A geometrical approach to the Littlewood-Richardson rule

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Abstract. In this talk we will give a geometrical interpretation to a formula equivalent to the Littlewood-Richardson rule, using a result of Steinberg which interprets the Robinson-Schensted correspondence in terms of flags.

1 The Littlewood-Richardson Rule

Let S_n be the symmetric group, $S_k \times S_l$ a Young subgroup where $k + l = n$. If λ is a partition of n (written $\lambda \vdash n$) let ζ^λ be the irreducible character of S_n corresponding to λ. Now let $\mu \vdash k$, $\nu \vdash l$. Then the induced character $Ind_{S_k \times S_l}^{S_n}(\zeta^\mu \times \zeta^\nu)$ can be decomposed as $\sum_\lambda c_{\mu,\nu}^\lambda \zeta^\lambda$ where the $c_{\mu,\nu}^\lambda$ are given by the well-known Littlewood- Richardson formula: $c_{\mu,\nu}^\lambda$ is the number of skew tableaux T of shape $\lambda - \mu$ and weight ν such that the word $w(T)$ of T is a lattice permutation.

An alternative way of computing the Littlewood-Richardson coefficients was given by Remmell and Whitney [3]. They used the interpretation of the Littlewood-Richardson coefficients as the multiplication constants given by the multiplication of Schur functions, and they proved their rule by using work of D. White [7] which relates the Littlewood-Richardson formula and the Robinson- Schensted correspondence. A rule similar to that of Remmell and Whitney has also been given by Robinson in his book ([4], p.61).

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The rule given by Remmell and Whitney is equivalent to the following: For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ let R_λ be the Young tableau with $1, 2, \ldots, \lambda_1$ in the first row, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in the second row and so on. Given the tableaux R_μ, R_ν on symbols $1, 2, \ldots, k$ and $k+1, \ldots, n$ respectively where $\mu \vdash k$, $\nu \vdash l$ as above we define the set $T(\mu, \nu)$ to be the set of tableaux T on the symbols $1, 2, \ldots, n$ satisfying:

(i) If $a, a+1$ are in the same row of R_μ or R_ν then $a+1$ is strictly east and weakly north of a in T, and

(ii) If a, b are at the ends of adjacent rows in R_μ or R_ν then $b - i$ is strictly south and weakly west of $a - i$, for any $a - i$ in the row of a and $b - i$ in the row of b.

Then the tableaux T are precisely those which correspond to characters ζ^λ of S_n such that $c^\lambda_{\mu,\nu}$ is not zero, and $c^\lambda_{\mu,\nu}$ is the number of T of shape λ.

2 Unipotent elements of $GL(n, C)$ and flags

A reference for the material in this section is [5] or [6].

Let $G = GL(nC))$ acting on a vector space V of dimension n over C. Two complete flags $F = \{V_0 \subset V_1 \subset \ldots \subset V\}$ and $F' = \{V'_0 \subset V'_1 \subset \ldots \subset V\}$ are in relative postition w, where $w \in S_n$, if there is a basis $\{v_1, v_2, \ldots v_n\}$ of V such that $\{v_1, v_2, \ldots v_j\}$ is a basis of V_j and $\{v_{w1}, v_{w2}, \ldots v_{w_j}\}$ is a basis of V'_j. Let u be a unipotent element in G, and let $\mathcal{B}_u$ be the variety of flags fixed by u. Let the conjugacy class of u correspond to $\lambda \vdash n$. Then the components of $\mathcal{B}_u$ are in bijection with the standard tableaux of shape λ as follows: If F is as above, to F we attach a tableau T of shape λ such that the subtableau containing $\{1, 2, \ldots k\}$ has the shape of $u \mid V_k$. The map $F \to T$ gives a bijection $\Sigma(u) \to T_\lambda$, where $\Sigma(u)$ is the set of components of $\mathcal{B}_u$ and T_λ is the set of standard tableaux of shape λ, such that each fibre is a dense open part of the corresponding component. We then have the following theorem.

Theorem (Steinberg). Let $u \in G$ be a unipotent element with Jordan form given by $\lambda \vdash n$. Let T, T' be standard tableaux of shape λ, and let F, F' be complete flags, "generic" in their components, such that F, F' correspond to T, T' respectively. Then, if F, F' are in relative position w, w corresponds to the pair (T, T') under the Robinson-Schensted correspondence.

We then have a bijection between S_n and the set of triples (u, F, F') where u runs over a set of representatives of the unipotent classes of G, and F, F' are generic representatives of the components of $\Sigma(u)$.

3 The connections

For an account of the Kazhdan-Lusztig theory we refer to [1], or to [2] for the combinatorial aspects of the theory in the case of S_n.

We return to the setup where we consider the induced representation $Ind^{S_n}_{S_k \times S_l}(\zeta^\mu \times \zeta^\nu)$, and interpret the Littlewood-Richardson rule as follows.

A (right) cell in S_n can be regarded as the set of all $w \in S_n$ having the same right-hand tableau under the Robinson-Schensted correspondence. We start with a cell representation of $S_k \times S_l$ and wish to describe the cells contained in the induced representation. An analysis of these cells leads to the following question: Let $w' \in S_k \times S_l$, and let $w = w'd \in S_n$, where d is a distinguished coset representative for $S_k \times S_l$ in S_n. If w', w correspond to triples $(v, K, K'), (u, F, F')$ respectively where v, u are unipotent elements in $GL(k, C) \times GL(l, C)$, $GL(n, C)$ respectively and K, K', F, F' are flags, what is the relationship between the triples (v, K, K') and (u, F, F')? An answer to this question leads to the Remmel-Whitney rule. This is the geometrical interpretation mentioned in the title.

References

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Computing the Hilbert-Poincaré series of monomial ideals, applications to Gröbner bases*

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Abstract

The Hilbert-Poincaré series of an homogeneous ideal, or of the homogenization of an affine ideal, can be computed through the associated staircase of a Gröbner basis of the ideal. In this paper we review some recent results on algorithms to compute the Hilbert-Poincaré series of a staircase, see [BCRT], and some applications of the computation of the Hilbert-Poincaré series to the computation of Gröbner bases, see [GT], [Ca].

1 The computation of the Hilbert-Poincaré series.

This section is a summary of the paper [BCRT], to which we refer for complete proofs and results.

In the computation of the Hilbert-Poincaré series of an homogeneous ideal I, the known algorithms, [MM1], [MM3], [KP], [BS], [BCR] have a first algebraic step coinciding with the computation of the associated Gröbner basis w.r.t. any term-ordering and the corresponding initial ideal (the *associated staircase*), and a second combinatorial step that from the staircase computes the Hilbert-Poincaré series.

The algorithms of [MM1] and [MM3] use techniques similar to the computation of a resolution; the algorithms of [KP] and [BCR] proceed by induction on the dimension; the algorithm of [BS] proceeds by induction on the number of generators of the initial ideal (the cogenerators of the staircase).

Usually, combinatorial algorithms can be speeded by a "Divide and Conquer" approach: splitting the problem into two smaller problems of approximately the same size. In successful cases this trades a linear step for a logarithmic step, and can reduce from exponential to polynomial complexity.

Our approach explains how to split a staircase through the choice of a monomial (the *pivot*), then we discuss how to design a strategy for the choice of the pivot. The worst case complexity is not improved, since in some extreme cases every splitting is bad, (the computation of Hilbert-Poincaré series is at least as difficult as a NP-complete problem in the number of variables, see [BS]) but in several practical cases the situation is much better; in particular, our algorithm in the best case has a complexity that is a linear factor better than the best case of [BS], and can be specialized, with a choice of the splitting strategy, to the algorithm of [BCR]. In practice, a simple random strategy is quite good, avoids the costly computations involved in choice of an optimal variable of [BCR], and marginally improves the performance even in the optimal Borel-normed case.

The algorithms have been implemented, both in CoCoA, [GN] and AlPi, [TD]. Some test cases are given.

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