \frac{-1}{a(1-q)}\right). \tag{2.2}$$

Since the generating function of the $U_n(x,b)$ is known [4], we see that

$$\sum_{n=0}^{\infty} C_n(x,a;q)\frac{t^n}{(q)_n} = \frac{(at)_\infty(-\frac{t}{1-q})_\infty}{(t(x-\frac{1}{1-q}))_\infty}. \tag{2.3}$$

This gives the explicit formula

$$C_n(x,a;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{n-k}{2}} \prod_{i=0}^{k-1}(x - [i]_q). \tag{2.4}$$

Clearly, we want a q-version of [15], which gives the Charlier polynomials as a generating function of weighted *partial permutations*, i.e. pairs (B,σ), where $B \subseteq \{1,2,\cdots,n\} = [n]$, and σ is a permutation on $[n] - B$. Thus we need only interpret the individual terms in (2.4) for a combinatorial interpretation. The inside product can be expanded in terms of the q-Stirling numbers of the first kind. We let $cyc(\sigma)$ be the number of cycles of a permutation σ and $inv(\sigma)$ be the number of inversions of σ written as a product of disjoint cycles (increasing minima, minima first in a cycle).

$$\prod_{i=0}^{k-1}(x - [i]_q) = \sum_{\sigma \in S_k} (-1)^{k-cyc(\sigma)} q^{inv(\sigma)} x^{cyc(\sigma)}.$$

For the sum over k in (2.4), we sum over all $(n-k)$ subsets $B \subseteq [n]$. Let

$$inv(B) = q^{\sum_{b\in B}(b-1)},$$

so that the generating function for these subsets is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}.$$

We have established the following theorem.

Theorem 1. *The q-Charlier polynomials are given by*

$$\begin{aligned} C_n(x,a;q) &= \sum_{B\subseteq[n]} \sum_{\sigma\in\mathfrak{S}_{n-B}} x^{cyc(\sigma)} q^{inv(\sigma)+inv(B)} (-1)^{n-cyc(\sigma)} a^{|B|}, \\ &= \sum_{B\subseteq[n]} \sum_{\sigma\in\mathfrak{S}_{n-B}} \omega_q(B,\sigma) x^{cyc(\sigma)}. \end{aligned}$$

A combinatorial proof of the three-term recurrence relation (2.1) can be given using Theorem 1. An involution is necessary. For more details, we refer the reader to [5].

3. The moments.

An explicit measure for the q-Charlier polynomials is known, [4]. It is not hard to find the n^{th} moment of this measure explicitly. The result is a perfect q-analogue of (1.1)

$$\mu_n = \sum_{k=1}^{n} S_q(n,k) a^k, \tag{3.1}$$

where $S_q(n,k)$ is the q-Stirling number of the second kind, given by the recurrence

$$S_q(n,k) = S_q(n-1,k-1) + [k]_q S_q(n-1,k), \tag{3.2}$$

where $S_q(0,k) = \delta_{0,k}$. In fact, one sees that [13]

$$S_q(n,k) = \frac{1}{(1-q)^{n-k}} \sum_{j=0}^{n-k} \binom{n}{k+j} \begin{bmatrix} k+j \\ j \end{bmatrix}_q (-1)^j. \tag{3.3}$$

Clearly (3.1) suggests that there is some statistic on set partitions, whose generating function is μ_n. This statistic, rs, arises from the Viennot theory of Motzkin paths associated with the three-term recurrence (2.1) [19]. We do not give the details of the construction here.

However, let us quickly review some combinatorial facts about q-Stirling numbers that shall be useful to us. Set partitions of $[n] = \{1,2,\ldots,n\}$ can be encoded as *restricted growth functions* (or *RG-functions*) as follow: if the blocks of π are ordered by increasing minima, the RG-function $w = w_1w_2\ldots w_n$ is the word such that w_i is the block where i is located. For example, if $\pi = 147|28|3|569$, $w = 123144124$. Note that set partitions on any set A can be encoded as RG-functions as long as A is a totally ordered set.

In [20], Wachs and White investigated four natural statistics on set partitions, called ls, lb, rs and rb. They are defined as follow:

$$\begin{aligned} ls(\pi) = ls(w) &= \sum_{i=1}^{n} |\{j : j < w_i, j \text{ appears to the left of position } i\}|, \\ lb(\pi) = lb(w) &= \sum_{i=1}^{n} |\{j : j > w_i, j \text{ appears to the left of position } i\}|, \\ rs(\pi) = rs(w) &= \sum_{i=1}^{n} |\{j : j < w_i, j \text{ appears to the right of position } i\}|, \\ rb(\pi) = ls(w) &= \sum_{i=1}^{n} |\{j : j > w_i, j \text{ appears to the right of position } i\}|. \end{aligned}$$

Thus in the example, $ls(\pi) = 13$, $lb(\pi) = 7$, $rs(\pi) = 7$ and $rb(\pi) = 11$. They showed, using combinatorial methods, that each had the same distribution (up to a constant) on the set $RG(n,k)$ of all restricted growth functions of length n and maximum k, and that their generating function was indeed $S_q(n,k)$ for rs and lb (respectively $q^{\binom{k}{2}}S_q(n,k)$ for ls and rb).

We also use another encoding of set partitions in terms of *0–1 tableaux*. A 0–1 tableau is a pair $\varphi = (\lambda, f)$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of an integer $m = |\lambda|$ and $f = (f_{ij})_{1\leq j\leq\lambda_i}$ is a "filling" of the corresponding Ferrers diagram of shape λ with 0's and 1's such that there is exactly one 1 in each column. 0–1 tableaux were introduced by Leroux in [17] to establish a q-log concavity result conjectured by Butler [3] for Stirling numbers of the second kind.

There is a natural correspondence between set partitions π of $[n]$ in k blocks and 0–1 tableaux with $n-k$ columns of length less than or equal to k. Simply write the RG-function $w = w_1 w_2 \ldots w_n$ associated to π as a $k \times n$ matrix, with a 1 in position (i,j) if $w_i = j$, and 0 elsewhere. The resulting matrix is row-reduced echelon, of rank k, with exactly one 1 in each column. A 0–1 tableau (in the third quadrant) is then obtained by removing all the pivot columns and the 0's that lie on the left of a 1 on a pivot column. Figure 1 illustrates these manipulations for $\pi = 1247|39(12)|568(11)|(10)$.

$$\begin{pmatrix} 1&1&0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&1&1&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0 \end{pmatrix}$$

$$\Updownarrow$$

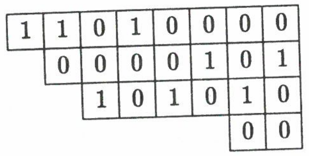

Figure 1: Correspondence between partitions and 0–1 tableaux

We define two statistics on 0–1 tableaux φ: first, the *inversion number*, $inv(\varphi)$, which is equal to the number of 0's below a 1 in φ; and the *non-inversion number*, $nin(\varphi)$, which is equal to the number of 0's above a 1 in φ. For example, for φ in figure 1, $inv(\varphi) = 7$ and $nin(\varphi) = 8$. Note that an easy involution on the columns of 0–1 tableaux sends the inversion number to the non-inversion number and vice-versa. We call this map the *symmetry involution*.

It is not hard to see that the inversion number (respectively non-inversion number) on 0–1 tableaux corresponds to the statistic lb (resp. $ls - \binom{k}{2}$) on set partitions.

Similarly, permutations σ of $[n]$ in k cycles can be encoded as 0–1 tableaux with $n-k$ columns of distinct lengths less than or equal to $n-1$. The correspondence is defined by recurrence on n. Suppose σ is written as a standard product of cycles. If $n = 1$, then $\sigma = (1)$ corresponds to the empty 0–1 tableau $\varphi = \emptyset$. Otherwise, let $\sigma \in \mathfrak{S}_{n+1}$ and let φ denote the 0–1 tableau associated to the permutation σ in which $(n+1)$ has been erased. There are two cases. If $(n+1)$ is the minimum of a cycle in σ, then σ corresponds to φ. If $(n+1)$ is not the minimum of a cycle, then it appears in σ at a certain position i, $2 \le i \le n+1$. The permutation σ then corresponds to the 0–1 tableau φ plus a column of length n with a 1 in the $(i-1)$-th position (from top to bottom). For example, $\sigma = (1,3,4,7,2)(5,6)(8)$ corresponds to the following 0–1 tableau.

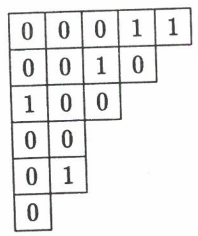

Figure 2: Correspondence between permutations and 0–1 tableaux

It is not hard to see that under this transformation, the inversion number on 0–1 tableaux corresponds to the inversion number on permutations, as defined in section 2. Thus, their generating functions are the q-Stirling numbers of the first kind $c_q(n,k)$.

In [6], de Médicis and Leroux investigated q and p,q-Stirling numbers from the point of view of the unified 0–1 tableau approach. In particular, they proved combinatorially or algebraically a certain number of identities involving q-Stirling numbers.

Now, for the combinatorial interpretation of the moments of the q-Charlier polynomials in terms of set partitions π, we need two statistics. The number of blocks $\#blocks(\pi)$ is one, and the other statistic is $rs(\pi)$.

Theorem 2. *The n^{th} moment for the q-Charlier polynomials is given by*

$$\mu_n = \sum_{\pi \in P(n)} a^{\#blocks(\pi)} q^{rs(\pi)}.$$

As we mentioned, many other q-Stirling distributed statistics have been found [20]. It is surprising that the Viennot theory naturally gives a so-called "hard" statistic (rs), not an easy one (e.g. lb, [20]). Other variations on the rs-statistic can be given from the Motzkin paths, although the lb-statistic is not among them. It can be derived from the Motzkin paths associated with the "odd" polynomials for (2.1).

4. The orthogonality relation and the linearization of products.

Let L be the linear functional on polynomials that corresponds to integrating with respect to the measure for the Charlier polynomials. The orthogonality relation is

$$L(C_n^a(x)C_m^a(x)) = a^n n!\ \delta_{m,n}. \tag{4.1}$$

The q-version of (4.1) is

$$L_q(C_n(x,a;q)C_m(x,a;q)) = a^n q^{\binom{n}{2}} [n]!_q \delta_{m,n}. \tag{4.2}$$

Since the polynomials $C_n(x,a;q)$ and L_q have combinatorial definitions from Theorems 1 and 2, it is possible to restate (4.2) as a combinatorial problem. We can give an involution which then proves (4.2) in this framework.

A more general question is to find $L(C_{n_1}^a(x)C_{n_2}^a(x)\cdots C_{n_k}^a(x))$ for any k. A solution is equivalent to finding the coefficients a_{n_k} in the expansion

$$C_{n_1}^a(x)C_{n_2}^a(x)\cdots C_{n_{k-1}}^a(x) = \sum_{n_k} a_{n_k} C_{n_k}^a(x).$$

This had been done bijectively for some classes of Sheffer orthogonal polynomials in [7], [9], [10]. Moreover, in the q-case of Hermite polynomials, some remarkable consequences have been found [14].

For the Charlier polynomials, it is easy to see that

$$\sum_{n_1,\cdots,n_k=0}^{\infty} L(C_{n_1}^a(x)C_{n_2}^a(x)\cdots C_{n_k}^a(x))\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_k^{n_k}}{n_k!} = e^{a(e_2(t_1,\cdots,t_k)+\cdots+e_k(t_1,\cdots,t_k))}, \tag{4.3}$$

where e_i is the elementary symmetric function of degree i, [18]. In this case $L(C_{n_1}^a C_{n_2}^a \cdots C_{n_k}^a)$ is a polynomial in a with positive integer coefficients; a combinatorial interpretation of this coefficient has been given ([12] and [22]). For $k=3$, (4.3) is equivalent to

(4.4)

$$L(C_{n_1}^a(x)C_{n_2}^a(x)C_{n_3}^a(x)) = \sum_{l=0}^{\lfloor (n_1+n_2-n_3)/2 \rfloor} \frac{a^{n_3+l} n_1! n_2! n_3!}{l!(n_3-n_2+l)!(n_3-n_1+l)!(n_1+n_2-n_3-2l)!}.$$

One can hope that $L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))$ is simply a weighted version, with an appropriate statistic, of the $q=1$ case. However this is false. For example,

$$L_q(C_2(x)C_2(x)C_1(x)) = q(q^2+2q+1)a^2 + q(q^3+q^2-q-1)a^3.$$

Nonetheless, we have an exact formula for $L_q(C_{n_1}(x,a,q)C_{n_2}(x,a,q)C_{n_3}(x,a,q))$, which is equivalent to one of Al-Salam-Verma [1].

Theorem 3. *Let $n_3 \geq n_1 \geq n_2 \geq 0$. Then*

$$L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x)) = \sum_{l=0}^{n_1+n_2-n_3} \sum_{j=0}^{l} a^{n_3+l} q^K (q-1)^{l-j} \frac{[n_1-j]!_q}{[n_1-l]!_q} \begin{bmatrix} n_2 \\ l-j \end{bmatrix}_q$$

$$(4.5) \qquad [n_3]!_q \begin{bmatrix} n_1 \\ j \end{bmatrix}_q \begin{bmatrix} n_2-l+j \\ n_3-n_1+j \end{bmatrix}_q \frac{[j]!_q[n_1-j]!_q}{[n_3-n_2+l]!_q} \begin{bmatrix} n_1+n_2-n_3-l \\ j \end{bmatrix}_q,$$

where

$$K = \binom{l-j}{2} + \binom{n_1}{2} + j(-n_3-j+1) + \binom{j}{2} + \binom{n_2-l+j}{2} + (n_3-n_1+j)(n_3-n_2+l) + j(n_3-n_2+l).$$

Sketch of the proof. Define

$$L_q(n_1,n_2,n_3) = \{((B_i,\sigma_i);\pi) = ((B_1,\sigma_1),(B_2,\sigma_2),(B_3,\sigma_3);\pi) \mid (B_i,\sigma_i) \text{ is a partial permutation on the set } \{i\} \times [n_i], \text{ and } \pi \text{ is a partition on the cycles of } \sigma_1, \sigma_2 \text{ and } \sigma_3 \}.$$

Note that the lexicographic order on pairs (i,j) induces a total order on the cycles of σ_1, σ_2 and σ_3, according to their minima. Therefore we can talk about RG-functions. Let w denote the RG-function associated to π. The first $cyc(\sigma_1)$ letters of w correspond to the positions of cycles of color 1 in π, the next $cyc(\sigma_2)$ to the positions of cycles of color 2, and the last $cyc(\sigma_3)$ letters to the positions of cycles of color 3. We will denote by w_a, w_b and w_c respectively these portions of $w = w_a w_b w_c$. Finally, we will use the notation $Supp(w)$ to denote the underlying set of letters of a word w.

From Theorems 1 and 2, we deduce that

$$(4.6) \qquad L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x)) = \sum_{((B_i,\sigma_i);\pi) \in L_q(n_1,n_2,n_3)} \omega_q((B_i,\sigma_i);\pi),$$

where

$$(4.7) \qquad \omega_q((B_i,\sigma_i);\pi) = \omega_q(B_1,\sigma_1)\omega_q(B_2,\sigma_2)\omega_q(B_3,\sigma_3) q^{rs(\pi)} a^{\#blocks(\pi)}.$$

The weight of each $((B_i,\sigma_i);\pi)$ is a signed monomial in the variables a and q. For $q=1$, the negative coefficients of a are counterbalanced by the positive coefficients of a, and (4.6) is a polynomial with positive coefficients. Indeed, in that case, it is not hard to find a weight-preserving sign-reversing involution on $L_q(n_1,n_2,n_3)$ (cf [5]) whose fixed points $((B_i,\sigma_i);\pi)$ are characterized by

i) $B_i = \varnothing$ and $\sigma_i =$ Identity, for $i = 1,2,3$;
ii) the word w_a (respectively w_b and w_c) contains all distinct letters, and $Supp(w_a) \subseteq Supp(w_b w_c)$ (respectively $Supp(w_b) \subseteq Supp(w_a w_c)$ and $Supp(w_c) \subseteq Supp(w_a w_b)$).

Identity (4.4) easily follows from ω_1-counting these fixed points.

However, the general q-case is much harder, and although most negative weights can still be annihilated, some negative weights remain. The sign of $\omega_q((B_i,\sigma_i);\pi)$ comes from the cardinalities of the sets B_i and the signs of the permutations σ_i. In our proof, we successively apply five weight-preserving sign-reversing involutions Φ_i to $L_q(n_1,n_2,n_3)$, each one acting on the fixed points of the preceding one. The final set of fixed points, $Fix\Phi_5$, contains no $((B_i,\sigma_i);\pi)$ such that B_1 or B_3 is non-empty or such that any permutation σ_i is not equal to the identity. Hence the negative part of (4.6) is due only to $B_2 \neq \varnothing$.

We do not describe the involutions here due to lack of space. Let us mention though that some of them are quite straight forward, while some others use more sophisticated techniques, such as interpolating between the statistics rs and ls on partitions and encoding as 0–1 tableaux. Interpolating statistics were studied at length by White in [21]. We need a bijection Ψ_S from this paper in order to describe the exact set of fixed points $Fix\Phi_5$. More precisely, define $\Psi_i : RG(n,k) \to RG(n,k)$, $1 \leq i \leq k-1$ as follow:

i) if $w \in RG(n,k)$ has a letter i to the right of the first occurrence of $(i+1)$, then the rightmost letter i is switched to $(i+1)$ and any $(i+1)$ to its right is changed to i. For example, $\Psi_1(111212332122) = 111212332211$.
ii) if w doesn't have a letter i to the right of the first occurrence of $(i+1)$, then all $(i+1)$'s to its right are switched to i's. For example, $\Psi_1(1112232) = 1112131$.

For convenience, we will set $\Psi_k : RG(n,k) \to RG(n,k)$ to be the identity. Now, given $S = \{s_1 < s_2 < \ldots < s_m\} \subseteq [k]$, Ψ_S is defined as follow:

$$\Psi_S = (\Psi_k \circ \Psi_{k-1} \circ \ldots \circ \Psi_{s_1}) \circ (\Psi_k \circ \Psi_{k-1} \circ \ldots \circ \Psi_{s_2}) \circ \ldots \circ (\Psi_k \circ \ldots \circ \Psi_{s_m}).$$

The final set of fixed points $Fix\Phi_5$ contains all $((B_i,\sigma_i);\pi) \in L_1(n_1,n_2,n_3)$ such that

Fix.1 $B_1 = B_3 = \varnothing$,
Fix.2 $\sigma_i = Id$ for $i = 1,2,3$,
Fix.3 w_a (respectively w_c) has all distinct letters and $Supp(w_a) \subseteq Supp(w_b w_c)$ (respectively $Supp(w_c) \subseteq Supp(w_a w_b)$),
Fix.4 for $S = [\#blocks(\pi)] - Supp(w_c)$, the word $\tilde{w}_b = \tilde{b}_1 \tilde{b}_2 \ldots \tilde{b}_{n_2-|B_2|}$ in $\Psi_S(w_a w_b) = w_a \tilde{w}_b$ has all $\tilde{b}_i$'s $\geq i$.

Clearly, for such elements, the weight (as was defined in (4.7)) reduces to

$$\omega_q((B_i,\sigma_i):\pi) = (-1)^{|B_2|} q^{inv(B_2)+rs(\pi)} a^{|B_2|+\#blocks(\pi)}.$$

By ω_q-counting this fixed point set, we find

$$(4.8) \qquad L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x)) = \sum_{l=0}^{n_1+n_2-n_3} \sum_{s=0}^{l} \sum_{j=0}^{s} a^{n_3+l}(-1)^{l-s} q^{KK} [n_3]!_q \begin{bmatrix} n_2 \\ l-s \end{bmatrix}_q$$

$$\begin{bmatrix} n_1 \\ j \end{bmatrix}_q \begin{bmatrix} n_3-n_1+s \\ s-j \end{bmatrix}_q \begin{bmatrix} n_2-l+s \\ n_3-n_1+s \end{bmatrix}_q \frac{[j]!_q[n_1-j]!_q}{[n_3-n_2+l]!_q} \begin{bmatrix} n_1+n_2-n_3-l \\ j \end{bmatrix}_q,$$

where

$$KK = \binom{n_1}{2} + \binom{l-s}{2} + j(-n_3-s+1) + \binom{j}{2} - \binom{s-j}{2} - (s-j)(n_3-n_1+j) + \binom{n_2-l+s}{2} + (n_3-n_1+s)(n_3-n_2+l) + j(n_3-n_2+l).$$

Evaluating the s-sum by the q-binomial theorem (which has a simple bijective proof) gives (4.5) and thus Theorem 3. □

Note that if we take $n_2 = 0$, the set $Fix\Phi_5$ is easily seen to be empty unless $n_1 = n_3$. This proves the orthogonality relation (4.2) and Theorem 2.

5. The classical q-Charlier polynomials.

We contrast the results of the previous sections with those for the classical q-Charlier polynomials [11, p. 187]

$$c_n(x;a;q) = {}_2\phi_1(q^{-n}, x; 0; q, -q^{n+1}/a). \tag{5.1}$$

The monic form of these polynomials, $cc_n(x;a;q)$ satisfies

$$cc_{n+1}(x;a;q) = (x - b_n)cc_n(x;a;q) - \lambda_n cc_{n-1}(x;a;q),$$

where

$$\lambda_n = -aq^{1-2n}(1-q^{-n})(1+aq^{-n}), \quad b_n = aq^{-1-2n} + q^{-n} + aq^{-2n} - aq^{-n}.$$

A calculation (see [11, p. 187]) shows that the moments for these polynomials are

$$\mu_n = \prod_{i=1}^{n}(1 + aq^{-i}).$$

We need to rescale x and a so that b_n and λ_n are q-analogues of $a + n$ and an respectively. If we put $x = 1 + z(1-q)$, and multiply a by $(1-q)$, and call the resulting monic polynomials $\hat{C}_n(z;a;q)$, the explicit formula from (5.1) is

$$\hat{C}_n(z;a;q) = q^{-n^2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^{n-k} q^{\binom{k+1}{2}} \prod_{i=0}^{k-1}(q^i z - [i]_q) \tag{5.2}$$

The three term recurrence relation coefficients are

$$b_n = q^{-n}[n]_q(1 + a(1-q)q^{-n}) + aq^{-1-2n}, \quad \lambda_n = aq^{1-3n}[n]_q(1 + a(1-q)q^{-n}). \tag{5.3}$$

A calculation using the measure in [11, p.187] gives

$$\mu_n = \sum_{j=1}^{n} q^{-\binom{j}{2}-n} S_{1/q}(n,j)a^j. \tag{5.4}$$

Again we find q-Stirling numbers for the moments. Zeng [23] has also derived (5.2) and (5.3) from the continued fraction for the moment generating function.

We see that the individual terms in (5.3) do not have constant sign. This means that the Viennot theory must involve a sign-reversing involution for its combinatorial versions of (5.3) and (5.4). Nonetheless we can give combinatorial interpretations of (5.2) and (5.4), but have no perfect analog of Theorem 3.

6. Remarks.

We obtain some corollaries of Theorem 3.

Corollary 4. *Let $n_1 \geq n_2 \geq \ldots \geq n_k$. The coefficient of the lowest power of a, a^{n_1} in $L_q(C_{n_1}C_{n_2}\ldots C_{n_k})$ is a polynomial in q with positive coefficients.*

Corollary 5. *Let $n_3 \geq n_1 \geq n_2$. The coefficient of $a^{n_1+n_2-i}$ in $L_q(C_{n_1}C_{n_2}C_{n_3})$ is equal to $(q-1)^{n_1+n_2-n_3-2i}$ times the coefficient of a^{n_3+i}, for $0 \leq i \leq \lfloor (n_1+n_2-n_3)/2 \rfloor$.*

Our proof of Corollary 5 is analytical, but we would like to have a combinatorial explanation of this "symmetry" property.

Note that $Fix\Phi_5$ is not an optimal set of fixed points, in the sense that there are still some terms that cancel each other when we proceed to ω_q-counting of $Fix\Phi_5$. For example, for $n_1 = n_2 = n_3 = 2$, the two elements of $Fix\Phi_5$ such that $B_2 = \{(2,2)\}$, $w = 12121$ and $B_2 = \varnothing$, $w = 123123$ have weight $-a^3q^3$ and a^3q^3 respectively. However, we do not believe that an attempt to reduce $Fix\Phi_5$ would be worthwhile.

Corollary 6. *Let $n_1 \geq n_2 \geq \ldots \geq n_k$. If $q = 1 + r$, $L_q(C_{n_1}C_{n_2}\ldots C_{n_k})$ is a polynomial in r with positive coefficients.*

Finally, it can be shown from Theorem 3 that

$$\sum_{n_1,n_2,n_3} L_q(C_{n_1}(x)C_{n_2}(x)C_{n_3}(x))\frac{t_1^{n_1}}{[n_1]!_q}\frac{t_2^{n_2}}{[n_2]!_q}\frac{t_3^{n_3}}{[n_3]!_q} =$$

$$(-t_3;q)_\infty(-at_1t_2(1-q);q)_\infty\, {}_2\Phi_1\left(\begin{matrix} at_1(1-q),\ at_2(1-q) \\ -at_1t_2(1-q) \end{matrix}; q, -t_3\right). \tag{6.1}$$

Letting $q \to 1$ in (6.1) gives back (4.2) for $k = 3$.

Acknowledgement. Theorems 1 and 2 were originally found in joint work with Mourad Ismail.

References

1. W. Al Salam and D. Verma, private communication (1988).
2. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs Amer. Math. Soc. **319** (1985).
3. L. Butler, *The q-log concavity of q-binomial coefficients*, J. of Comb. Theory A **54** (1990), 53–62.
4. T.S. Chihara, *An Introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.
5. A. de Médicis, *Aspects combinatoires des nombres de Stirling, des polynômes orthogonaux de Sheffer et de leurs q-analogues*, ISBN 2-89276-114-X, vol. 13, Publications du LACIM, UQAM, Montréal, 1993.
6. A. de Médicis and P. Leroux, *A unified combinatorial approach for q-(and p,q-)Stirling numbers*, J. of Stat. Planning and Inference **34** (1993), 89–105.
7. M. de Sainte-Catherine and G. Viennot, *Combinatorial interpretation of integrals of products of Hermite, Laguerre and Tchebycheff polynomials*, Polynômes Orthogonaux et Applications, Lecture Notes in Math., vol. 1171, Springer-Verlag, 1985, pp. 120–128.
8. D. Foata, *Combinatoire des identités sur les polynômes orthogonaux*, Internat. Congress Math. (1983), Warshaw, Poland.
9. D. Foata and D. Zeilberger, *Laguerre polynomials, weighted derangements and positivity*, SIAM J. Discrete Math. **1** (1988), 425–433.
10. ———, *Linearization coefficients for the Jacobi polynomials*, Actes 16e Séminaire Lotharingien **341/S-16** (1987), 73–86, I.R.M.A., Strasbourg.
11. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of mathematics and its applications, vol. 35, Cambridge University Press, New York, 1990.
12. I. Gessel, *Generalized rook polynomials and orthogonal polynomials*, q-Series and Partitions (D. Stanton, ed.), IMA Volumes in Math. and its Appl., vol. 18, Springer-Verlag, New York, 1989, pp. 159–176.
13. H.W. Gould, *The q-Stirling Numbers of First and Second Kinds*, Duke Math. J. **28** (1961), 281–289.
14. M. Ismail, D. Stanton and X.G. Viennot, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, Europ. J. Comb. **8** (1987), 379–392.
15. J. Labelle and Y.N. Yeh, *The combinatorics of Laguerre, Charlier and Hermite polynomials*, Studies in Applied Math. **80** (1989), 25–36.
16. ———, *Combinatorial proofs of some limit formulas involving orthogonal polynomials*, Discrete Math. **79** (1989), 77–93.
17. P. Leroux, *Reduced matrices and q-log concavity properties of q-Stirling numbers*, J. of Comb. Theory A **54** (1990), 64–84.
18. I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford, 1979.
19. X.G. Viennot, *Une Théorie Combinatoire des Polynômes Orthogonaux*, Lecture Notes, Publications du LACIM, UQAM, Montréal, 1983.
20. M. Wachs and D. White, *p,q-Stirling Numbers and Set Partition Statistic*, J. Comb. Theory Ser. A **56** (1991), 27–46.
21. D. White, *Interpolating Set Partition Statistics*, preprint (1992).
22. J. Zeng, *Weighted derangements and the linearization coefficients of orthogonal Sheffer polynomials*, Proc. London Math. Soc. **65** (1992), 1–22.
23. ———, *The q-Stirling numbers, continued fractions and the q-Charlier and q-Laguerre polynomials*, preprint (1993).

School of Mathematics, University of Minnesota, Minneapolis, MN 55455.