SOME STUDIES OF FACTORS OF INFINITE WORDS GENERATED BY INVERTIBLE SUBSTITUTION

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ABSTRACT

In this paper, we introduce a class of infinite words generated by invertible substitutions and establish some elementary properties of factors of these infinite words. As a consequence, we prove these words are Sturmian words.

Keywords: Invertible substitution,singular word, sesquipower.

1. Introduction

Let $S = \{a, b\}$ be the alphabet of two letters. Let S^* and $\tilde{S}$ be respectively the free monoid with empty word ϵ as neutral element and the free group generated by S.

A morphism $\pi : S^* \to S^*$ is called a substitution over S. If π is also in $Aut(\tilde{S})$, then π is called an invertible substitution, where $Aut(\tilde{S})$ denotes the group of automorphisms over $\tilde{S}$. It is known [6,7] that $Aut(\tilde{S})$ is generated by the following three special automorphisms: $\sigma = (ab, a), \phi = (b, a), \psi = (a, b^{-1})$, where $\tau = (u, v)$ means that $\tau(a) = u, \tau(b) = v$.

Let $\sigma = (ab, a)$ be defined as above, then the infinite Fibonacci word is obtained by iterating σ starting with the letter a which is also a fixed point of σ [1]. The combinatorial properties of the infinite Fibonacci word are studied extensively by many authors, (see Berstel[1], Mignosi[5] and the references therein.) Notice that σ is an invertible substitution, we are led to consider a natural generalization of the infinite Fibonacci word: the infinite words generated by the invertible substitutions.

Brown [2], Kosa [4], Séébold [9] have considered the invertible substitutions of $\langle \phi, \Phi_1, \Phi_2 \rangle$, where $\Phi_1 = (a, ab), \Phi_2 = (a, ba)$, $\langle \phi, \Phi_1, \Phi_2 \rangle$ denotes the semigroup generated by ϕ, Φ_1, Φ_2. They studied both combinatorial and arithmetic properties of the infinite words generated by these invertible substitutions.

The purpose of this note is to study the properties of factors of infinite words generated by the invertible substitutions. We recall first some preliminaries. Then in section 1, we discuss the local isomophism of two invertible substitutions, by using the result, instead of studying the general case, we only need to study a special class of invertible substitutions. In section 2, after establishing some elementary properties of factors, we introduce singular factors and discuss their properties. As an example, by using singular factors, we determine the special factors and prove that all these infinite words are Sturmian words.

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In this paper, we shall use the following terminology.

Let $w \in S^*$ be a word. We denote by $|w|$ the length of w, and we denote by $|w|_a$ (resp.$|w|_b$)the number of letters a (resp.b) in the word w, we denote by $L(w)$ the vector $(|w|_a, |w|_b)$.

A word v is a factor of a word w and we write $v \prec w$, if there exists $u, u' \in A^*$, such that $w = uvu'$. We say that v is a left (resp.right) factor of a word w and we note $v \triangleleft w$ (resp. $v \triangleright w$), if there exists $u \in A^*$ such that $w = vu$ (resp. $w = uv$). The notions of factor, left factor are extended in a natural way to an infinite word.

Let $w = x_1, x_2 \dots x_n \in S^*$, the reversed word $\overline{w}$ of w is defined as $\overline{w} = x_n \dots x_2 x_1$. A word w is called palindrome if $w = \overline{w}$ and we denote by $\mathcal{P}$ the set of palindrome words.

Let $w = x_1 x_2 \dots x_n$. For $1 \le k \le n$, we define the kth conjugation of w as $C_k(w) = x_{k+1} \dots x_n x_1 \dots x_k$, and we denote by $C(w) = \{C_k(w); 1 \le k \le |w|\}$.

A word $w \in S^*$ is called primitive if $w = u^p, u \in S^*, p > 0$ implies $w = u$.

Let $\tau : S^* \to S^*$ be a substitution of S^*. We denote by F_τ the fixed point of τ [8] (if it exists).

Let w be a word, we denote by $\Omega_n(w)$ the set of factors of w of the length n, where $|w| \ge n$, and we note simply $\Omega_n := \Omega_n(F_\tau)$.

Let $w = x_1 x_2 \dots x_n \in S^*$, we denote by w^{-1} the inverse word of w, that is $w^{-1} = x_n^{-1} \dots x_2^{-1} x_1^{-1}$. Let $w = uv$, then $wv^{-1} = u$ by convention.

1. Local isomorphism of invertible substitutions

Lemma 1.1. *Let*

$$\mathcal{M} = \left\{ M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}, p, q, r, s \in \mathbf{N},\ det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1 \right\}$$

Then $\mathcal{M} = \langle \alpha, \beta \rangle$, *where* $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. Let $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, g_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then it is easy to check that $g_1 = \alpha^2, g_2 = \beta\alpha, g_3 = \alpha\beta, g_4 = \alpha\beta\alpha$.

Now let $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ be an element of $\mathcal{M}$, such that at least one entry is larger than 1. By using α, we can make p the largest entry of the matrix. Moreover, $r \ge s$ (otherwise $|detM| > 1$).

Let $M' = \begin{pmatrix} p-q & r-s \\ q & s \end{pmatrix}$, then $M = \beta\alpha M'$. By the same argument as above, we can transform M' by α such that it satisfies the same condition as M.

After a finite number of steps, the matrix M will be transformed, by the actions of α and β , to one of the six matrices: $\alpha, \beta, g_1, g_2, g_3, g_4$, which may be generated by α, β.

From the definition of the substitutive matrix, we can prove easily

Lemma 1.2. *Let* $\tau = \tau_1 \tau_2 \dots \tau_n$, *then*

$$M_\tau = M_{\tau_1} M_{\tau_2} \dots M_{\tau_n},$$

where M_τ *and* M_{τ_j} *are the substitutional matrices of* τ *and* τ_j *respectively.*

Lemma 1.3. *For any* $M \in \mathcal{M}$, *there is a* $\tau \in \langle \sigma, \phi \rangle$, *such that* $M_\tau = M$.

Proof. By lemma 1.1, for any $M \in \mathcal{M}$, there exist $M_1, M_2, \dots M_n \in \{\alpha, \beta\}$ such that $M = M_1 M_2, \dots M_n$.

Note that $M_\sigma = \beta$, $M_\phi = \alpha$, so if we take $\tau = \tau_1 \tau_2 \dots \tau_n$, where $\tau_j = \phi$ or σ according to $M_j = \alpha$ or β, then by lemma 1.2, we get

$$M_\tau = M_{\tau_1} M_{\tau_2} \dots M_{\tau_n} = M_1 M_2 \dots M_n = M.$$

Lemma 1.4. [8] Let τ_1, τ_2 be two invertible substitutions , then the following two assertions are equivalent:

1). $M_{\tau_1} = M_{\tau_2}$;

2). there exists a $w \in S^*$, such that $\tau_1 = w\tau_2 w^{-1}$, (or $\tau_1 = w^{-1}\tau_2 w$).

Let $u = u_1 u_2 \dots u_n \dots$ and $v = v_1 v_2 \dots v_n \dots$ be two infinite words over the alphabet S. We say that u and v are locally isomorphic if any factor (or its mirror image) of u is also factor of v and vice versa.

If u and v are locally isomorphic, we shall note $u \simeq v$.

Lemma 1.5. *Let* τ_1, τ_2 *be two invertible substitutions and let* F_{τ_1} ,F_{τ_2} *be the fixed points of* τ_1 *and* τ_2 *respectively. If* $M_{\tau_1} = M_{\tau_2}$, *then* $F_{\tau_1} \simeq F_{\tau_2}$.

Proof. By lemma 1.4, $\exists w \in S^*$, such that

$$\tau_1(a) = w\tau_2(a)w^{-1}.$$

Furthermore for any n

$$\tau_1^n(a) = \tau_1^{n-1}(w)\tau_1^{n-2}(w) \dots w\tau_2^n(a)w^{-1} \dots \tau_2^{n-1}(w^{-1})\tau_2^n(w^{-1})$$

That is, $\exists w_n \in S^*$, such that $\tau_1^n(a)w_n = w_n\tau_2^n(a)$. Thus $\tau_1^n(a)$ and $\tau_2^n(a)$ are conjugate, (see [3]), that is, there exist u_n and v_n, such that $\tau_1^n(a) = u_n v_n$, and $\tau_2^n(a) = v_n u_n$.

This means that, both u_n and v_n are factors of $\tau_1^n(a)$ and $\tau_2^n(a)$. Since $|\tau_1^n(a)|$ tends to infinity, so does $|u_n|$ (or $|v_n|$.) Let w be a factor of F_{τ_1}, by proposition 4.12 of [8], $w \prec u_n$ (or $w \prec v_n$) for n large enough, thus $w \prec F_{\tau_2}$. Inversely, by the same argument, if $w \prec F_{\tau_2}$, then $w \prec F_{\tau_1}$. We thus prove that $F_{\tau_1} \simeq F_{\tau_2}$.

The case of $\tau_1(a) = w^{-1}\tau_2(a)w$ can be treated in the same manner.

Remark 1.6. By lemma 1.3 and 1.5, we see that to study the properties of factors of an infinite word generated by an invertible substitution, it suffices to consider the elements of $\langle \sigma, \phi \rangle$, that is, the invertible substitutions of the following form:

$$\sigma^{n_k} \circ \phi \circ \sigma^{n_{k-1}} \circ \phi \dots \sigma^{n_2} \circ \phi \circ \sigma^{n_1},$$

where $n_1, n_k \ge 0; n_2, \dots, n_{k-1} \ge 1$.

Throughout this paper, we only discuss the following form:

$$\sigma^{n_k} \circ \phi \circ \sigma^{n_{k-1}} \circ \phi \dots \sigma^{n_2} \circ \phi \circ \sigma^{n_1},$$

where $n_j \geq 1, 1 \leq j \leq k$, and we denote by $\mathcal{G}$ the set of these invertible substitutions.

The other three cases ($n_k \geq 1, n_1 = 0; n_k = 0, n_1 \geq 1; n_k = n_1 = 0$.) can be discussed in the same way.

Lemma 1.1 and lemma 1.3 give an algorithm for finding an invertible substitution of which the substitutional matrix is equal to a given matrix of $\mathcal{M}$. Now we give an explicit algorithm by continued fractions.

The following lemma can be checked easily.

Lemma 1.7. *For any $w \in S^*$, we have*
1). $L(\sigma(w)) = (|w|_a + |w|_b, |w|_a)$, $L(\phi(w)) = (|w|_b, |w|_a)$;
2). $L((\sigma\phi)^n(w)) = (|w|_a + n|w|_b, |w|_b)$, $L((\phi\sigma)^n(w)) = (n|w|_a + |w|_b, |w|_a)$.

Lemma 1.8. *Let $(p,q) = 1, q \neq 1$. Then there exists a unique pair (r,s) (resp. (r',s')), $0 \leq r < p, 0 \leq s < q$ (resp. $0 \leq r' < p, 0 \leq s' < q$), such that $ps - rq = 1$ (resp. -1).*

Proof. The existence of a such pair is well known. Now suppose that there is another pair (r^*, s^*) satisfying the condition, then we have

$$p(s - s^*) = q(r - r^*).$$

Since $s - s^* < q, r - r^* < p$, we get a contradiction to the hypothesis for $(p,q) = 1$.

Remark 1.9. If $q = 1$, the conclusion holds trivially.

Now let $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathcal{M}$ with p the largest entry of M. Let $t_0, t_1, ... t_n$ be the continued fraction expansion of p/q. Then by the lemmas 1.7 and 1.8, it is not difficult to prove that

Proposition 1.10. *With the notations above. Let*

$$\begin{aligned} \tau_1 &= (\sigma\phi)^{t_0 - 1}\sigma(\sigma\phi)^{t_1 - 1}\sigma(\sigma\phi)^{t_{n-1}-1}\sigma(\sigma\phi)^{t_n - 1}\sigma, \\ \tau_2 &= (\sigma\phi)^{t_0 - 1}\sigma(\sigma\phi)^{t_1 - 1}\sigma(\sigma\phi)^{t_{n-1}-1}\sigma(\sigma\phi)^{t_n - 1}\phi\sigma. \end{aligned}$$

If $\det M = 1$, then $M_{\tau_1} = M$ if n is odd and $M_{\tau_2} = M$ if n is even,
If $\det M = -1$, then $M_{\tau_1} = M$ if n is even and $M_{\tau_2} = M$ if n is odd.

2. Studies of factors

In this section, we first establish some elementary properties of the infinite words generated by invertible substitution, then we introduce singular words and study their properties. As we shall see, the singular words play an important role in the study of the properties of the factors.

If $\alpha\beta$ appears in the context, we mean that $\alpha, \beta \in S$ with $\alpha \neq \beta$. Moreover if we write $\alpha\beta \triangleright \tau(b)$, we assume $|\tau(b)| \geq 2$.

The following lemma can easily be proved:

Lemma 2.1 *Let $\tau \in \mathcal{G}$, then*

1). $(\sigma\phi)^n(a) = a$, $(\sigma\phi)^n(b) = a^n b$;
2). If $\sum_{j=1}^k n_j + k \in 2\mathbf{N}+1$, then $ab \triangleright \tau(a)$; if $\sum_{j=1}^k n_j + k \in 2\mathbf{N}$, then $ba \triangleright \tau(a)$;
3). $\tau(b) \triangleleft \tau(a)$, $\tau(a) \triangleleft \tau^n(a)$, $\tau(b) \triangleleft \tau^n(b)$, $n \geq 2$;
4). If $|\tau(b)| = 1$, then $\tau = (\sigma\phi)^n\sigma$ for some n; if $|\tau(b)| = 2$, then $\tau = \sigma(\sigma\phi)^n\sigma$ for some n; if $|\tau(a)| = 2$, then $\tau = \sigma$;
5). If $|\tau(b)| > 1$ and $\alpha\beta \triangleright \tau(a)$, then $\beta\alpha \triangleright \tau(b)$.

Lemma 2.2 *let $\pi = (\sigma\phi)^n\sigma^2$ and let $w \in S^*$. Then*
1). $a^{n+1}b \triangleleft \pi(w)$;
2). $a \triangleright w \Rightarrow ba \triangleright \pi(w)$, $b \triangleright w \Rightarrow a^{n+1}b \triangleright \pi(w)$, $ab \triangleright w \Rightarrow a^{n+2}b \triangleright \pi(w)$;
3). The highest power of a appearing in $\pi(w)$ is a^{n+2} if $aab \prec \sigma^2(w)$ and a^{n+1} otherwise; The factor of $\pi(w)$ situated between two adjacent b is either a^{n+1} or a^{n+2};
4). The factor of $\pi(w)$ situated between two adjacent b is either a^{n+1} or a^{n+2}.

Proof. Let $\sigma^2(w) = x_1 x_2 ... x_{m-1} x_m$. Since $\sigma^2(a) = aba, \sigma^2(b) = ab$, it is easy to see that: 1. $x_1 x_2 = ab$; 2. $x_{m-1}x_m = ab$ or ba according to $b \triangleright w$ or $a \triangleright w$; 3. $a^3 \not\prec \sigma^2(w), b^2 \not\prec \sigma^2(w)$; 4. the factor of $\sigma^2(w)$ situated between two adjacent b is either a^2 or a. Since

$$\pi(w) = (\sigma\phi)^n\sigma^2(w) = (\sigma\phi)^n(x_1)(\sigma\phi)^n(x_2)...(\sigma\phi)^n(x_{m-1})(\sigma\phi)^n(x_m),$$

the lemma follows from the discussions above and lemma 2.1.1.

Lemma 2.3 *Let $k(\tau(a))$ (resp. $k(\tau(b))$) be the integer such that $a^{k(\tau(a))}$ (resp. $b^{k(\tau(b))}$) is the highest power of a (resp. b) appearing in $\tau(a)$. (resp. $\tau(b)$) Then*
1). $k(\tau(b)) \leq k(\tau(a))$;
2). $a^{p(\tau(a))}b \triangleleft \tau(a)$, where $p(\tau(a))$ is $k(\tau(a))$ or $k(\tau(a)) - 1$;
3). Either $a^{p(\tau(a))}b \triangleright \tau(a)$, or $ba \triangleright \tau(a)$;
4). $a^{p(\tau(a))+1} \prec F_\tau$, $a^{p(\tau(a))+2} \not\prec F_\tau$, $b^2 \not\prec F_\tau$;
5). Any factor of F_τ situated between two adjacent b is either $a^{p(\tau(a))+1}$ or $a^{p(\tau(a))}$.

Proof. Let $\tau = \sigma^{n_k}\phi\sigma^{n_{k-1}}\phi...\sigma^{n_2}\phi\sigma^{n_1}$.

If $n_j = 1$ for all j, $1 \leq j \leq k$, then $\tau = (\sigma\phi)^k\sigma$. In this case, the lemma can be proved easily.

Now suppose that there exists at least one index j, such that $n_j > 1$. Let j_0 be the largest index of these indices. Then

$$\tau = (\sigma\phi)^{k-j_0}\sigma^2(\sigma^{n_{j_0}-2}...\sigma^{n_2}\phi\sigma^{n_1}). \tag{1}$$

Hence the proof of this lemma follows from the lemma 2.2 by treating the different cases of (1) (the discussion is tedious, but not difficult, and we omit the details.)

We now determine the length of $\tau(a)$ and $\tau(b)$. Let $\mu, \nu \in \mathbf{N}$, we denote by $F_n(\mu,\nu)$ the Fibonacci number with initial condition $F_{-1}(\mu,\nu) = \nu, F_0(\mu,\nu) = \mu$. Thus for $w \in S^*$, it is readily seen that

$$L(\sigma^n(w)) = (F_n(|w|_a, |w|_b), F_{n-1}(|w|_a, |w|_b)), L(\phi(w)) = (|w|_b, |w|_a).$$

Thus it is not difficult to get

Lemma 2.4. *Let $\tau = \sigma^{n_k}\phi...\sigma^{n_2}\phi\sigma^{n_1}$, then*

$$\begin{aligned} |\tau(a)| &= F_{n_k+1}(\mu_{k-1},\nu_{k-1}), |\tau(b)| = F_{n_k}(\mu_{k-1},\nu_{k-1}); \\ L(\tau(a)) &= (F_{n_k}(\mu_{k-1},\nu_{k-1}), F_{n_k-1}(\mu_{k-1},\nu_{k-1}), \\ L(\tau(b)) &= (F_{n_k-1}(\mu_{k-1},\nu_{k-1}), F_{n_k-2}(\mu_{k-1},\nu_{k-1})); \end{aligned}$$

$$M_\tau = \begin{pmatrix} F_{n_k}(\mu_{k-1},\nu_{k-1}) & F_{n_k-1}(\mu_{k-1},\nu_{k-1}) \\ F_{n_k-1}(\mu_{k-1},\nu_{k-1}) & F_{n_k-2}(\mu_{k-1},\nu_{k-1}) \end{pmatrix},$$

where $\mu_0 = 1, \nu_0 = 0$, $\mu_{k-1} = F_{n_{k-1}}(\mu_{k-2},\nu_{k-2})$, $\nu_{k-1} = F_{n_k}(\mu_{k-2},\nu_{k-2})$.

Lemma 2.5. *$\forall\gamma \in S$, the word $\tau(\gamma)$ is primitive.*

Proof. Suppose that $\tau(\gamma)$ is not primitive, then there will exist a word $w \in S^*$ and an integer $p \geq 2$, such that $\tau(\gamma) = w^p$. Hence $L(\tau(\gamma))) = (p|w|_a, p|w|_b)$, thus $det(M_\tau)$ will have factor p, which is in contradiction with the fact $det(M_\tau) = \pm 1$.

Now we are going to establish an important factorization of $\tau(a)$ which will be used in the following.

Since $\tau(b) \triangleleft \tau(a)$, we can write $\tau(a) = \tau(b)u_n^*$. Let $u \triangleleft \tau(a)$ with $|u| = |u_n^*|$, we can write also $\tau(a) = uv$ with $|v| = |\tau(b)|$.

Lemma 2.6. *Let $\tau(a) = \tau(b)u^* = uv$ as above. Let $\alpha\beta \triangleright \tau(a)$, $|\tau(b)| \geq 2$. Then*

1). $u = u^*$;

2). $\tau(b)\alpha^{-1}\beta^{-1} = v\beta^{-1}\alpha^{-1}$;

3). $\tau(a) = \tau(b)u = u\tau(b)\alpha^{-1}\beta^{-1}\alpha\beta$.

Proof. Since $\alpha\beta \triangleright \tau(a)$, $\beta\alpha \triangleright \tau(b)$.

We prove the lemma by induction with respect to the length of τ in $\Sigma := \{\sigma, \phi\}$.

1). If $\tau = \sigma$, then $\sigma(a) = ab = \tau(b)b$, and the statement is true.

Suppose that for $\tau = \tau_n...\tau_1$, we have

$$\tau_n...\tau_1(a) = \tau_n...\tau_1(b)u_n^* = u_nv_n \qquad (*)$$

with $u_n^* = u_n$,(so $|v_n| = |\tau_n...\tau_1(b)|$.)

Let $\tau = \tau_{n+1}\tau_n...\tau_1$ and let

$$\tau(a) = \tau_{n+1}\tau_n...\tau_1(a) = \tau_{n+1}\tau_n...\tau_1(b)u_{n+1}^* = u_{n+1}v_{n+1}$$

with $|u_{n+1}^*| = |u_{n+1}|$. We have to prove $u_{n+1}^* = u_{n+1}$.

If $\tau_{n+1} = \phi$, the proof is trivial. Now let $\tau_{n+1} = \sigma$, we have by $(*)$

$$\begin{aligned} \sigma(\tau_n...\tau_1(a)) &= \sigma(u_nv_n) \\ &= \sigma(u_n)\sigma(v_n) = \sigma(\tau_n...\tau_1(b))\sigma(u_n^*) \end{aligned}$$

Thus $u_{n+1}^* = \sigma(u_n^*, u_{n+1} = \sigma(u_n)$.

Since $u_n^* = u_n$ by hypothesis of induction, we get $u_{n+1}^* = u_{n+1}$.

2.) The case of $|\tau(b)| = 2$ can be checked directly. Let $\tau = \tau_n...\tau_1$, then by 1) we have

$$\tau_{n+1}\tau_n...\tau_1(a) = \tau_{n+1}\tau_n...\tau_1(b)u_{n+1} = u_{n+1}v_{n+1}.$$

Let $\alpha\beta \triangleright \tau_n...\tau_1(a)$ and suppose that $v_n\beta^{-1}\alpha^{-1} = \tau_n...\tau_1(b)\alpha^{-1}\beta^{-1}$. We shall prove

$$v_{n+1}\alpha^{-1}\beta^{-1} = \tau_{n+1}\tau_n...\tau_1(b)\beta^{-1}\alpha^{-1}.$$

As in 1), it suffices to consider $\tau_{n+1} = \sigma$. Since $\alpha\beta \triangleright \tau_{n+1}\tau_n...\tau_1(b)$, $\beta\alpha \triangleright \sigma(v_{n+1})$, we have

$$\begin{aligned} v_{n+1} = \sigma(v_n) &= \sigma(\tau_n...\tau_1(b)\alpha^{-1}\beta^{-1}\alpha\beta) \\ &= \sigma(\tau_n...\tau_1(b))(\sigma(\beta\alpha))^{-1}\sigma(\alpha\beta). \end{aligned}$$

Notice that $\sigma(\alpha\beta) = a\beta\alpha$ for any $\alpha \neq \beta$, and therefore we obtain

$$v_{n+1}\alpha^{-1}\beta^{-1} = \tau_{n+1}\tau_n...\tau_1(b)\beta^{-1}\alpha^{-1}.$$

3). The conclusion follows immediately from 1) and 2).

Lemma 2.7. *Let $w \in \mathcal{P}$, then $\sigma(w)a, a^{-1}\sigma(w) \in \mathcal{P}$.*

Proof. Since $w \in \mathcal{P}$, we can write w in the following form:

$$a^{k_1}b^{l_1}a^{k_2}b^{l_2}...a^{k_m}\gamma a^{k_m}...b^{l_2}a^{k_2}b^{l_1}a^{k_1}$$

where $\gamma \in \{a, b, \epsilon\}$, $k_j, l_j \geq 1$,if $1 \leq j \leq m-1$, an d $k_m \geq 0$.

Thus we have

$$\begin{aligned} \sigma(w)a &= (ab)^{k_1}a^{l_1}...(ab)^{k_m}\sigma(\gamma)(ab)^{k_m}...a^{l_2}(ab)^{k_2}a^{l_1}(ab)^{k_1}a \\ &= (ab)^{k_1}a^{l_1}...(ab)^{k_m}\sigma(\gamma)a(ba)^{k_m}...a^{l_2}(ba)^{k_2}a^{l_1}(ba)^{k_1}, \end{aligned}$$

Since $\sigma(\gamma)a$ is equal to a, aba, aa respectively according to γ being ϵ, a, b, $\sigma(w)a$ is a palindrome with center a, b, ϵ respectively by the formula above.

Corollary 2.8. *Let $\beta \triangleright \tau(\gamma)$, $\gamma \in S$. Then $\alpha\tau(\gamma)\beta^{-1} \in \mathcal{P}$. In particular, if $|\tau(\gamma)| \geq 3$, and $\alpha\beta \triangleright \tau(\gamma)$, then $\tau(\gamma)\beta^{-1}\alpha^{-1} \in \mathcal{P}$.*

Proof. Let $\tau = \tau_n\tau_{n-1}...\tau_1$. (Notice that $\tau_1 = \sigma$ by the definition of τ.)

We prove this lemma by induction as in lemma 2.6. We only prove the case $\gamma = a$, the case of $\gamma = b$ can be proved in a similar way.

The case of $n = 1$ is evident. Now suppose that $\beta \triangleright \tau_n\tau_{n-1}...\tau_1(a)$ and $\alpha\tau_n\tau_{n-1}...\tau_1(a)\beta^{-1} \in \mathcal{P}$.

If $\tau_{n+1} = \phi$, notice that the word $\phi(\tau_n\tau_{n-1}...\tau_1(a))$ is obtained by exchanging the letters a and b in the word $\tau_n\tau_{n-1}...\tau_1(a)$. So by the induction hypothesis, we get $\beta\phi(\tau_n\tau_{n-1}...\tau_1(a))\alpha^{-1} \in \mathcal{P}$.

Now suppose that $\tau_{n+1} = \sigma$. Notice that $\alpha \triangleright \sigma\tau_n\tau_{n-1}...\tau_1(a)$. Thus

$$\begin{aligned} \beta\sigma\tau_n\tau_{n-1}...\tau_1(a)\alpha^{-1} &= \beta\sigma(\alpha^{-1}\alpha\tau_n\tau_{n-1}...\tau_1(a)\beta^{-1}\beta)\alpha^{-1} \\ &= \beta\sigma(\alpha^{-1})\sigma(\alpha\tau_n\tau_{n-1}...\tau_1(a)\beta^{-1})\sigma(\beta)\alpha^{-1}. \end{aligned}$$

By the induction hypothesis $\alpha\tau_n\tau_{n-1}...\tau_1(a)\beta^{-1} \in \mathcal{P}$. On the other hand, by a simple calculation, we have either $\beta\sigma(\alpha^{-1}) = a^{-1}$, $\sigma(\beta)\alpha^{-1} = \epsilon$, or $\beta\sigma(\alpha^{-1}) = \epsilon$, $\sigma(\beta)\alpha^{-1} = a$. Hence by lemma 2.7, we get $\beta\sigma\tau_n\tau_{n-1}...\tau_1(a)\alpha^{-1} \in \mathcal{P}$.

Let $\tau(a) = \tau(b)u$ be the factorization of lemma 2.6. Let $\tau = \sigma^{n_k}\phi...\sigma^{n_2}\phi\sigma^{n_1}$, then $u = \sigma^{n_k}\phi...\sigma^{n_2}\phi\sigma^{n_1-1}(b)$. Thus by lemmas 2.1.4, 2.6 and corollary 2.8, we get

Corollary 2.9. *Let $\tau \in \mathcal{G}$ and let $\alpha\beta \triangleright \tau(a)$.*

1). If $|\tau(b)| = 1$, then $\tau(a) = a^n b$ for some n, thus $\tau(a)$ is the product of palindromes a^n and b;

2). If $|\tau(b)| = 2$, then $\tau(a)$ is a palindrome;

3). If $|\tau(b)| > 2$, then $\tau(a)$ is the product of palindromes $\tau(b)\alpha^{-1}\beta^{-1}$ and $\beta\alpha u$.

Lemma 2.10. *Suppose that $w \in S^*$ is a primitive word and suppose that $w = u_1u_2$ is a product of two palindromes u_1 and u_2.*

1). If $|u_1|, |u_2| \in 2\mathbf{N}+1$, then for any k, $C_k(w) \notin \mathcal{P}$;

2). If $|u_1|, |u_2| \in 2\mathbf{N}$, then $C_{|u_1|/2}(w) \in \mathcal{P}$, $C_{|u_1|+|u_2|/2}(w) \in \mathcal{P}$. For other k, $C_k(w) \notin \mathcal{P}$;

3). If $|u_1| \in 2\mathbf{N}$, $|u_2| \in 2\mathbf{N}+1$, (or $|u_1| \in 2\mathbf{N}+1$, $|u_2| \in 2\mathbf{N}$,) then $C_{|u_1|/2}(w) \in \mathcal{P}$, (or $C_{|u_1|+|u_2|/2}(w) \in \mathcal{P}$.) For other k, $C_k(w) \notin \mathcal{P}$.

Proof. We only prove 1), the other cases can be discussed in the same way. Notice that at least one of $|u_1|, |u_2|$ is larger than 2 (otherwise u_1u_2 will be a^2 or b^2.) Suppose that without loosing generality $|u_1| \geq 3$. Then we can write $u_1 = xu_1'x$, where $x \in S$, $u_1' \in \mathcal{P}$. Thus $C_1(w) = C_1(u_1u_2) = (u_1')(xu_2x)$ is a product of two palindromes, by the proposition 9 of [3], $C_1(u_1u_2) \notin \mathcal{P}$. Since $|u_1'|, |xu_2x| \in 2\mathbf{N}+1$ and $u_1', xu_2x \in \mathcal{P}$, we can repeat the discussion above which follows the proof .

Lemma 2.11. *Suppose that $|\tau(a)| \in 2\mathbf{N}$. If $\tau(a) = u_1u_2$ is a product of two palindromes,then $|u_1|, |u_2| \in 2\mathbf{N}+1$.*

Proof. If $|u_1|, |u_2| \in 2\mathbf{N}$, then $|u_1|_a, |u_1|_b, |u_2|_a, |u_2|_b \in 2\mathbf{N}$. Thus both components of $L(\tau(a))$ are even which follows that $|det(M_\tau)| \in 2\mathbf{N}$. But we know that $det(M_\tau)$ must be ± 1.

From corollary 2.9, lemma 2.10, lemma 2.11 and the propositions 9,10,11 of [3], we obtain

Proposition 2.12. *Let $\tau \in \mathcal{G}$.*

1). Any conjugate of $\tau(a)$ is primitive;

2). $|C(\tau(a))| = |\tau(a)|$. That is, all conjugates of $\tau(a)$ are distinct;

3). Any conjugation of $\tau(a)$ (containing $\tau(a)$ itself) is either a palindrome, or a product of two palindromes. In the later case, the factorization is unique;

4). $C(\tau(a)) = \overline{C(\tau(a))}$, where $\overline{C(\tau(a))} = \{\overline{w}; w \in C(\tau(a))\}$.

5). If $|\tau(a)| \in 2\mathbf{N}$, there is no palindrome in $C(\tau(a))$; if $|\tau(a)| \in 2\mathbf{N}+1$, there is only one palindrome in $C(\tau(a))$.

Let $F_\tau = \tau(F_\tau) = x_1x_2...x_k...$ be the fixed point of τ. Then for any $n \in \mathbf{N}$, we have

$$F_\tau = \tau^n(F_\tau) = \tau^n(x_1)\tau^n(x_2)...\tau^n(x_k)... \qquad (**)$$

We now discuss the properties of the factors of the infinite word F_τ.

We denote by $A_n = \tau^n(a), B_n = \tau^n(b)$, and $l_{n,a} = |A_n|, l_{n,b} = |B_n|$.

Notice that if $\tau \in \mathcal{G}$, then for any $n \geq 2,$, $\tau^n \in \mathcal{G}$. Therefore the conclusions about τ which we have obtained above hold also for τ^n.

Let w be a factor of F_τ of length $|A_n|$. By $(**)$, w will be contained in one of the following four words: $A_nA_n, A_nB_n, B_nA_n, A_nB_nA_n$. By lemma 2.1.3, $B_n \triangleleft A_n$, so $A_nB_n \prec A_nA_n$.

Let $w \prec A_nB_nA_n$,write $w = w_1B_nw_2$.Then $w_1 \triangleright A_n, w_2 \triangleleft A_n$. Let $A_n = B_nu_n$ be the factorization as in lemma 2.6. Then $u_n \triangleleft A_n$ by lemma 2.6.3. Since

$|w| = |A_n| = |B_n| + |u_n|$, $|w_2| < |u_n|$. Consequently $w_2 \triangleleft u_n$, and hence $w = w_1B_nw_2 \prec A_nB_nu_n = A_nA_n$.

By the discussions above, any factor of length $|A_n|$ of F_τ will be contained either in A_nA_n or B_nA_n which we are going to study respectively.

Lemma 2.13. *Let $w \prec A_nA_n$ with $|w| = |A_n|$, then $w \in C(A_n)$.*

Proof. Since $w \prec A_nA_n$, and $|w| = |A_n|$, we can write $w = uv$ with $u \triangleright A_n, v \triangleleft A_n$, and $|u| + |v| = |A_n|$, which follows that $A_n = vu$. That is $w = C_{|v|}(A_n)$.

Since all conjugates of A_n are distinct by proposition 2.12.2, we get

Corollay 2.14. *The set of the factors of length $|A_n|$ of $A_n^k, k \geq 2$ is exactly $C(A_n)$.*

Now we study the factors of the word B_nA_n. For this aim, we introduce singular words of F_τ as follows:

Let $\alpha \triangleright A_n$ (so $\beta \triangleright B_n$). Then the nth singular words with respect a or b of F_τ are defined respectively as:

$$w_{n,a} = \beta A_n\alpha^{-1}, w_{n,b} = \alpha B_n\beta^{-1}.$$

From the definition of the singular words, we get immediately

Lemma 2.15. *With the notations above, we have*

$$L(w_{n,a}) = \begin{cases} (|A_n|_a + 1, |A_n|_b - 1) & \text{if } \alpha = b \\ (|A_n|_a - 1, |A_n|_b + 1) & \text{if } \alpha = a \end{cases}$$

$$L(w_{n,b}) = \begin{cases} (|B_n|_a + 1, |B_n|_b - 1) & \text{if } \alpha = a \\ (|B_n|_a - 1, |B_n|_b + 1) & \text{if } \alpha = b \end{cases}$$

Since $L(w) = L(A_n)$ for any $w \in C(A_n)$, we get from lemma 2.15,

Corollay 2.16. $\forall n \in \mathbf{N}, w_{n,a} \notin C(A_n), w_{n,b} \notin C(B_n)$.

Let $\alpha\beta \triangleright A_n$,(so $\beta\alpha \triangleright B_n$.) By lemma 2.6.3,

$$B_nA_n = B_nu_nB_n\alpha^{-1}\beta^{-1}\alpha\beta = A_nB_n\alpha^{-1}\beta^{-1}\alpha\beta,$$

Since $A_nB_n\alpha^{-1}\beta^{-1} \prec A_nB_n \prec A_nA_n$, we see that the first $|B_n| - 2$ factors of length $|A_n|$ of B_nA_n are those of A_nA_n which are distinct from each other by corollary 2.14, and the $(|B_n| - 1)th$ factor is exactly $w_{n,a}$ which is not in $C(A_n)$ by corollary 2.16. We get from the analysis above

Lemma 2.17. *Let $w \prec B_nA_n$ with $|w| = |A_n|$. Then w is either in $C(A_n)$ or equal to $w_{n,a}$. In particular, as a factor, $w_{n,a}$ appears only once in B_nA_n.*

From proposition 2.12.4, lemma 2.13, corollary 2.14, and lemma 2.17, we obtain

Proposition 2.18. *Let $\tau \in \mathcal{G}, n \in \mathbf{N}$. Then*

1). $\Omega_{l_{n,a}} = C(A_n(a)) \cup w_{n,a}$, $\Omega_{l_{n,b}} = C(B_n(b)) \cup w_{n,b}$;

2). $|\Omega_{l_{n,a}}| = l_{n,a} + 1$, $|\Omega_{l_{n,b}}| = l_{n,b} + 1$;
3). $\Omega_{l_{n,a}} = \overline{\Omega_{l_{n,a}}}$, $\Omega_{l_{n,b}} = \overline{\Omega_{l_{n,b}}}$.

We now discuss the properties of the singular words.

Proposition 2.19. $\forall n \in \mathbf{N}$, $w_{n,a}, w_{n,b} \in \mathcal{P}$.

Proof. This follows immediately from the definition of the singular words and lemma 2.8.

Lemma 2.20. $w_{n,a} \triangleright w_{n+2,a}$, $w_{n,b} \triangleright w_{n+2,b}$.

Proof. Let $\alpha\beta \triangleright \tau^n(a)$,then $\alpha\beta \triangleright \tau^{n+2}(a)$ by lemma 2.1.4, thus

$$w_{n+2,a} = \alpha\tau^{n+2}(a)\beta^{-1} = \alpha\tau^n(\tau^2(a))\beta^{-1}.$$

By lemma 2.1.2,$a \triangleleft \tau^2(a)$, so $\tau^2(a) = ua$ for some $u \in S$. Thus $w_{n+2,a} = \alpha\tau^n(u)\tau^n(a)\beta^{-1}$ which yields the conclusion of the lemma.

Lemma 2.21. *Either* $a^{p(\tau(a))+1} \triangleright w_{n,a}$, $a^{p(\tau(a))+1} \triangleleft w_{n,a}$, *or* $b \triangleright w_{n,a}, b \triangleleft w_{n,a}$, *where* $p(\tau(a))$ *is defined as in lemma 2.3.2.*

This follows from lemma 2.3.2, proposition 2.18 and the definition of $w_{n,a}$.

Proposition 2.22. $\forall n \in \mathbf{N}$, *if* $w_{n,a}$ *is not a power of* a, *then* $w_{n,a}$ *is primitive.*

Proof. Suppose that $w_{n,a}$ is not a power of a. If $w_{n,a} = w^p$ for some $w \in S^*$ and $p \geq 2$, then $w_{n,a}$ will be equal to one of its conjugates. But by lemma 2.21, we will then have either $b^2 \prec F_\tau$ or $a^{2p(\tau(a))+2} \prec F_\tau$. This is impossible because of lemma 2.2.4.

Let $F_\infty = x_1 ... x_n ...$ be an infinite word over S. Let $w \prec F_\infty$ be a word which appears infinitely many times in F_∞. If any two adjacent $w's$ appearing in F_∞ are separated by a factor of F_∞ , (this assertion is equivalent to $\nexists u \prec F_\infty$ such that $u = xyz$ with $w = xy = yz$.) we say that the word w posseses the positive separation property, and the factor is called a separate factor (with respect to w). If any separate factor is not equal to ϵ, we say that w is strongly separated, (otherwise, we say that w is weakly separated.)
A word $w \in S^*$ of the form $w = (uv)^k u$, where $k > 0$, uv is a primitive word, is called a sesquipower [3]. The positive integer k is said to be the order of the sesquipower. A sesquipower of order larger than 1 is called a strong sesquipower.

Let $n \geq 1$ and let $\alpha\beta \triangleright A_n$, from lemma 2.3, lemma 2.13, lemma 2.17, the formula (**) and the definition of $w_{n,a}$, we can write F_τ as

$$F_\tau = (A_n^{p(\tau(a))} B_n \alpha^{-1}) w_{n,a} z_1 w_{n,a} z_2 ... z_k w_{n,a} z_{k+1} ...$$

where z_j is either $\beta A_n^{p(\tau(a))} B_n \alpha^{-1}$ or $\beta A_n^{p(\tau(a))+1} B_n \alpha^{-1}$ and $p(\tau(a))$ is defined as in lemma 2.2.2.
Let $A_n = B_n u_n$ be the factorization as in lemma 2.6, then for any $m \geq 1$,

$$\beta A_n^m B_n \alpha^{-1} = ((\beta B_n \alpha^{-1})(\alpha u_n \beta^{-1}))^m (\beta B_n \alpha^{-1})$$

by corollary 2.9, $\beta B_n \alpha^{-1}, \alpha u_n \beta^{-1} \in \mathcal{P}$. Thus $\beta A_n^m B_n \alpha^{-1} \in \mathcal{P}$.

From the discussions above, we obtain

Theorem 2.23. *Let* $\tau \in \mathcal{G}$, *if* $\alpha\beta \triangleright \tau(a)$, *then for any* n,
1). $w_{n,a}$ *is strongly separated;*
2). The separate factor is either $\beta A_n^{p(\tau(a))+1} B_n \alpha^{-1}$, *or* $\beta A_n^{p(\tau(a))} B_n \alpha^{-1}$;
3). The separate factors are sesquipower palindromes of order $p(\tau(a))$ and $p(\tau(a)) + 1$ respectively.

Berstel [2] introduced the special words of an infinite word F_∞ as follows : if $ua, ub \prec F_\infty$, then the word u is called a special word of F_∞. Since $\{w, w \prec F_\tau\} = \{\overline{w}, w \prec F_\tau\}$, the special words of F_τ are equivalent to $au, bu \prec F_\tau$.
From corollary 2.14, lemma 2.17, theorem 2.23, by an analogous argument with that the theorem 5 of [10], we have the following result which has been found by Séébold [9. Proposition 4.7],(but he does not determine the special words).

Theorem 2.24 *Let* $\tau \in \mathcal{G}$. *Then for any* $n \geq 1$, *there is a unique special word of length* n. *Moreover* w *is a special word if and only if* $w \triangleleft F_\tau$.

Final remarks. As said in the introduction, the motivation of this paper is to generalize the properties of the factors of the infinite Fibonacci word to the case of all invertible substitutions. By comparing with the results of [10], we see that one main difference is the structure of the singular words (and the singular decomposition of the infinite word according to the singular words). In fact, the structure of the singular words of a general invertible substitution is much more complicated than that of the Fibonacci substitution (for example, all singular words and separate words of the Fibonacci word are sesquipowers of order 1, i.e, weak sesquipowers. On the other hand, those of a general invertible substitution are strong sesquipowers.) But as we shall see in another paper, the singular words still play an important role in the studies of the factors, such as power of factors, overlap of factors and local isomophism.

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SUR LES SYMÉTRIES TERNAIRES LIÉES AUX NOMBRES DE GENOCCHI

Par

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Abstract. — In [Du2] DUMONT stated several conjectures about some symmetric polynomial sequences which are the refinements of the Genocchi numbers. In this paper we shall prove all of his conjectures. We first show that some special cases of his conjecture can be readily derived from a result of Wall and then prove this conjecture by computing Hankel determinants. Finally we present a new symmetric model for the Dumont-Foata polynomials.

1. Introduction. — Un *escalier* F de taille $n \geq 1$ est le graphe d'une application *surjective* f de $[2n] := \{1, 2, \ldots, 2n\}$ sur $\{2, 4, 6, \ldots, 2n\}$ telle que pour tout k, $f(k) \geq k$. Un point $(k, f(k))$, $1 \leq k \leq 2n-2$, de F est dit *maximal* si $f(k) = 2n$, est dit *point fixe* s'il n'est pas maximal et si $f(k) = k$ (ce qui implique que k est pair), est dit *surfixe* s'il n'est pas maximal et si $f(k) = k+1$ (ce qui implique que k est impair). On note $m(F)$ le nombre de ses points maximaux, $f(F)$ le nombre de ses points fixes, et $s(F)$ le nombre de ses points surfixes. Si l'on note E_n l'ensemble des escaliers de taille n, il est alors bien connu (voir [Du1, Du-Fo, Ha, Vi]) que le cardinal de E_n est le nombre de Genocchi G_{2n+2}, qui peuvent être définis par leur fonction génératrice :

$$\frac{2t}{e^t+1} = t - \frac{t^2}{2!} + \frac{t^4}{4!} - 3\frac{t^6}{6!} + 17\frac{t^8}{8!} + \cdots + (-1)^n G_{2n}\frac{t^{2n}}{(2n)!} + \cdots$$

DUMONT et FOATA [Du-Fo] ont introduit une suite de polynômes $F_n(x,y,z)$ qui sont définis par $F_1(x,y,z) = 1$ et

$$F_n(x,y,z) = (x+y)(x+z)F_{n-1}(x+1,y,z) - x^2 F_{n-1}(x,y,z).$$

Ils ont montré que ces polynômes sont symétriques dans les variables x, y, z et raffinent les nombres de Genocchi en ce sens que $F_n(1,1,1) = G_{2n+2}$. On connait plusieurs interprétations combinatoires de ces polynômes (voir [Du1, Du-Fo, Ha, Vi]), mais il semble que celle la plus simple, en ce sence que la vérification de la récurrence est facile, est le résultat de HAN [Ha] suivant :

(1) $$F_n(x,y,z) = \sum_{F \in E_n} x^{m(F)} y^{f(F)} z^{s(F)}.$$

De là, DUMONT [Du2] a récemment proposé une extension des polynômes $F_n(x,y,z)$ de la manière suivante. Pour un escalier F de taille n, un point $(k, f(k))$ de F est dit *doublé* s'il n'est pas seul sur sa ligne, c'est-à-dire s'il existe $j \neq k$ tel que $f(j) = f(k)$. On note $\mathrm{fd}\, F$ (*resp.* $\mathrm{fnd}\, F$) le nombre de ses points *fixes doublés* (*resp. fixes non doublés*), $\mathrm{sd}\, F$ (*resp.* $\mathrm{snd}\, F$) le nombre de ses points *surfixes doublés* (*resp. surfixes*