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A TWO-DIMENSIONAL PICTORIAL PRESENTATION OF BERELE'S INSERTION ALGORITHM FOR SYMPLECTIC TABLEAUX

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ABSTRACT

Our purpose is to give a new presentation of Berele's correspondence [1]. S. Fomin [2] showed that the Robinson-Schensted correspondence can be presented as a doubly inductive application of "local rules", which are based on the properties of Young's lattice $\mathcal{P}$ (the poset of all partitions, ordered by containment of diagrams) as a differential poset. T. Roby [7], [8] generalized this interpretation to several variants of the Robinson-Schensted correspondence. S. Fomin's analysis is based on a certain poset invariant. M. A. A. van Leeuwen [5] analyzed that a direct investigation of the bumping procedure, which is used in the original definition of the Robinson-Schensted correspondence [9] (also [6]), can lead to the same presentation. Fomin, and later van Leeuwen, also gave a similar presentation of Schützenberger's jeu de taquin or sliding algorithm [10].

Our presentation of Berele's correspondence incorporates these two ingredients.

1. REVIEW ON BERELE'S CORRESPONDENCE BY INSERTION

The usual Robinson-Schensted correspondence gives a bijection from permutations of a certain alphabet to pairs of same-shape standard Young tableaux. A generalization of Schensted gives a bijection from permutations with repetitions to pairs of tableaux of the same shape, one column-strict, one standard. The latter may be viewed from the standpoint of representation theory as giving the decomposition of the action of the group $GL(n) \times S(n)$ on the k-fold tensor power of the natural representation $\otimes^k \square_{GL(n)}$. Berele's correspondence, originally conceived to explain a similar representation theoretic phenomenon for the symplectic group, gives a bijective map from the set of words on a certain alphabet of size $2n$ to pairs of tableaux, one "symplectic", the other "up-down".

First we recall the basic notion of a partition. Let $\mathbb{N}$ denote the set of **natural numbers** $\{0, 1, 2, \dots\}$. When we need to exclude 0 we will use the notation $\mathbb{N}^+$. A **partition** λ is a sequence of natural numbers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ such that the terms are weakly decreasing, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ and only a finite number of the terms are nonzero. The nonzero terms are called the **parts** of λ. The number of parts is called the **length** of λ and is written $\ell(\lambda)$. The sum of the parts is called the **weight**, and we write $|\lambda| = \sum_{i \geq 1} \lambda_i$. When writing concrete paritions, we generally surpress parentheses, commas, and trailing zeros; since no parts greater than 9 occur in any of the examples, no confusion should result. The unique partition of weight 0 is denoted by 0. If the weight of λ is n, then call λ a **partition of n** and write $|\lambda| = n$. Let $\mathcal{P}_n$ denote the set of all partitions of n, and $\mathcal{P}$ denote the union $\cup_{n\in\mathbb{N}}\mathcal{P}_n$.

The **diagram** or **shape** of a partition λ is the set $D_\lambda = \{(i,j) \in \mathbb{N}^2 : 1 \le j \le \lambda_i\}$, which is called the **Young diagram** of λ. The elements of the Young diagram are called its **squares**, and we may visualize it by placing the square (i,j) in the ith row and jth column, using the same convention as matrices. A glance at one of the examples below should make our convention clear.

The **conjugate** of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ whose diagram is given by $D_{\lambda'} = \{(i,j) \in \mathbb{N}^2 : (j,i) \in D_\lambda\}$. In other words, the diagram of λ' is obtained from that of λ by transposition.

Define a partial order $\subseteq$ on partitions by $\mu \subseteq \lambda$ if and only if $D_\mu \subseteq D_\lambda$. This partial order is easily seen to be a distributive lattice $\mathcal{P}$, called **Young's Lattice**. We say that "λ covers μ" and write $\lambda \mathrel{\dot{\supset}} \mu$ if $\mu \subseteq \lambda$ and they differ by exactly one square. We call such a square a **corner** of λ and a **cocorner** of μ (following van Leeuwen).

Fix a positive integer n, and let Γ_n denote the totally ordered set $\{1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}\}$. An **$Sp(2n)$-tableau** or an **$n$-symplectic tableau** of shape λ, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a partition with at most n parts, is a filling T of the Young diagram of λ with elements of Γ_n, which satisfies the following three conditions:

(1) $T(i,1) \le T(i,2) \le \cdots \le T(i,\lambda_i)$ for $1 \le i \le l$,
(2) $T(1,j) < T(2,j) < \cdots < T(\lambda'_j, j)$ for $1 \le j \le \lambda_1$,
(3) $T(i,j) \le \bar{i}$ for $1 \le i \le l$, $j \le \lambda_i$.

Here $T(i,j)$ denotes the entry of the cell at the position (i,j)—the intersection of the ith row, counted from the top, and the jth column, counted from the left. λ'_j is the length of the jth column of the Young diagram of λ. More generally, if Γ_n is replaced by *any* totally ordered set, and T is a filling satisfying the first two conditions above, then T is called a **column-strict** tableau. The shape of T will sometimes be denoted $sh(T)$.

Let $m_T(\gamma)$ denote the number of occurrences of the letter γ appearing in T. Then the sum of the weight monomials of T: $x_1^{m_T(1)-m_T(\bar{1})} x_2^{m_T(2)-m_T(\bar{2})} \cdots x_n^{m_T(n)-m_T(\bar{n})}$, for all $Sp(2n)$-tableaux T of a given shape λ, equals the character $\lambda_{Sp(2n)}$ of the irreducible representation of $Sp(2n,\mathbb{C})$ (see [4]).

To define Berele insertion we first need to define "ordinary insertion" in the sense of Schensted and Knuth. The description we give here will be somewhat informal; a more formal version can be found in [3].

Given a column-strict tableau T of shape λ and a letter γ, we determine a pair $(T \leftarrow \gamma, \lambda')$ as follows. First insert γ into the first row of T, where it replaces the smallest letter strictly larger itself, or gets placed at the end of the row if none exists. In the following steps, we proceed row by row. As long as some letter α

has been displaced from the row above, it is similarly inserted into the row below. At some iteration, the currently displaced number will come to rest at the end of some row (perhaps creating a new row at the bottom). This is the tableau $T \leftarrow \gamma$, and its shape is λ', which covers λ. Examples of this procedure are contained in the example of Berele insertion given below.

To define Berele insertion we also need the notion of a jeu de taquin slide, due originally to Schützenberger. Define a **punctured shape** (λ, h) to be the diagram of a partition from which one square h, called the **hole** has been removed. Define a **punctured tableau** (T,h) of shape (λ,h) to be a filling of the squares of (λ,h) which satisfies the same inequalities (1) and (2) given in the above definition of symplectic tableau, except that no value is assigned to the hole h so any inequalities involving h are ignored.

A **slide** is a transformation $\xi : (T,h) \mapsto (T',h')$ between punctured tableaux defined as follows. Compare the value of T at the two squares below and to the right of $h = (i,j)$. If $T(i+1,j) \le T(i,j+1)$, then set $T'(i,j) = T(i+1,j)$, $h' = (i+1,j)$, and set T' to be identical to T elsewhere. Otherwise set $T'(i,j) = T(i,j+1)$, $h' = (i,j+1)$, and set T' to be identical to T elsewhere. Informally, we simply slide the lesser letter adjacent to h into the hole h and make the vacated square the new hole. This insures that (except for the hole) T' satisfies the conditions of being a column-strict tableaux.

Now it is clear that one can apply the above procedure inductively until the original hole h has become a corner of the transformed tableau T'. At this point one can forget the hole and consider T' to be a column-strict tableau of shape $sh(T')\backslash h'$. We use this procedure below.

Berele insertion is a explicitly given bijection from the set of pairs (T,γ), where T is an $Sp(2n)$-tableau of a given shape λ, and $\gamma \in \Gamma_n$, to the set of $Sp(2n)$-tableau whose shape either covers λ or is covered by λ (in the poset $\mathcal{P}$). If the ordinary row insertion of γ into T yields a valid $Sp(2n)$-tableau, then it is also the result of the Berele insertion of γ into T. In this case the resulting shape covers λ. On the other hand, if the result of the row insertion violates condition (3), then at some row k, the symbol $\bar{k}$, which was in row k in T, was bumped by a k. Capture the earliest such occurrence, and at this point erase both the k and the $\bar{k}$ involved in this bumping, leaving the position formerly occupied by the $\bar{k}$ as a hole. After this, apply the sliding algorithm until the hole moves to a corner, and then forget the hole. This is the result of the Berele insertion, and the resulting shape if covered by λ. Let $T \underset{\mathcal{B}}{\leftarrow} \gamma$ denote the result of the Berele insertion of γ into T.

The weighted enumerative identity following from this bijection represents the decomposition of the tensor product of the irreducible representation $\lambda_{Sp(2n)}$ and $\square_{Sp(2n)}$ (the natural representation or the vector representation).

Berele's correspondence, as we call it in this abstract, is a bijection from the set of words $w = w_1 w_2 \dots w_f$ in the alphabet Γ_n of fixed length f to the set of pairs (P,Q), where P is an $Sp(2n)$-tableau of some shape λ, and Q is an up-down tableau of length f with initial shape $\varnothing$ and final shape λ and consisting of partitions with at most n parts; namely $Q = (\varnothing = \kappa^{(0)}, \kappa^{(1)}, \ldots, \kappa^{(f)} = \lambda)$, $\kappa^{(i)} \in \mathcal{P}$, $\ell(\kappa^{(i)}) \leq n$, and for each i either $\kappa^{(i-1)} \dot{\subset} \kappa^{(i)}$ or $\kappa^{(i-1)} \dot{\supset} \kappa^{(i)}$, where $\lambda \dot{\supset} \mu$ means λ covers μ. If w is such a word, then for $0 \leq i \leq f$ put $P_i = (\cdots((\varnothing \underset{\mathcal{B}}{\leftarrow} w_1) \underset{\mathcal{B}}{\leftarrow} w_2) \underset{\mathcal{B}}{\leftarrow} \cdots) \underset{\mathcal{B}}{\leftarrow} w_i$, and let $\kappa^{(i)}$ be the shape of P_i. Put $P = P_f$ and $Q = (\kappa^{(0)}, \kappa^{(1)}, \ldots, \kappa^{(f)})$. Then Berele's correspondence takes w to this pair (P, Q).

Example. Applying Berele insertion to the word $w = \bar{3}1\bar{2}\bar{3}3\bar{1}12\bar{3}\bar{1}\bar{2}223\bar{2}\bar{1}22\bar{3}12$ yields the following sequence P_i of symplectic tableaux:

$$\begin{array}{|c|}\hline \bar{3} \\ \hline\end{array}\,,\quad \begin{array}{|c|}\hline 1 \\ \hline \bar{3} \\ \hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & \bar{2} & 3 \\ \hline \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & \bar{2} & 3 \\ \hline \bar{3} & \bar{3} \\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & \bar{1} & 3 \\ \hline \bar{2} & \bar{3} \\ \cline{1-2} \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \bar{2} & \bar{3} \\ \hline \bar{3} \\ \cline{1-1}\end{array}\,,$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & 3 \\ \hline \bar{3} & \bar{3} \\ \hline\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & 2 & \bar{3} \\ \hline \bar{2} & 3 \\ \cline{1-2} \bar{3} & \bar{3} \\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & \bar{1} & 3 \\ \hline 3 & \bar{3} \\ \cline{1-2} \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & \bar{1} & \bar{2} \\ \hline 3 & \bar{3} & \bar{3} \\ \hline \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & \bar{1} & 2 \\ \hline \bar{2} & \bar{3} & \bar{3} \\ \hline\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & 2 & 3 \\ \hline \bar{2} & \bar{3} & \bar{3} \\ \cline{1-3}\end{array}\,,$$

$$\begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & 2 & \bar{2} \\ \hline \bar{2} & 3 & \bar{3} \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & \bar{2} \\ \hline 3 & \bar{3} \\ \cline{1-2} \bar{3} \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & 2 \\ \hline \bar{2} & \bar{3} \\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & 2 & 2 \\ \hline \bar{2} & \bar{3} \\ \cline{1-2}\end{array}\,,$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & \bar{1} & \bar{1} & 2 & 2 & \bar{3} \\ \hline \bar{2} & 3 \\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|c|c|}\hline 1 & \bar{1} & 2 & 2 & \bar{3} \\ \hline \bar{2} & \bar{3} \\ \cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|c|c|}\hline 1 & \bar{1} & 2 & 2 & 2 \\ \hline \bar{2} & \bar{3} & \bar{3} \\ \cline{1-3}\end{array}\,.$$

The final output is the pair (P, Q) where

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & \bar{1} & 2 & 2 & 2 \\ \hline \bar{2} & \bar{3} & \bar{3} \\ \cline{1-3}\end{array} \qquad \text{and} \qquad Q = (0, 1, 11, 21, 31, \ldots, 52, 53)$$

is the sequence of shapes of the P_i's.

The enumerative identity following from the whole Berele correspondence represents the decomposition of the f-fold tensor product of the natural representation of $Sp(2n)$ according to the action of $Sp(2n)$ and the Brauer algebra. For more information about related matters, including the case of orthogonal groups, we refer the interested reader to [11]. There an interesting connection between up-down tableaux and stadard tableaux (a connection between a parametrization of the $Sp(2n)$-decomposition of $\bigotimes^f \mathbb{C}^{2n}$ and that of the $GL(2n)$-decomposition, so to speak) is also explained.

In order to give a pictorial interpretation, we need to "standardize" our word. Let w be as above. Let $\tilde{w}$ denote the word obtained from w by replacing, for each $\gamma \in \Gamma_n$, the occurrences of γ in w by the symbols $\gamma_1, \gamma_2, \ldots, \gamma_{m_w(\gamma)}$ from the left to

the right, where $m_w(\gamma)$ is the number of such occurrences. Let $\tilde{\Gamma}_w$ denote the totally ordered set $1_1 < 1_2 < \cdots < 1_{m_w(1)} < \bar{1}_1 < \bar{1}_2 < \cdots < \bar{1}_{m_w(\bar{1})} < \cdots\cdots < n_1 < n_2 < \cdots < n_{m_w(n)} < \bar{n}_1 < \bar{n}_2 < \cdots < \bar{n}_{m_w(\bar{n})}$ and $\tilde{\gamma}_w : [1, f] \to \tilde{\Gamma}_w$ be the unique poset isomorphism. Following Schützenberger, we call $\tilde{w}$ the **standardization** of w.

Now we can define a slightly modified Berele correspondence for standardized words. All our insertions and slides occur according to the usual rules (though in this case all the letters are distinct). Violations of the symplectic condition are still determined by the ignoring the subscript on γ_i. Then we have the following

Lemma. *Let $w = w_1 w_2 \ldots w_f$ be a word on the alphabet Γ_n, and $\tilde{w}$ be its standardization. If $\tilde{w}$ corresponds to a pair (P, Q) by the modified Berele described above, then $w \underset{\mathcal{B}}{\longleftrightarrow} (P', Q)$ where P' is obtained from P by deleting all the subscripts.*

For the usual Schensted correspondence, the case of words with repeated letters can be reduced to that of permutation by means of a similar standardization. (In other words, standardization commutes with Schensted insertion.) The present situation is more complicated, in that one cannot simply replace the standardized word with a permutation. One need to know the maximum row into which a given letter can be bumped before it is will cause a cancellation and sliding. So our standardized words are really equivalent to "weighted permutations".

2. Berele's Correspondence by Local Rules

Next we explain our pictorial approach, which is a two-dimensional presentation of Berele's algorithm based on a modified set of local rules in the spirit of Fomin. We draw an $f \times f$ lattice as in Fig. 1. We employ the matrix coordinate system, and the vertices are labelled (i, j) with $0 \leq i \leq f$, $0 \leq j \leq f$. The square region cornered by the four vertices $(i-1, j-1)$, $(i-1, j)$, (i, j), $(i, j-1)$ will be called the cell at (i, j). The region $\tilde{\gamma}_w^{-1}(\gamma_1) - 1 < i \leq \tilde{\gamma}_w^{-1}(\gamma_{m_w(\gamma)})$ will be called the γ-zone. We will refer to this partitioning of the lattice as its **stratification**. The picture of w is obtained by writing a $\times$ inside the cells at $(\tilde{\gamma}_w^{-1}(\tilde{w}_j), j)$ for $1 \leq j \leq f$.

Now let $\tilde{w}(i, j)$ denote the word in $\tilde{\Gamma}_w$ obtained from the section of the picture to the left and above the vertex (i, j), i.e., $\tilde{w}(i, j)$ is the subword of $\tilde{w}_1 \tilde{w}_2 \cdots \tilde{w}_j$ consisting of letters $\leq \tilde{\gamma}_w(i)$. Let $w(i, j)$ denote the word in Γ_n obtained from $\tilde{w}(i, j)$ by forgetting the subscripts of the letters. Let $\Lambda(i, j)$ denote the shape of the $Sp(2n)$-tableau obtained by applying Berele's correspondence to $w(i, j)$. Then we have the following:

Theorem. 1) *For any square at (i, j), let $A = (i-1, j-1)$, $B = (i-1, j)$, $C = (i, j-1)$, and $D = (i, j)$. Then the quadruple $(\Lambda(A), \Lambda(B), \Lambda(C), \Lambda(D))$ falls into exactly one of the following cases.*

(The $\Longrightarrow$ sign in the following sentences distinguising the cases represents the fact that these can actually be used as "local rules" to determine $\Lambda(D)$ from $\Lambda(A)$, $\Lambda(B)$, and $\Lambda(C)$, as well as the position of the square and whether or not the square

contains a $\times$, *as claimed in the next statement. At this point, these should only be read as "and".)*

($\times$) *the cell at* (i,j) *contains a* $\times$, $\Lambda(A)=\Lambda(B)=\Lambda(C) \implies \Lambda(D)$ *grows from* $\Lambda(C)$ *in the first row.*

In the remaining cases, the cell at (i,j) *does not contain a* $\times$.

( ) $\Lambda(A)=\Lambda(B)=\Lambda(C) \implies \Lambda(D)=\Lambda(C)$

( ) $\Lambda(A)=\Lambda(B),\ \Lambda(A)\neq\Lambda(C) \implies \Lambda(D)=\Lambda(C)$

( ) $\Lambda(A)\neq\Lambda(B),\ \Lambda(A)=\Lambda(C) \implies \Lambda(D)=\Lambda(B)$

(M) $\Lambda(A)\dot{\subset}\Lambda(B),\ \Lambda(A)\dot{\subset}\Lambda(C),\ \Lambda(B)\neq\Lambda(C) \implies \Lambda(D)=\Lambda(B)\cup\Lambda(C)$

(R) $\Lambda(A)\dot{\subset}\Lambda(B)=\Lambda(C)$, $\Lambda(B)$ *grows from* $\Lambda(A)$ *in the* k*th row,* $k+1\le\tilde{\gamma}_w(i)$ $\implies \Lambda(D)$ *grows from* $\Lambda(C)$ *in the* $k+1$*st row*

(○) $\Lambda(A)\dot{\subset}\Lambda(B)=\Lambda(C)$, $\Lambda(B)$ *grows from* $\Lambda(A)$ *in the* k*th row,* $k+1>\tilde{\gamma}_w(i)$ $\implies \Lambda(D)=\Lambda(A)$

($\bar{\mathrm{J}}$) $\Lambda(A)\dot{\subset}\Lambda(B),\ \Lambda(A)\dot{\supset}\Lambda(C),\ \Lambda(B)/\Lambda(C)$ *is a domino* $\implies \Lambda(D)=\Lambda(A)$

(J) $\Lambda(A)\dot{\subset}\Lambda(B),\ \Lambda(A)\dot{\supset}\Lambda(C),\ \Lambda(B)/\Lambda(C)$ *is not a domino* $\implies \Lambda(C)\dot{\subset}\Lambda(D)\dot{\subset}\Lambda(B),\ \Lambda(D)\neq\Lambda(A)$

($\bar{\mathrm{J}}'$) $\Lambda(A)\dot{\supset}\Lambda(B),\ \Lambda(A)\dot{\subset}\Lambda(C),\ \Lambda(C)/\Lambda(B)$ *is a domino* $\implies \Lambda(D)=\Lambda(A)$

(J′) $\Lambda(A)\dot{\supset}\Lambda(B),\ \Lambda(A)\dot{\supset}\Lambda(C),\ \Lambda(C)/\Lambda(B)$ *is not a domino* $\implies \Lambda(B)\dot{\subset}\Lambda(D)\dot{\subset}\Lambda(C),\ \Lambda(D)\neq\Lambda(A)$

(W) $\Lambda(A)\dot{\supset}\Lambda(B),\ \Lambda(A)\dot{\supset}\Lambda(C),\ \Lambda(B)\neq\Lambda(C) \implies \Lambda(D)=\Lambda(B)\cap\Lambda(C)$

(Я) $\Lambda(A)\dot{\supset}\Lambda(B)=\Lambda(C)$, $\Lambda(B)$ *shrinks from* $\Lambda(A)$ *in the* k*th row,* $k\ge 2 \implies \Lambda(D)$ *shrinks from* $\Lambda(C)$ *in the* $k-1$*st row*

$k=1$ *never happens in the immediately above case.*

2) *Knowing* $\Lambda(A)$, $\Lambda(B)$, $\Lambda(C)$, *the position (including the zone division), and whether or not the square contains a* $\times$ *is sufficient to determine to which case the square belongs. So we can recover the whole array of* $\Lambda(i,j)$ *from the picture of* w *by starting from the empty shapes at the top and the leftmost edges and applying these local rules inductively.*

3) *Knowing* $\Lambda(B)$, $\Lambda(C)$, $\Lambda(D)$, *the position (including the zone division) is sufficient to determine to which case it belongs, so we can recover* w *and the whole array from the bottom and the rightmost edges. This means that the map from* w *to the information on the bottom and the rightmost edges is injective.*

4) *The sequence of shapes on the bottom edge equals the up-down tableau of length* f *obtained from* w *by Berele's algorithm.*

5) *Let* $1\le k\le n$. *The rightmost edge shape sequence in the* k*-zone represents a horizontal strip growing from the left to the right.*

6) *The rightmost edge shape sequence in the* $\bar{k}$*-zone represents a shrink by a horizontal strip (shrinking from the right to the left) followed by a growth by another horizontal strip, from the left to the right. Moreover, if one puts* $\lambda^{(k)}=\Lambda(\tilde{\gamma}_w^{-1}(k_1)-1,f)$ *and* $\mu^{(k)}=\Lambda(v_k,f)$, *where* v_k *is the row coordinate of the turning point from the shrink to the growth in the rightmost edge of the* $\bar{k}$*-zone, then* $\mu^{(k)}/\lambda^{(k)}$ *is also*

a horizontal strip.

7) *The tableau of shape* $\Lambda(f,f)$ *in which* $\mu^{(k)}/\lambda^{(k)}$ *is filled by the symbol* k *and* $\lambda^{(k+1)}/\mu^{(k)}$ *is filled by the symbol* $\bar{k}$ $(k=1,\ 2,\ \ldots,\ n$, *where* $\lambda^{(f+1)}$ *is understood to be* $\Lambda(f,f))$ *is the* $Sp(2n)$*-tableau obtained from* w *by Berele's correspondence.*

This shows that the result of applying Berele's correspondence to a word w can be completely determined by the "local rules" listed in 1).

3. The Reverse Correspondence

As stated in 3), if we know the up-down tableau on the bottom and the shape sequence on the rightmost edge, we can recover the whole array of shapes as well as the word w. On the other hand, suppose we are given a pair (P,Q) of an $Sp(2n)$-tableau P and an up-down tableau Q which lie in the range of Berele's correspondence. We can put Q along the bottom edge; however, the tableau P only tells us the rightmost shape at each border between the zones labeled $\overline{k-1}$ and k, and the minimal shape in each $\bar{k}$-zone where the sequence of shrinks becomes a sequence of growths. This of itself is insufficient to determine the rightmost edge. We do not know directly from P and Q how many k-$\bar{k}$ cancellations occur in the $\bar{k}$-zone, nor what shape should be put at the border of k and $\bar{k}$. Nonetheless, Berele's correspondence in its original form is reversable, so it should be possible to see how to do this from our pictorial point of view.

Suppose we are given a pair (P,Q) as above, which by the usual Berele correspondence corresponds to a word w. If we apply our algorithm to this word, we obtain Q along the bottom edge, and an updown tableau $T=(0=\tau_0,\tau_1,\ldots,\tau_f)$ which (along with the stratification) determines P as described in 6) and 7) of the theorem above. In fact, T may be viewed not merely as a sequence of shapes, but in fact as a sequence of symplectic tableaux $T=(0=T_0,T_1,\ldots,T_f=T)$; here T_i is the symplectic tableau determined by the updown tableau $(0=\tau_0,\tau_1,\ldots,\tau_i)$ as above. (Note that this sequence of symplectic tableaux is certainly not the same as the sequence obtained in the usual correspondence, whose shapes are given by the updown tableau Q.)

We give the following inductive procedure, which we conjecture to work in all cases. Given the pair (T_f,Q) we wish to recover T_{f-1}. Put $Q=(0=\kappa_0,\kappa_1,\ldots,\kappa_f=sh(P))$. First we note that we can recover the information of which zone k the vertex T_f lies in as follows. Set the maximum length of the shapes appearing in Q to be l. By the definition of symplectic tableau, there must be some letters $\ge l$ in w. On the other hand, any letters $m>l$ which appear in w must also appear in T_f, since they could only have been cancelled out *after* being pushed in the mth row. Therefore, the maximum letter in w, i.e., the zone of T_f is the maximum of l and m.

If T_f contains the letter $\bar{k}$, then we obtain T_{f-1} from T_f by deleting the rightmost occurrence of $\bar{k}$ in T_f, because the numbers are added in a horizontal strip from left to right by 6).

If T_f does not contain the letter $\bar{k}$, then we first check to see whether in Q there is a "k-shrink", i.e., an adjacent pair of shapes $\kappa_i \dot{\supset} \kappa_{i+1}$ which differ in the kth row. If not, then as in the determination of the zone we conclude that T_f must contain a letter k; by deleting the rightmost occurence of k we obtain T_{f-1}.

Now assume that T_f does not contain the letter $\bar{k}$, but Q does contain a k-shrink. Here we need a more elaborate procedure to test whether the assumption that $\tau_{f-1} \dot{\supset} \tau_f$ will yield an appropriate row of our diagram. This assumption determines the shape of T_{f-1} as follows: add the letter k to the first column of T_f which does not already contain a k to get T_{f-1}. The location of this k will be a cocorner of T_f, say at position (r, c). Note that given τ_f and the row number r, τ_{f-1} is uniquly determined.

For convenience we use the following shorthand. We assign Q to the bottom edge of our diagram as follows: κ_i is assigned to the vertex (f, i). Let κ_i' denote the value which would be assigned to the vertex $(f-1, i)$ by the local rules under the assumption that $\tau_{f-1} \dot{\supset} \tau_f$. Let s_i be the row in which κ_i and κ_{i-1} differ if $\kappa_{i-1} \dot{\subset} \kappa_i$; let $-s_i$ be the row in which they differ if $\kappa_{i-1} \dot{\supset} \kappa_i$. (So $s_i = -j$ indicates a j-shrink.) Define $s_i = 0$ if $\kappa_{i-1} = \kappa_i$. Similarly, define r_i to be the row in which κ_i and κ_i' differ, counting shrinks as negative. (So $r_f = -r$, where r is as above.) We will claim that the local rules determine the numbers r_j for $f \geq j \geq 1$ according to the following rules. Here we only need to use the subset of the local rules which include the case where the r_j are negative.

(M,J,J̄) If $s_i \neq r_i, -r_i - 1$, then set $r_{i-1} = r_i$.
(Я) If $s_i = r_i \neq -k$, then set $r_{i-1} = r_i - 1$.
(J+) If $s_i = -r_i - 1$ and κ_i has the same row lengths in rows s_i and s_{i+1}, then set $r_{i-1} = r_i + 1$.
(◯) If $s_i = r_i = -k$, then set $r_{i-1} = k$. Here we would label the cell at (f, i) with a circle and conjecturally take our assumption that $\tau_{f-1} \dot{\supset} \tau_f$ was correct.

It may happen that we never reach the last case, which means that when we work backwards by local rules along the row in question from right to left under our assumption, we do not obtain a trivial (identically zero) left edge. This is because all the other local rules preserve the sign of r as being negative. In this case, our orginal assumption is untenable and we are *forced* to take $\tau_{f-1} \dot{\subset} \tau_f$.

In fact, the latter assumption that $\tau_{f-1} \dot{\subset} \tau_f$ will always yield an acceptable row working backwards by local rules for the following reason. All the local rules we use in the case where r_i is positive will yield a positive r_{i-1}, with the exception of $(\times)$, which sets $r_{i-1} = 0$. If we reach the latter case, the reverse computation of the row is complete, since the local rules ( ) insist that $r_j = 0$ for all $0 < j < i$; thus, we get $r_0 = 0$, which means that $\kappa_0' = \kappa_0 = 0$. Now, if by the time we reach the cell $(f, 1)$ we still have $r_1 > 0$, then since $\kappa_0 = 0$ and $\kappa_1 = 1$, we must have the shape at vertex $(f-1, 1)$ is 0. Hence, the cell $(f, 1)$ falls into case $(\times)$. This means that we always will obtain exactly one cell in case $(\times)$ if we work backwards from a

growth on the right edge by local rules. Similarly, if we start with a shrink on the right edge and working backwards find case (◯), then we we also obtain exactly one cell in case $(\times)$ to the left of the circle.

Our conjecture is as follows.

Conjecture. *Let (P, Q) be a pair consisting of an $Sp(2n)$-tableau P and an up-down tableau Q of the same shape. The procedure described above computes the shapes $\Lambda(i,j)$ for $0 \leq i, j \leq f$ via reverse local rules.*

The only step left to be proven is in the ambiguous case described at length above. It is not yet clear that if our assumption that $\tau_{f-1} \dot{\supset} \tau_f$ yields an acceptable row, i.e., one containing a circle and a $\times$, then it is indeed the correct row in the two-dimensional picture working forwards.

An example of the full correspondence, which coincides with the example we give earlier by bumping, follows at the end of this article.

4. Open Questions and Remarks

Although the current pictorial viewpoint allows some additional insight into the workings of Berele's correspondence, it is not yet the major simplification that one could hope for. In particular, the difficulty of running the algorithm backwards is still less than satisfactory. It would also be nice if Berele's correspondence could be seen as a particular case of some more general correspondence between pairs of updown tableau and certain kinds of permutation-like objects. This is currently under investigation.

The pictorial version of Schensted's algorithm is connected with a certain poset invariant due to Greene and Kleitman. It is a natural question to try to generalize this to the current case, but all efforts to date have failed. Or it might be meaningful to find an alternative correspondence which would yield the same enumerative identity but also admit a clearer combinatorial interpretation.

5. A Complete Example

Example. Fig. 1 on the next page shows the pictorial presentation of the same example that was previously carried out by bumping and sliding in §1.

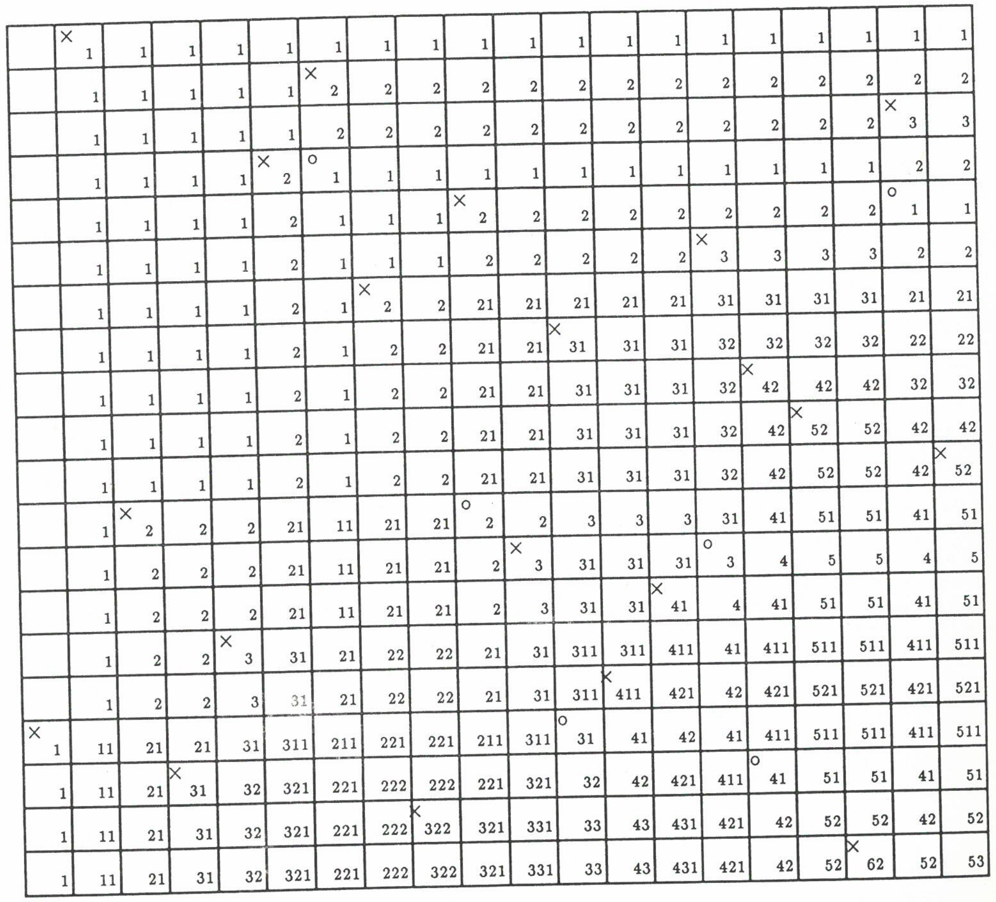

Fig. 1

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