When is an S-Series Generated by $\prod_i \frac{1\pm x_i^n}{1\pm x_i^m}$ Multiplicity-Free?

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Introduction.

In [5], it was shown that for any two infinite power series $p(x)$ and $q(x)$ with integer coefficients, the symmetric product $\prod_i \frac{p(x_i)}{q(x_i)}$ generates an infinite series of Schur functions (S-series)

$$\prod_i \frac{p(x_i)}{q(x_i)} = \sum_\lambda c_\lambda s_\lambda(x)$$

with integer coefficients. We call such series *multiplicity-free* if the coefficients $c_\lambda = 0, \pm 1$ only.

In [5], we developed a general combinatorial method for evaluating c_λ using *bi-special rim hook tabloids* which are fillings of the shape λ with special and transposed special rim hooks. Roughly speaking, c_λ is given by the sum of weights (depending on p and q) of certain bi-special rim hook tabloids of shape λ. Based on this construction, we proved in [5] that an S-series generated by $\prod_i(\frac{1-x^n}{1-x^m})^{\pm 1}$ is multiplicity-free if $m = 2$, or n is a multiple of m.

The purpose of this paper is to extend our methods to determine which series of the form $\prod_i \frac{1\pm x_i^n}{1\pm x_i^m}$ are multiplicity free where we allow all possible choices of the $+, -$ signs and arbitrary values of m and n. In this case, the weight of each bi-special rim hook tabloid reduces to either 1 or -1. For series of the form $\prod_i(\frac{1-x^n}{1-x^m})^{\pm 1}$ where $m = 2$ or n is a multiple of m, it was shown in [5] that there is at most one bi-special rim hook tabloid of shape λ associated to the series for any given λ. However for general series of the form $\prod_i(\frac{1-x^n}{1-x^m})^{\pm 1}$, there may be many bi-special rim hook tabloid of shape λ associated with the series. The main new tool introduced in this paper is to use the class of transformations generated by adjacent switches of rim hooks as defined in [2] on the set of all bi-special rim hook tabloids of a fixed shape associated with the given series to define certain weight preserving involutions on this set. Then for example, we can show that for all series of the form $\prod_i(\frac{1-x^n}{1-x^m})^{\pm 1}$ our involutions will have at most one fixed point for any given shape and hence all such series are multiplicity free.

Our main result is the following:

Theorem 0.1 *Let m, n be any positive integers, and $\gcd(m,n)$ their greatest common divisor. Then,*

1. $\prod_i \frac{1+x_i^n}{1+x_i^m}$ *is multiplicity-free iff* $\frac{m-n}{\gcd(m,n)}$ *is even.*
2. $\prod_i \frac{1-x_i^n}{1+x_i^m}$ *is multiplicity-free iff* $\frac{n}{\gcd(m,n)}$ *is even.*
3. $\prod_i \frac{1+x_i^n}{1-x_i^m}$ *is multiplicity-free iff* $\frac{m}{\gcd(m,n)}$ *is even.*
4. $\prod_i \frac{1-x_i^n}{1-x_i^m}$ *is multiplicity-free.*

Moreover in the case when an S-series generated by $\prod_i \frac{1\pm x_i^n}{1\pm x_i^m}$ is not multiplicity-free, we show that the coefficient c_λ of $s_\lambda(x)$ is equal to

$$\pm 2^k,$$

where 2^k is the order of the class of transformation generated by adjacent switches of rim hooks on the set of all possible fillings of shape λ_{red} with special rim hooks of length m and n only. Here $\lambda_{red} \subseteq (n^m)$ is the reduced shape of λ, which is obtained uniquely from λ by peeling off certain rim hooks from λ. Finally, we show that the largest possible value of c_λ for such series is given by $2^{\gcd(m,n)}$.

In the following sections, we shall briefly outline the main steps in the proof of Theorem 1.

1 Expansions of Series of the form $\prod_i(1+\sum_{k\geq 1} f_k x_i^k)^{\pm 1}$

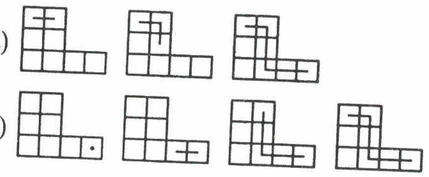
In this section we shall breifly review some of the general methods of [5] for expanding series of the form $F = \prod_i(1+\sum_{k\geq 1} f_k x_i^k)^{\pm 1}$ and closely related series . For convenience, we will write the coefficient of $S_\lambda(x)$ in a S-function series F as $< S_\lambda(x), F >$.

First we need to define the notion of special and transposed special rim hook tabloids. Given a Ferrers diagram λ, a *rim hook* h of λ is a consecutive sequence of cells along the northeast boundary of λ such that any two consecutive cells of h share and edge and the removal of the cells of h from λ results in a Ferrers diagram corresponding to another partition. We let $r(h)$ denote the number of rows of h and $c(h)$ denote the number of columns of h. We say that h is *special* if h has a cell in the first column of λ and h is *transposed special* (t-special) if h has cells in the first row of λ. For example, Figure 1.1(a) pictures all special rim hooks of $\lambda = (2,2,4)$ and Figure 1.1(b) pictures all t-special rim hooks of $\lambda = (2,2,4)$.

Figure 1.1

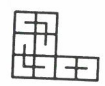
(a)

(b)



This given, a *rim hook tabloid* T of shape λ and type $\mu = (\mu_1, \ldots, \mu_k)$ is a filling of the Ferrers diagram of λ with rim hooks $(h_1, \ldots, h_k)$ such that $(|h_1|, \ldots, |h_k|)$ is a rearrangement of $(\mu_1, \ldots, \mu_k)$ where $|h_i|$ denotes the number of cells of h_i. To be more precise, one can think of a rim hook tabloid T as a sequence of shapes $\{\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$ such that for all $i \geq 1$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a rim hook of $\lambda^{(i)}$ and $(|\lambda^{(1)}/\lambda^{(0)}|, |\lambda^{(2)}/\lambda^{(1)}|, \ldots, |\lambda^{(k)}/\lambda^{(k-1)}|)$ is a rearrangement of μ. T is called a *special (t-special) rim hook tabloid* if for all $i \geq 1$, $\lambda^{(i)}/\lambda^{(i-1)}$ is a special (t-special) rim hook of $\lambda^{(i)}$. We emphasize however that the rim hook tabloid T of shape λ is the filling of the Ferrers diagram and is not the sequence of shapes $\{\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$. That is, Figure 1.2 pictures a rim hook tabloid T of shape $\lambda = (2,2,4)$ and type (2,3,3) whose sign is $(-1)^{2-1}(-1)^{2-1}(-1)^{1-1} = 1$ and gives the two sequence of shapes that can be associated to it. Of course, if T is a special rim hook tabloid or a t-special rim hook tabloid, then there is a unique sequence of shapes $\{\phi = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(k)} = \lambda\}$ that can be associated to T.

Figure 1.2

$T =$ 

$$= \{\phi \subset (1,2) \subset (2,2,2) \subset (2,2,4)\}$$
$$= \{\phi \subset (1,2) \subset (1,4) \subset (2,2,4)\}.$$

Special and transposed special rim hook tabloids can be defined for arbitrary skew shapes λ/μ in an analogous manner. The only difference is that a rim hook h of λ which does not intersect the shape μ is called special (t-special) for the shape λ/μ if h starts in the northwest (southeast) corner of the the shape λ/μ. A bi-special rim hook tabloid (bi-SRHT) B of shape λ/α and type $< \nu, \gamma >$ is a pair (T, H) where T is a t-SRHT of shape μ/α with $\alpha \leq \mu \leq \lambda$ and H is a SRHT of shape λ/μ. Let $SRHT(\lambda,\mu)$ and $t\text{-}SRHT(\lambda,\mu)$ denote the set of SRHT's and t-SRHT's of shape λ and type μ. Let $bi\text{-}SRHT(\lambda/\alpha, < \nu, \gamma >)$ denote the set of all bi-SRHT's of shape λ/α and type $< \nu, \gamma >$.

First, suppose that we start with two polynomials $p_m(x) = 1 + p_1 x + \cdots + p_m x^m$ and

$q_n(x) = 1 + q_1 x + \cdots + q_n x^n$ of degree m and n respectively. We can write

$$p_m(x) = \prod_{i=1}^{m} (1 - xs_i) \tag{1}$$

and

$$q_n(x) = \prod_{i=1}^{n} (1 - xt_i) \tag{2}$$

where $p_m(\frac{1}{s_i}) = 0$, and $q(\frac{1}{t_j}) = 0$ for all i, j. Denote the sequence $(s_1, \cdots, s_m)$ of inverses of the roots of $p_m(x)$ by $IR[p_m(x)]$ and the sequence $(t_1, \cdots, t_n)$ of inverses of the roots of $q_n(x)$ by $IR[q_n(x)]$ respectively. Next, let us recall the generalized Cauchy's identity, see [1]:

$$\prod_{i,j} \frac{1}{1 - x_i s_j} \prod_{i,j} \frac{1}{1 - y_i t_j} \prod_{i,j} (1 + x_i t_j) \prod_{i,j} (1 + y_i s_j) = \sum_{\lambda} HS_\lambda(x; y)\, HS_\lambda(s; t) \tag{3}$$

where

$$\begin{aligned} HS_\lambda(x; y) &= \sum_{\mu \subseteq \lambda} S_\mu(x) S_{\lambda/\mu}(y) \\ &= \sum_{\mu \subseteq \lambda} S_\mu(y) S_{\lambda'/\mu'}(x) \end{aligned}$$

is the hook (or super) Schur-function. Now by applying the identity to the product of polynomials $p_m(x_i)$ and $q_n(x_j)$ over i and j, we have:

Theorem 1.1 *Let $IR[p_m(x)] = (s_1, s_2, \cdots, s_m)$ and $IR[q_n(x)] = (t_1, t_2, \cdots, t_n)$ be the set of inverse roots of the polynomials $p_m(x)$ and $q_n(x)$ of degree m and n given in (1) and (2) respectively, then*

(i) $\prod_i \frac{q_n(-x_i) p_m(-y_i)}{p_m(x_i) q_n(y_i)} = \sum_\lambda HS_\lambda(x; y) HS_\lambda(IR[p_m(x)]; IR[q_n(x)])$

(ii) $\prod_i \frac{q_n(-x_i)}{p_m(x_i)} = \sum_\lambda S_\lambda(x) HS_\lambda(IR[p_m(x)]; IR[q_n(x)])$

(iii) $\prod_i \frac{p_m(-x_i)}{q_n(x_i)} = \sum_\lambda S_{\lambda'}(x) HS_\lambda(IR[p_m(x)]; IR[q_n(x)])$.

From the above theorem, the computation of the various series reduces to the problem of computing $HS_\lambda(IR[p_m(x)]; IR[q_n(x)])$. Using the combinatorial interpretation of the entries inverse Kostka matrix in terms of SRHT's due to Egecioglu and Remmel [2], the following was proved in [5].

Theorem 1.2 *Let $IR[p_m(x)]$, $IR[q_n(x)]$ be the same as in Theorem 1.1.*

$$HS_{\lambda/\alpha}(IR[p_m(x)]; IR[q_n(x)]) = \sum_{B=(T,H) \in bi-SRHT(\lambda/\alpha)} \omega_{p,q}(B),$$

where

$$\begin{aligned} \omega_{p,q}(B) &= \omega_q(T) \bar{\omega}_q(H), \\ \omega_p(T) &= \prod_{t \in T} (-1)^{r(t)} p_{|t|}, \\ \bar{\omega}_q(H) &= \prod_{h \in H} (-1)^{r(h)} q_{|h|}, \end{aligned}$$

$|t|$ and $|h|$ are the length of t and h respectively.

Consider now the infinite series $p(x) = 1 + \sum_{i \geq 1} p_i x^i$ and $q(x) = 1 + \sum_{i \geq 1} q_i x^i$. Note that if $|\lambda| = n$, then $HS_\lambda(IR[p(x)], IR[q(x)])$ depends only on the coefficients $p_1, p_2, \cdots, p_n$, and $q_1, q_2, \cdots, q_n$. That is, they are independent of p_i or q_i for $i \geq n+1$ since any srh or tsrh which is used in the filling of B can have length at most $|\lambda|$, which is equal to n. It then follows that

$$\begin{aligned} < \prod_i \frac{q(-x_i) p(-y_i)}{p(x_i) q(y_i)}, HS_\lambda(x, y) > &= < \prod_i \frac{q_n(-x_i) p_n(-y_i)}{p_n(x_i) q_n(y_i)}, HS_\lambda(x, y) > \\ &= HS_\lambda(IR[p_n(x)], IR[q_n(x)]) \end{aligned}$$

where $p_n(x)$ and $q_n(x)$ are polynomials obtained from $p(x)$ and $q(x)$ by truncation. Combining Theorem1.2 with Theorem 1.1, the following is proved in [5].

Theorem 1.3 *Let $p(x) = 1 + \sum_{i \geq 1} p_i x^i$ and $q(x) = 1 + \sum_{i \geq 1} q_i x^i$ be two power series. Then*

(i) $\prod_i \frac{q(-x_i) p(-y_i)}{p(x_i) q(y_i)} = \sum_\lambda HS_\lambda(x; y) \sum_{B=(T,H) \in bi-SRHT(\lambda)} \omega_{p,q}(B)$

(ii) $\prod_i \frac{q(-x_i)}{p(x_i)} = \sum_\lambda S_\lambda(x) \sum_{B=(T,H) \in bi-SRHT(\lambda)} \omega_{p,q}(B)$

(iii) $\prod_i \frac{p(-x_i)}{q(x_i)} = \sum_\lambda S_{\lambda'}(x) \sum_{B=(T,H) \in bi-SRHT(\lambda)} \omega_{p,q}(B)$

where

$$bi-SRHT(\lambda) = \cup_\mu t-SRHT(\mu) \times SRHT(\lambda/\mu),$$

and $\omega_{p,q}(B)$ is as given in Theorem 1.2.

2 Shape Reduction

Theorem 1.3 provides us with the following combinatorial procedure for finding the coefficient of $S_\lambda(x)$ in the expansion of the series $\prod_i \frac{q_i(-x)}{p_i(x)}$: (1) Divide the Ferrers diagram F_λ into two complementary parts: F_μ and $F_{\lambda/\mu}$ for some $\emptyset \subseteq \mu \subseteq \lambda$. (2) Fill the outer diagram $F_{\lambda/\mu}$ with srh's h to get $H \in SRHT(\lambda/\mu)$ and the inner diagram F_μ with tsrh t_i to get $T \in t-SRHT(\mu)$ to obtain pairs $(H, T) = B \in bi-SRHT(\lambda)$. (3) Add up the weight $\omega_{p,q}(B) = \bar{\omega}_q(H)\omega_p(T)$ over all possible fillings B and all partitions μ.

Although not very practical to use, this procedure does give us a very simple criterion for predicting zero coefficients. It is clear from the weight formula in Theorem 1.3 that

$$\bar{\omega}_q(h) = 0 \quad \text{if} \quad q_{|h|} = 0,$$

where q_k is the coefficient of x^k in $q(x)$, and $|h|$ is the length of the srh. Similarly,

$$\bar{\omega}_q(t) = 0 \quad \text{if} \quad p_{|t|} = 0.$$

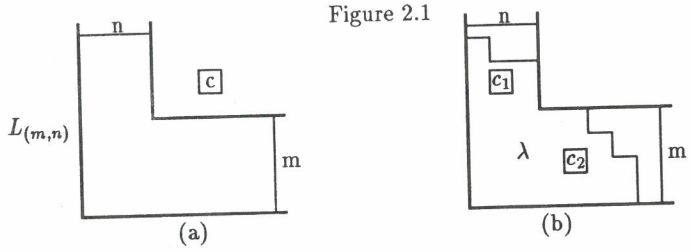
Given $q(x)$ and $p(x)$, we say a srh h is *legal* if $q_{|h|} \neq 0$ and a t-srh t is *legal* if $p_{|t|} \neq 0$. A bi-SRHTB associated with $p(x)$ and $q(x)$ is *legal* if it is filled with legal srh's and tsrh's. Then clearly the coefficient of $S_\lambda(x)$ in the expansion of $\prod_i \frac{q_i(-x)}{p_i(x)}$ is zero if it is impossible to fill the diagram of F_λ with legal hooks, i.e. if there is no legal B of shape λ..

When $p(x)$ and $q(x)$ are polynomials of degree m and n, respectively, the *maximum* hook length is $|t| = m$ and $|h| = n$. This condition greatly restricts the shape λ associated with non-zero coefficients. To state our results more precisely, we introduce the following notation. Let a cell in $Z^+ \times Z^+$ be labeled by (i,j) , where i-axis is assumed to increase to the right, and j-axis increase up the page. Denote by $L_{(m,n)}$ the set of cells $\{(i,j) : 1 \le j \le \infty$ for $1 \le i \le m; 1 \le j \le n$ for $m < i \le \infty\}$. Geometrically, $L_{(m,n)}$ is as an open L-shape, where the horizontal and vertical line of 'L' both extend to infinity, and the thickness of the lines are m and n respectively, as shown in Figure 2.1.(a).

We say λ fits inside $L_{(m,n)}$ if $F_\lambda \subset L_{(m,n)}$ as a subset. This given, it is easy to prove the following.

Theorem 2.1 *Suppose* $p(x) = 1+p_1x+p_2x^2+\cdots+p_mx^m$ *and* $q(x) = 1+q_1x+q_2x^2+\cdots+q_nx^n$ *are polynomials of degree* m *and* n *respectively. Then, the coefficient of* $S_\lambda(x)$ *in the expansion of* $\prod_i \frac{q(-x_i)}{p(x_i)}$ *is zero if* F_λ *does not fit inside* $L_{(m,n)}$.

Figure 2.1



We will now concentrate on series of the form $\prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m}$. Let $p(x) = 1 \pm x^m$, $q(-x) = 1 \pm x^n$. By Theorem 2.1,

$$< S_\lambda(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >= 0 \quad \text{if} \quad \lambda \not\subset L_{(m,n)}.$$

In this case, the only legal hook lengths are $|t| = 0$ or m and $|h| = 0$ or n. Since $p_{|t|}, q_{|h|} \in \{0, \pm 1\}$, we have $\omega_{p,q}(B) \in \{0, 1, -1\}$. Moreover the following holds.

Theorem 2.2 *For any* λ *such that* $\lambda \in L_{(m,n)}$ *and* $\lambda \not\subset (n^m)$, *if there is at least one legal* $B \in SRHT(\lambda)$ *associated with* $p(x) = 1 \pm x^m$, *and* $q(-x) = 1 \pm x^n$, *then there exists a unique partition* $\lambda_{\text{red}} \subset \lambda \subseteq (n^m)$ *such that*

$$< S_\lambda(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >= \pm < S_{\lambda_{\text{red}}}(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} > .$$

Note: if no legal B exists, then the coefficient of $S_\lambda(x)$ is zero.

Proof: We will assume the most general case, i.e. λ has more than m rows and more than n columns, as shown in Figure 2.1 (b).

Step 1: First we use Theorem 3.1(ii) with $q(-x) = 1 \pm x^n$ and $p(x) = 1 \pm x^m$. Thus if B is a legal bi-SRHT for p and q, the the t-srh's of B are of size m and the shr's of B are of size n. Now consider the cells like $c_1(i,j)$, which lies above the mth row, i.e. $i > m$. Obviously, this cell can not be reached by a t-srh of length $|t| = m$, the only legal non-zero t-srh. Hence it can only be covered by some srh h of length n starting at the first column of F_λ. In particular, every legal B must contain a srh h_1 of length n stating at the northwest corner of F_λ. Removing h_1 from each B, and denote the remaining partition by λ/h_1, we have, by Theorem 1.3

$$< S_\lambda(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >= \bar{\omega}_q(h_1) < S_{\lambda/h_1}(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} > .$$

If λ/h_1 still has $> m$ rows, by repeating the same argument, we can remove a second srh h_2 of length n from λ/h_1, starting at its northwest corner. Denote the remaining partition as $\lambda/(h_1h_2)$. We can continue this process until the remaining partition has $\le m$ rows to get:

$$< S_\lambda(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >= \prod_{i=1}^{a} \bar{\omega}_q(h_i) < S_{\lambda_{\le m}}(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >,$$

where $\lambda_{\le m} = \lambda/(h_1h_2\cdots h_a)$, and a is the minimum number of srh's that must be removed in order that $\lambda/(h_1h_2\cdots h_a)$ has no more than m rows.

Step 2: Next we use Theorem 3.1(iii) with $p_1(-x) = 1 \pm x^n$ and $q_1(x) = 1 \pm x^m$. In this case, we must consider legal bi-SRHT's of shape $\lambda'_{\le m}$ for p_1 and q_1 where the t-srh's are of size n and the srh's are of size m. Now consider the cells like $c_2(i,j)$ which lies beyond the nth column in $\lambda_{\le m}$, i.e. $j > n$, and hence lie above row n in $\lambda'_{\le m}$. Since it can not be reached by a tsrh of length n in $\lambda'_{\le m}$, it can only be covered by a srh of length $|t| = m$ in $\lambda'_{\le m}$. Thus by an argument which is similar to the one used in step 1,

$$< S_{\lambda_{\le m}}, \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >= \pm \prod_{i=1}^{b} \omega_p(t_i) < S_{\lambda_{\le m, \le n}}(x), \prod_i \frac{1 \pm x_i^n}{1 \pm x_i^m} >,$$

where $\lambda_{\leq m, \leq n} = \lambda_{\leq m}/(t_1, t_2, \cdots, t_b)$, b is the least number of srh's which can be removed from $\lambda'_{\leq m}$ so that $\lambda'_{\leq m}/(t'_1, t'_2, \cdots, t'_b)$ has no more than n rows. Here we let t_i denote the rim hook which results from transposing t'_i. Combining the two steps, we have

$$\lambda_{\text{red}} = \lambda_{\leq m, \leq n} = \lambda/(h_1, h_2, \cdots, h_a; t_1, t_2, \cdots, t_b) \in (n^m),$$

and the result follows since

$$\prod_{i=1}^{a} \bar{\omega}_q(h_i) \prod_{i=1}^{b} \omega_p(t_i) = \pm 1.$$

3 The Switching Operator

Let $(n_1, n_2, \cdots, n_\ell)$ be a sequence of integers. For $1 \leq i < j \leq \ell$, we define *switching operator* $\sigma_{i,j}$ on the sequence by

$$(n_1, n_2, \cdots, n_\ell)\sigma_{i,j} = (n_1, \cdots, n_j + j - i, \cdots, n_i - j + i, \cdots, n_\ell) \tag{4}$$

Clearly, $\sigma_{i,j}$ is an involution, $\sigma_{i,j}^2 = 1$. Denote the adjacent switch $\sigma_{i,i+1}$ by σ_i. It is easy to check that σ_i satisfy the Coxter relations:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| > 1$
2. ${\sigma_i}^2 = 1$
3. $\sigma_{j-1}\sigma_j\sigma_{j-1} = \sigma_j\sigma_{j-1}\sigma_j$

The adjacent switching operators σ_i behave in the same way as the adjacent transpositions t_i in the symmetric group S_ℓ. We define a switching operator corresponding to a permutation $\pi \in S_\ell$ as follows. Let $t_{i_1}t_{i_2}\ldots t_{i_k} = \pi$ be any decomposition of π in terms of adjacent transpositions. The multiplication is assumed to be from left to right. Then, define σ_π by

$$\sigma_\pi = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}.$$

where

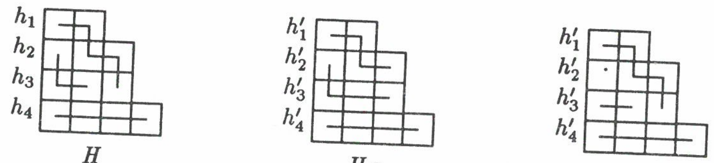
$$(n_1, n_2, \cdots, n_\ell)\sigma_{i_1}\sigma_{i_2} = ((n_1, n_2, \cdots, n_\ell)\sigma_{i_1})\sigma_{i_2}.$$

If we have two decompositions $\pi = t_{i_1}t_{i_2}\cdots t_{i_k} = t_{j_1}t_{j_2}\cdots t_{j_m}$, then $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k} = \sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_m} = \sigma_\pi$ since σ_i satisfy the Coxter relations.

The switching operator defined above is motivated by the description of switching of two srh's in a SRHT given by Egecioglu and Remmel in [2]. A SRHT H of shape λ with ℓ rows can be uniquely represented by a sequence of nonnegative integers $H = (|h_1|, |h_2|, \cdots, |h_\ell|)$ where $|h_i|$ is the length of the srh in H starting from the ith row of F_λ, counted from top to bottom of

the diagram. For convenience, we will write h for $|h|$. Suppose $h_i \neq 0$, then, it has been shown in [2] that $(h_1, h_2, \cdots, h_\ell)\sigma_i = (h_1, \cdots, h_{i+1} + 1, h_i - 1, \cdots, h_\ell)$ corresponds to a SRHT of shape λ obtained by switching the tails of h_i and h_{i+1}. An example of the action of σ_i on a SRHTis shown in Figure 3.1. Here $H = (5,3,0,4)$, $H\sigma_1 = (4,4,0,4)$ and $H\sigma_2 = (5,1,2,4)$. Note that $H\sigma_3 = (5,3,5,-1)$ does not correspond to a SRHT, since it contains a negative part.

Figure 3.1



H $\quad$ $H\sigma_1$ $\quad$ $H\sigma_2$

Two questions naturally arise: (1) When does a non-negative sequence of integers correspond to a SRHT? (2) If we apply an arbitrary switching operator σ_π on SRHT, will the result be another SRHT?

Let λ be a partition with ℓ rows. We have the following results:

Proposition 3.1 *Let $H = (h_1, h_2, \cdots, h_\ell)$ be a sequence of nonnegative numbers, $\Delta_\ell = (0, 1, 2, \cdots, \ell - 1)$, and $H + \Delta_\ell$ be their vector sum. Let order$(h_1, h_2, \cdots, h_\ell)$ denote the sequence of h_i's arranged in weakly increasing order. Then H is a SRHT of shape λ iff order$(H + \Delta_\ell) - \Delta_\ell = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$.*

Equivalently, $H \in SRHT(\lambda)$ iff $(h_1, h_2 + 1, \cdots, h_\ell + \ell - 1)$ is a permutation of $(\lambda_1, \lambda_2 + 1, \lambda_3 + 2, \cdots, \lambda_\ell + \ell - 1)$ or iff there is a switching operator σ_π such that $H\sigma_\pi = \lambda$.

Example 3.1 Suppose $H = (5,3,0,4)$. Then, *order*$(H + \Delta_4) - \Delta_4 = (2,4,5,7) - (0,1,2,3) = (2,3,3,4)$, which is a partition. Hence H corresponds to a SRHT of shape $(2,3,3,4)$ H is shown in Figure 3.1. On the other hand, $(3,1,0,5,2)$ does not correspond to a SRHT, since *order*$(H + \Delta_5) - \Delta_5 = (2,1,1,3,4)$, not a partition.

Proposition 3.2 *Let $H \in SRHT(\lambda)$ and π be any permutation on ℓ objects. Suppose $H' = H\sigma_\pi = (h'_1, h'_2, \cdots, h'_\ell)$. If h'_i is nonnegative for $1 \leq i \leq \ell$, then $H' \in SRHT(\lambda)$.*

Proposition 3.3 *The action of switching operators σ_π on $SRHT(\lambda)$ is transitive. In other words, let H and $H' \in SRHT(\lambda)$, then there exists $\pi \in S_\ell$ such that*

$$H' = H\sigma_\pi.$$

Proposition 3.4 *Suppose $H' = H\sigma_\pi$, and $H, H' \in SRHT(\lambda)$. There exists at least one decomposition $\sigma_\pi = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ such that $(H\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{j-1}})\sigma_{i_j} \in SRHT(\lambda)$ for $1 \le j \le k$.*

Proposition 3.5 *Suppose $(n_1, n_2, \cdots, n_\ell)\sigma_\pi = (n'_1, n'_2, \cdots, n'_\ell)$. Then,*

$$n'_{\pi(i)} = n_i + i - \pi(i) \tag{5}$$

for $1 \le i \le \ell$.

Given a SRHT H, let h be a srh in H, and $r(h)$ be the number of rows it covers, i.e. the leg-length of h. Define

$$r\!-\!sgn(H) = \prod_{h\in H} r\!-\!sgn(h), \quad \text{where} \quad r\!-\!sgn(h) = (-1)^{r(h)-1}.$$

Similarly,

$$-r\!-\!sgn(H) = \prod_{h\in H} -r\!-\!sgn(h), \quad \text{where} \quad -r\!-\!sgn(h) = (-1)^{r(h)}.$$

Proposition 3.6 *Suppose $H, H' \in SRHT(\lambda)$, and $H' = H\sigma_\pi$. Then,*

$$r\!-\!sgn(H') = \epsilon(\pi) r\!-\!sgn(H).$$

and

$$-r\!-\!sgn(H') = \epsilon(\pi)(-1)^{n_0} - r\!-\!sgn(H).$$

where n_0 is the number of hooks of length zero produced by the switching operator σ_π on H and $\epsilon(\pi) = (-1)^{\ell(\pi)}$ where $\ell(\pi)$ is the length of the decomposition of π in terms of adjacent transpositions.

This result is crucial in determining multiplicity-free series.

Example 3.2 Let $H = (3,0,3,3)$ and $H' = (6,0,3,0)$. It is easy to check that H, H' are both SRHT of shape $\lambda = (1,2,3,3)$, as shown in Figure 3.2, and $H' = H\sigma_\pi$, where $\sigma_\pi = \sigma_1\sigma_2\sigma_3$. In this case, $\ell(\pi) = 3$, $n_0 = 1$, since H' has one more zero hook than H. We have $r\!-\!sgn(H') = (-1)^3 r\!-\!sgn(H) = 1$, and $-r\!-\!sgn(H') = (-1)^{3-1} - r\!-\!sgn(H) = 1$.

Figure 3.2

4 Multiplicity-free Series

For convenience, we will write the four series $\prod_i \frac{1\pm x_i^n}{1\pm x_i^m}$ as $\prod_i \frac{1+tx_i^n}{1+bx_i^m}$ where $t, b = \pm 1$. When we apply Theorem 1.3 (ii), we have several choices in picking $p(x)$ and $q(x)$. For instance, $p(x) = 1 + bx^m$ and $q(-x) = 1 + tx^n$, or, $p(x) = (1+tx^n)^{-1}$ and $q(-x) = (1+bx^m)^{-1}$. Each choice will give us a different set of legal bi-SRHT's and different weights $\omega_{p,q}(B)$. For our purpose, we will chose $p(x) = 1$ and $q(-x) = \frac{1+tx^n}{1+bx^m}$, and assume $n \le m$ ($n < m$ if $b = t$). For if not, then we can always evaluate the coefficient of $S_{\lambda'}(x)$ in the conjugate series $\prod_i \frac{1+(-x_i)^m}{1+t(-x_i)^n}$, which is equal to the coefficient of $S_\lambda(x)$ in the original series. To determine the legal srh's, we expand $q(x) = \frac{1+t(-x)^n}{1+b(-x)^m}$ as power series:

$$\begin{aligned} q(x) &= (1+t(-x)^n)\sum_{k\ge 0}(-1)^k b^k (-x)^{mk} && (6)\\ &= \sum_{k\ge 0} b^k(-1)^{k(m+1)}x^{km} + \sum_{k\ge 0} tb^k(-1)^{k(m+1)+n}x^{km+n}. && (7)\end{aligned}$$

Hence a legal srh must have length $|h| = km$ or $km+n$, for $k \ge 0$. On the other hand, a legal tsrh must have length $|t| = 0$, since $p(x) = 1$. In this case, a legal bi-SRHT $B = (T, H) = H \in SRHT(\lambda)$, since T is empty. By Theorem 1.3, we have

$$\omega_{p,q}(B) = \prod_{h\in H} \bar{\omega}_q(h) = \prod_{h\in H} (-1)^{c(h)} q_{|h|} \tag{8}$$

By Proposition 2.2, we only need to determine if the coefficient of every $\lambda \subseteq (n^m)$ belongs to $\{0, \pm 1\}$ in order to determine if the series $\prod_i \frac{1+tx_i^n}{1+bx_i^m}$ is multiplicity-free. In the following discussions, we will assume (i) $\lambda \subseteq (n^m)$ and (ii) $n \le m$ ($n < m$ if $t = b$). We will use h to represent both the srh and its length $|h|$.

Since the longest rim hook that can fit inside (n^m) whose diagram is a $m \times n$ box, is $m+n-1$, the only legal srh's that can fit inside $\lambda \subseteq (n^m)$ are $|h| = 0, n$ or m. By (7) (8) and the fact $|h| = r(h) + c(h) - 1$, we have

$$\bar{\omega}_q(h) = \begin{cases} 1 & \text{if } h = 0\\ t(-1)^{r(h)-1} & \text{if } h = n\\ b(-1)^{r(h)} & \text{if } h = m\\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

Denote the set of all SRHT's of shape $\lambda \subseteq (n^m)$, filled with srh's of lengths m, n or 0 only by $SRHT(\lambda, m, n)$. Fix $H \in SRHT(\lambda, m, n)$. For convenience, we will write $H = (h_1, h_2, \cdots, h_m)$ with the understanding that $h_i = 0$ for $\ell(\lambda) < i \le m$. The set of permutations π such that $H\sigma_\pi \in SRHT(\lambda, m, n)$ form the automorphism class of $SRHT(\lambda, m, n)$ which we will denote by $Auto(\lambda, m, n)$. In general $Auto(\lambda, m, n)$ will not be a group, however one can completely analyze the structure of $Auto(\lambda, m, n)$. For example, it is easy to see from Proposition 3.5 that the only way a srh of length n can be switched to a legal hook is by either lossing n or gaining $m-n$ cells to become a srh of length 0 or m, respectively. Similarly, the only way a srh of length m or 0 can be switched to a legal srh is by lossing $m-n$ cells, or gaining n cells respectively, to become a srh of length n. However, if $i \leq n$ ($i > n$), then, it is impossible to have a srh of length 0 (length m) switched to a srh of length n since there is not enough room in F_λ to do so.

By writing $H\sigma_\pi = order(h'_{\pi(1)}, h'_{\pi(2)}, \cdots, h'_{\pi(i)}, \cdots, h'_{\pi(m)}) = (h'_1, h'_2, \cdots, h'_m)$, we have that if $\pi(i) \neq i$, then

$$h'_{\pi(i)} = \begin{cases} 0 \text{ or } m & \text{if } h_i = n \\ 0 & \text{if } h_i = m \text{ and } i \leq n \\ n & \text{if } h_i = 0 \text{ and } i > n \end{cases} .$$

From the above result, the permutation π can be determined uniquely by the relations $\pi(i) = h_i - h'_{\pi(i)} + i$, for $1 \leq \pi(i) \leq m$, which follow from Proposition 3.5.

Proposition 4.1 *Suppose $\pi \in Auto(\lambda, m, n)$, $H = (h_1, h_2, \cdots, h_m) \in SRHT(\lambda, m, n)$, and i is not a fixed point of π. Then*

$$\pi(i) = \begin{cases} i+n & \text{if } h_i = n \text{ and } i \leq m-n \\ i-(m-n) & \text{if } h_i = n \text{ and } i > m-n \\ i+(m-n) & \text{if } h_i = m \text{ and } i \leq n \\ i-n & \text{if } h_i = 0 \text{ and } i > n \end{cases} .$$

Further, it is impossible to have $h_i = 0$ with $i \leq n$, or $h_i = m$ with $i > n$ when i is not a fixed point of π.

Using proposition 4.1, one can completely determine the structures of cycles $\pi \in Auto(\lambda, m, n)$.

Proposition 4.2 *Let $\alpha = (i_1, i_2, \cdots, i_k, \cdots, i_d)$, where $1 \leq i_k \leq m$, $d \leq m$, be a cycle, and $\alpha \in Auto(\lambda, m, n)$. Suppose $H, H' \in SRHT(\lambda, m, n)$ and $H' = (h'_1, h'_2, \cdots, h'_m) = H\sigma_\alpha$. Then, either for $1 \leq k \leq d$,*

$$h_{i_k} = n \quad \text{and} \quad h'_{\alpha(i_k)} = \begin{cases} 0 & \text{if } \alpha(i_k) > n \\ m & \text{if } \alpha(i_k) \leq n \end{cases},$$

where the cycle α is given by

$$i_{k+1} = (i_k + n) \bmod (m) \quad \text{for } i \quad 1 \leq k \leq d. \tag{10}$$

or, for $1 \leq k \leq d$,

$$h_{i_k} = \begin{cases} 0 & \text{if } i_k > n \\ m & \text{if } i_k \leq n \end{cases} \quad \text{and} \quad h'_{\alpha(i_k)} = n.$$

where the cycle α is given by

$$i_{k+1} = (i_k + m - n) \bmod (m). \tag{11}$$

Further, the length of the cycle α, i.e. the number of distinct integers it contains, is given by $d = \frac{m}{\gcd(m,n)}$.

We will call α defined by (10) an *n-cycle*. Clearly, the cycle defined by (11) is the inverse of α, and we will denoted it by α^{-1}. The effect of α on H is to change the length of each srh h_{i_k} from n to m or 0, for every i_k in the orbit of α. The effect of α^{-1} is the opposite. It changes the length of each srh h_{i_k} from m or 0 to n for every i_k in the orbit of α^{-1}.

One can then show that different n-cycles and inverse n-cycles commute among themselves and with each other since they must be disjoint. Also if $\alpha \in Auto(\lambda, m, n)$ is a cycle of length greater than 2, then $\alpha^2 \notin Auto(\lambda, m, n)$ since the application of the switching operator σ_{α^2} on $H \in SRHT(\lambda, m, n)$ will produce some illegal srh's. From these facts, one can show that $Auto(\lambda, m, n)$ contains only products of distinct n-cycles or their inverses and that $|SRHT(\lambda, m, n)| = |Auto(\lambda, m, n)| = 2^c$ where $c \leq \gcd(m, n)$.

From the detailed analysis of the stucture of $Auto(\lambda, m, n)$, one can derive the results of Theorem 1.1. For example, suppose that we want to show that series $\prod_i \frac{1-x_i^n}{1-x_i^m}$ is multiplicity-free. By the comment earlier, we only need to consider the reduced shape $\lambda \subseteq (n^m)$. Assume $SRHT(\lambda, m, n)$ is not empty. Otherwise, the coefficient of $S_\lambda(x)$ is zero. By (9)

$$\bar{\omega}_q(h) = \begin{cases} -r-sgn(h) & \text{if } h = n \\ r-sgn(h) & \text{if } h = m \end{cases} \tag{12}$$

If $Auto(\lambda, m, n)$ is the trivial group, i.e. contains only the identity, then there is exactly one element H in $SRHT(\lambda, m, n)$ and the coefficient of $S_\lambda(x)$ is equal to $\bar{\omega}_q(H) = \pm 1$.

Now suppose that $Auto(\lambda, m, n)$ is nontrivial. Under the action of $\pi \in Auto(\lambda, m, n)$, there will be certain sign changes, $\bar{\omega}_q(H\sigma_\pi) = \pm\bar{\omega}_q(H)$. Let us first consider the action of a n-cycle $\alpha = (i_1, i_2, \cdots, i_d)$. In this case, the parity is $\epsilon(\alpha) = d-1$. Let $H \in SRHT(\lambda, m, n)$ and $h_{i_k} = n$ for each i_k on the orbit of α. Let n_0 and n_m be the number of h_{i_k}'s whose length changes to 0 and m respectively, under the action of α. From (12) and Proposition 3.6, it is clear that

$$\bar{\omega}_q(H\sigma_\alpha) = \epsilon(\alpha)(-1)^{n_0+n_m}\bar{\omega}_q(H)$$

Since by Lemma 4.2, $n_0 + n_m = d$, we have

$$\bar{\omega}_q(H\sigma_\alpha) = (-1)^{2d-1}\bar{\omega}_q(H) = -\bar{\omega}_q(H). \tag{13}$$

A general element $\pi \in Auto(\lambda, m, n)$ can be written as a product of k distinct n-cycles or inverse n- cycles $\pi = \alpha_{j_1}^{\pm 1}\alpha_{j_2}^{\pm 1}\cdots\alpha_{j_k}^{\pm 1}$ where $1 \leq k \leq c$, and c is the total number of different n-cycles or inverse n-cycles in $Auto(\lambda, m, n)$. Moroever since H is fixed, we must choose either α_j or α_j^{-1} but not both for any given j. Hence

$$\bar{\omega}_q(H\sigma_\pi) = \bar{\omega}_q(H\sigma_{\alpha_{j_1}^{\pm 1}}\sigma_{\alpha_{j_2}^{\pm 1}} \cdots \sigma_{\alpha_{j_k}^{\pm 1}}) = (-1)^k \bar{\omega}_q(H)$$

and finally, by Theorem 1.3, the coefficient

$$\begin{aligned} < S_\lambda(x), \prod_i \frac{1-x_i^n}{1-x_i^m} > &= \sum_{H \in SRHT(\lambda)} \bar{\omega}_q(H) \\ &= \sum_{\pi \in Auto(\lambda,m,n)^+} \bar{\omega}_q(H\sigma_\pi) \\ &= \bar{\omega}_q(H) + \sum_{1 \le j_1 < j_2 < \cdots < j_k; 1 \le k \le c} \bar{\omega}_q(H\sigma_{\alpha_{j_1}^{\pm 1}\alpha_{j_2}^{\pm 1} \cdots \alpha_{j_k}^{\pm 1}}) \\ &= \sum_{k=0}^{c} (-1)^k \binom{c}{k} \bar{\omega}_q(H) \\ &= (1-1)^c \bar{\omega}_q(H) = 0 \end{aligned}$$

So, if there are more than one element in $SRHT(\lambda, m, n)$, then the weights will cancell out. We have thus shown that the series is multiplicity-free.

The series $\prod_i \frac{1-x_i^n}{1+x_i^m}$, $\prod_i \frac{1+x_i^n}{1+x_i^m}$, and $\prod_i \frac{1+x_i^n}{1-x_i^m}$ can be analyzed in similar manner.

References

[1] A. Berele and J.B. Remmel, *Hook flag characters and their combinatorics*, J. of Pure and Applied Algebra, **35**, 225-245 (1985).

[2] O. Egecioglu and J.B. Remmel, *A Combinatorial Interpretation of the Inverse Kostka Matrix*, Linear and Multilinear Algebra, **26**, 59-84 (1990).

[3] D. E. Littlewood, *The Theory of Group Characters*, 2nd edition, Oxford University Press, Oxford (1950).

[4] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford (1979).

[5] J.B. Remmel and M. Yang, *Special Rim Hook Tabloids and Some New Multiplicity-free S-series*, SIAM J. Disc. Math. 4, pp. 253-274, 1991

THÉORIE COMBINATOIRE DES T-FRACTIONS ET APPROXIMANTS DE PADÉ EN DEUX POINTS

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INTRODUCTION

Nous présentons un modèle combinatoire pour les fractions continues de Thron [Th], ou T-fractions, qui s'écrivent sous la forme

$$\frac{1}{1-b_1t-} \frac{\lambda_1 t}{1-b_2t-} \cdots \frac{\lambda_k t}{1-b_{k+1}t-} \cdots \tag{1}$$

Ces fractions sont moins connues que les fractions continues de Jacobi ou de Stieltjes, dont les formes générales sont respectivement

$$J(t) = \frac{1}{1-b_0t-} \frac{\lambda_1 t^2}{1-b_1t-} \cdots \frac{\lambda_k t^2}{1-b_kt-} \cdots \quad \text{et} \quad S(t) = \frac{1}{1\ -} \frac{\lambda_1 t}{1\ -} \cdots \frac{\lambda_k t}{1\ -} \cdots \tag{2}$$

L'interprétation combinatoire des fractions de Jacobi et de Stieltjes a été initiée par Flajolet [Fl] avec des chemins valués dans le plan discret $\mathbf{Z} \times \mathbf{Z}$ (respectivement appelés chemins de Motzkin et chemins de Dyck) et est devenue un sujet classique (voir par exemple [Fr-Vi], [Go-Ja], [Vi]).

Dans le modèle développé en section 1 pour les T-fractions, nous utilisons à nouveau des chemins de Dyck valués, mais avec une règle de valuation différente de celle correspondant aux fractions de Stieltjes.

Si les réduites successives des fractions continues de Jacobi ou de Stieltjes fournissent des approximants de Padé "classiques", celles des T-fractions conduisent à des *approximants de Padé en deux points*, auxquels est consacrée la section 2. La notion de T-fraction *duale* d'une T-fraction apparaît alors naturellement et soulève de nouveaux problèmes combinatoires. Un exemple de cette dualité est donné à la section 4, qui conduit à de surprenants rapprochements relatifs à l'énumération de figures planaires appelés *polyominos*.

Enfin, en section 3, nous interprétons combinatoirement des développements en T-fraction relatifs à certaines séries hypergéométriques et à leurs q-analogues. Nous retrouvons ainsi des résultats de Dumont, Kreweras [Du-Kr] et Zeng [Ze]. L'interprétation repose sur une bijection entre permutations et certaines *histoires* combinatoires. Cette nouvelle bijection, exposée au paragraphe 3.1, joue pour les T-fractions le même rôle que celle de Françon et Viennot [Fr-Vi] pour les fractions de Jacobi et que celle de Foata et Zeilberger [Fo-Ze] pour les fractions de Stieltjes (voir [dM-Vi]).

Nous adoptons pour les fractions continues la notation "horizontale" utilisée par Jones et Thron dans [Jo-Th] *: ainsi, l'écriture*

$$\frac{a_0}{b_1+} \frac{a_1}{b_2+} \cdots \frac{a_{n-1}}{b_n+} (\cdots) \tag{3}$$

désigne la fraction continue (finie ou infinie)

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