10. Conclusions

We made some numerical experiments and obtained good agreement with the predicted theoretical results.

Without going into details, probabilistic counting resembles *coupon collecting* (compare [3]), and we think that there is some work to be done analyzing combinatorial parameters in the context of geometrically distributed random variables. We hope to report on some other problems and results in the future.

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Young-Derived Sequences of S_n-Characters and Their Asymptotics

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Given the sequence $\psi = \{\psi_n \text{ is an } S_n \text{ character }\}_{n\geq 0}$, we construct the Young derived sequence $y(\psi) = \{y_n(\psi)\}_{n\geq 0} : y_n(\psi) = \sum_{j=0}^{n} \psi_j \hat{\otimes} \chi_{(n-j)}$.

We study the relations between ψ and $y(\psi)$, and between deg ψ_n and deg $(y_n(\psi))$, when ψ is supported on a strip. Their asymptotics, as $n \to \infty$, leads to some interesting integration formulas. Part of the work reviewed here was done in collaboration with W. A. Beckner and with A. Berele.

Let S_n denote the symmetric group, and assume throughout that the characteristic of the base field is zero, so that the ordinary representation theory of S_n can be applied.

Consider a sequence $\psi = \{\psi_n\}_{n=0}^{\infty}$ where each ψ_n is an S_n character, $n = 0, 1, 2, \ldots$. One is interested in the decomposition of ψ : $\psi_n = \sum_{\lambda\vdash n} a(\lambda)\chi_\lambda$, where χ_λ is the irreducible S_n character which corresponds to the partition λ, and $a(\lambda)$ is its multiplicity in ψ_n.

Given such $\psi = \{\psi_n\}_{n\geq 0}$, we construct its "Young derived sequence $\phi = \{\phi_n\}_{n\geq 0}$ via $\phi_n = \sum_{j=0}^{n} \psi_j \hat{\otimes} \chi_{(n-j)}$. Here $\chi_{(\ell)}$ is the trivial S_ℓ character, and $\hat{\otimes}$ denotes the outer product. The terms $\psi_j \hat{\otimes} \chi_{(n-j)}$ can be calculated by Young's rule.

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Let $\phi = y(\psi)$, $\psi_n = \sum_{\lambda \vdash n} a(\lambda)\chi_\lambda$ and $\phi_n = \sum_{\mu \vdash n} b(\mu)\chi_\mu$, then in general, the $b(\mu)$'s are much more complicated to describe (even asymptotically) than the $a(\lambda)$'s. However, Young's rule provides a simple way to express the $b(\mu)$'s in terms of the $a(\lambda)$'s. Thus, a satisfactory description of the decompositions of ψ implies a satisfactory description of the decompositions of $\phi = y(\psi)$.

Some well known character sequences are Young derived. For example, the characters of the classical representations of S_n on $V^{\otimes n}$ form a sequence which is dim V times Young derived – from the trivial sequence [6, 1.4].

The cocharacter-sequence of a P.I. algebra (i.e., an algebra that satisfies polynomial identities) is always Young derived [3].

An interesting example of such sequences arise when $\phi = \phi_{(k)}$ is given by $\phi_{(k),n} = \sum_{\lambda \in \wedge_k(n+1)} (\chi_\lambda \otimes \chi_\lambda)_{S_n}$. Here $\otimes$ is the inner (Kronecker) product, and $\wedge_k(m) = \{\lambda = (\lambda_1, \lambda_2, \dots) \vdash m \mid \lambda_{k+1} = 0\}$.

The sequence $\phi_{(k)}$ provides a description of the polynomial identities of the $k \times k$ matrices!

For $k = 2$, $\phi_{(2)} = y(\psi)$ where $\psi_n = \sum_{\mu \in \wedge_3(n)} \chi_\mu$. This follows from the study of the trace identities of the 2×2 matrices [4]. A combinatorial proof (i.e., free from P.I. theory) was later given [5].

By studying the trace identities of $k \times k$ matrices it was recently shown that for all k, $\phi_{(k)}$ is Young derived [2]. A combinatorial proof of that fact is yet to be found.

We turn now to the asymptotics!

Again consider $\phi = y(\psi)$, where $\psi_n = \sum_{\lambda \in \wedge_k(n)} a(\lambda)\chi_\lambda$, so that $\phi = \sum_{\mu \in \wedge_{k+1}(n)} b(\mu)\chi_\mu$. As $n \to \infty$, the asymptotics of deg ϕ_n can be calculated in two different ways which, when compared, imply some intriguing equations between certain multi-integrals.

First, deg $\phi_n = \sum_{j=0}^{n} \binom{n}{j}$ deg ψ_j (deg $\chi_{(n-j)} = 1$), hence the asymptotics of deg ψ_n determine that of deg ϕ_n.

On the other hand, by Young's rule,

$$b(\mu) = b(\mu_1, \cdots, \mu_{k+1}) = \sum_{\lambda_1=\mu_2}^{\mu_1} \cdots \sum_{\lambda_k=\mu_{k+1}}^{\mu_k} a(\lambda) \approx$$

$$\approx \int_{\mu_2}^{\mu_1} dx, \cdots \int_{\lambda_{k+1}}^{\mu_k} dx_k \, a(x_1, \cdots, a_k),$$

and the $b(\mu)$'s clearly determine the asymptotics of deg ϕ_n. Comparing these two asymptotics we obtain

THEOREM [6, 3.7]. *Let $a(x_1, \cdots, x_k)$ be a polynomial in the $(x_i - x_j)$'s, homogeneous of degree d, such that*

$$\sum_{\lambda_1=\mu_2}^{\mu_1} \cdots \sum_{\lambda_k=\mu_{k+1}}^{\mu_k} a(\lambda_1, \cdots, \lambda_k) \approx$$

$$\approx \int_{\mu_2}^{\mu_1} dx_1, \cdots \int_{\mu_{k+1}}^{\mu_k} dx_k \, a(x_1, \cdots, x_k) \stackrel{\text{def}}{=} p(\mu_1, \cdots, \mu_{k+1}).$$

Denote $D_\ell(x) = \prod_{1 \le i < j \le \ell} (x_i - x_j)$. Then

$$\int_{\substack{z_1+\cdots+z_{k+1}=0 \\ z_1 \ge \cdots \ge z_{k+1}}} \cdots \int p(z_1, \cdots, z_{k+1}) \cdot D_{k+1}(x) \cdot \exp(-\frac{k+1}{2}(z_1^2 + \cdots + z_{k+1}^2))dz =$$

$$= c \cdot \int_{\substack{x_1+\cdots+x_k=0 \\ x_1 \ge \cdots \ge x_k}} \cdots \int a(x_1, \cdots, x_k) \cdot D_k(x) \cdot \exp(-\frac{k}{2}(x_1^2 + \cdots + x_k^2))dx,$$

where $c = \sqrt{\frac{2\pi}{k+1}} \cdot \left(\frac{k}{k+1}\right)^{\frac{1}{2}(d+k-\frac{1}{2}k(k-1))}$.

For example, if $a = 1$ then $p(z) = (z_1 - z_2)(z_2 - z_3) \cdots (z_k - z_{k+1})$, hence

$$\int_{\substack{z_1+\cdots+z_{k+1}=0 \\ z_1 \ge \cdots \ge z_{k+1}}} \cdots \int (z_1 - z_2) \cdots (z_k - z_{k+1}) D_{k+1}(z) \cdot \exp(-\frac{k+1}{2}(z_1^2 + \cdots + z_{k+1}^2))dz$$

is reduced to a "Mehta" (or "Selberg") integral and can be evaluated.

A different approach is taken in [1]. Increase the length of the columns of each partition λ by a fixed factor q : $q * \lambda = (\underbrace{\lambda_1, \cdots, \lambda_1}_{q}, \underbrace{\lambda_2, \cdots, \lambda_2}_{q}, \ldots)$.

Given the character sequence $\Theta = \{\Theta_n\}$, $\Theta_n = \sum_{\lambda \vdash n} a(\lambda)\chi_\lambda$, denote by $\psi = q * \Theta$ the following sequence $\{\psi_m\}$: $\psi_{qn} = \sum_{\lambda \vdash n} a(\lambda)\chi_{q*\lambda}$, and $\psi_m = 0$ if $m \not\equiv 0 (\mathrm{mod}\ q)$. Finally, let $\phi = y(\psi)$ and write $\phi_n = \sum_{\lambda \vdash n} b(\mu)\chi_\mu$.

Again, dim $\phi_n = \sum_{j=0}^{n} \binom{n}{j} \deg \psi_j$, but now $\deg \psi_j = 0$ if $j \not\equiv 0 (\mathrm{mod}\ q)$. Probabilistic methods are applied here to obtain the asymptotics of deg ϕ_n from those deg Θ_n. Again, that asymptotics of deg ϕ_n can also be found from the relations between the $a(\lambda)$'s and the $b(\mu)$'s. Comparing these asymptotics we obtain – for $q = 2$ – the following

THEOREM [1, 3.7]. *Let $g(x_1, \cdots, x_k)$ be a homogeneous polynomial of the $(x_i - x_j)$'s, then*

$$\underset{\substack{x_1+\cdots+x_k=0 \\ x_1\geq\cdots\geq x_k}}{\int \cdots \int} g(x_1, \cdots, x_k) \cdot (D_k(x_1, \cdots, x_k))^4 \cdot \exp(-\sum_{j=1}^{k} x_j^2) dx =$$

$$= \alpha \underset{\substack{x_1+\cdots+x_{2k+1}=0 \\ x_1\geq\cdots\geq x_{2k+1}}}{\int \cdots \int} g(x_2, x_4, \cdots, x_{2k}) D_{2k+1}(x_1, \cdots, x_{2k+1}) \exp(-\frac{1}{2} \sum_{j=1}^{2k+1} x_j^2) dx$$

where $\alpha = \frac{1}{\sqrt{\pi}} \binom{2k+1}{k}^{\frac{1}{2}}$.

Specializing to $g(x_1, \cdots, x_k) = (D_k(x_1, \cdots, x_k))^\ell$ we deduce

COROLLARY [1, 3.8].

$$\underset{\substack{x_1+\cdots+x_{2k+1}=0 \\ x_1\geq\cdots\geq x_{2k+1}}}{\int \cdots \int} (D_k(x_2, \cdots, x_{2k}))^\ell \cdot D_{2k+1}(x_1, \cdots, x_{2k+1}) \cdot \exp(-\frac{1}{2} \sum_{j=1}^{2k+1} x_j^2) dx =$$

$$= \frac{1}{\alpha} \underset{\substack{x_1+\cdots+x_k=0 \\ x_1\geq\cdots\geq x_k}}{\int \cdots \int} (D_k(x_1, \cdots, x_k))^{\ell+4} \cdot \exp(-\sum_{j=1}^{k} x_j^2) dx.$$

Notice that the second integrand is symmetric in $x_1, \cdots, x_k$, hence the domain of integration can be transformed into $\mathbb{R}^k$. Thus the second integral is a "Mehta" integral, and can be evaluated by the Selberg integral, which yields the value for the first integral.

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