A New Cubical h-Vector (Extended Abstract)

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Abstract

A new definition of an h-vector for cubical polytopes (and complexes) is introduced. It has many properties in common with the well-known h-vector for simplicial polytopes. In particular, it is symmetric, nonnegative and easily computable from a shelling of the polytope. Lower or upper bounds on its components imply corresponding bounds on the face numbers.

On introduit un nouveau h-vecteur défini pour des polytopes (ainsi que pour des complexes) cubiques. Celui-ci possède de nombreuses propriétés enjouies par le h-vecteur habituel des polytopes simpliciaux. Notamment, ce nouveau h-vecteur est symétrique et positif et se calcule facilement à partir d'un effeuillage du polytope. Des bornes inférieures et supérieures pour ses composantes entrainent des bornes pour le nombres de faces.

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1 Introduction

A **d-polytope** P is the convex hull of finitely many points which affinely span $\mathbf{R}^d$. The intersections of P with its various supporting hyperplanes are its (proper) **faces**, and are called **vertices**, **edges** or **facets** if they are of dimension 0, 1 or $d-1$, respectively. (The improper faces are P itself and the empty set.) A d-polytope is **cubical** if all its proper faces (equivalently: all its facets) are combinatorially equivalent to cubes (respectively, to $(d-1)$-cubes).

More generally, let C^{d-1} be the standard cube $[0,1]^{d-1}$ in $\mathbf{R}^{d-1}$, and let $V_0 = \operatorname{vert} C^{d-1}$ be its set of vertices. A (finite, pure, abstract) **cubical $(d-1)$-complex** consists of a finite nonempty set V together with a (finite) nonempty collection $\{\phi_\alpha\}_{\alpha\in I}$ of distinct injective maps $\phi_\alpha : V_0 \longrightarrow V$, such that $\phi_\alpha^{-1}(\phi_\beta(V_0))$ is the set of vertices of a (proper or improper) face of C^{d-1}, for all $\alpha, \beta \in I$. The images (under the various maps ϕ_α) of i-dimensional faces of C^{d-1} $(0 \le i \le d-1)$ are the **i-faces** (or **i-cubes**) of the complex. The (vertex-sets of) facets of a cubical d-polytope form a cubical $(d-1)$-complex.

Now let K be a cubical $(d-1)$-complex, and let f_i be the number of i-cubes in it $(0 \le i \le d-1)$. Use the convention $f_{-1} = 1$ to account for the empty set. The vector $(f_{-1}, \dots, f_{d-1})$ is known as the **f-vector** of K. Define a **cubical h-vector** $(h_0^{(c)}, \dots, h_d^{(c)})$ and a corresponding **cubical h-polynomial** $h_K^{(c)}(q)$ for the complex K by

$$h_K^{(c)}(q) = \sum_{i=0}^{d} h_i^{(c)} q^i \overset{\text{def}}{=} \sum_{j=0}^{d} f_{j-1}\phi_j(q), \tag{1}$$

where

$$\phi_0(q) = 2^{d-1}\frac{1-(-q)^{d+1}}{1+q} \tag{2}$$

and

$$\phi_j(q) = \frac{(2q)^j}{2} \cdot \frac{(1-q)^{d-j} + q(-2q)^{d-j}}{1+q} \qquad (1 \le j \le d). \tag{3}$$

Define also a **short cubical h-vector** $(h_0^{(sc)}, \dots, h_{d-1}^{(sc)})$ by

$$h_i^{(sc)} = h_i^{(c)} + h_{i+1}^{(c)} \qquad (0 \le i \le d-1), \tag{4}$$

or equivalently

$$h_K^{(sc)}(q) = \sum_{i=0}^{d-1} h_i^{(sc)} q^i \overset{\text{def}}{=} \sum_{j=0}^{d-1} f_j (2q)^j (1-q)^{d-1-j}. \tag{5}$$

These definitions are reminiscent of the defining equation for the h-vector of a **simplicial** $(d-1)$-complex Σ, which is

$$h_\Sigma^{(s)}(q) = \sum_{i=0}^{d} h_i^{(s)} q^i \overset{\text{def}}{=} \sum_{j=0}^{d} f_{j-1} q^j (1-q)^{d-j}. \tag{6}$$

The simplicial h-vector has some very appealing properties, and has been found to be an invaluable tool in the formulation and proof of results in the enumerative theory of simplicial convex polytopes. Typical examples are the proof by McMullen [Mc$_1$] of Motzkin's Upper Bound Conjecture [Mo], and the complete characterization result ("McMullen's g-conjecture") [Mc$_2$] proved by Stanley [S$_1$] (necessity) and Billera and Lee [BL] (sufficiency). The cubical h-vector introduced above shares some of these properties, and will hopefully find appropriate use in the (recently reviving) study of cubical polytopes.

2 Properties of the Cubical h-Vector

Let us first collect a few immediate observations.

Lemma 1
Let K be a cubical $(d-1)$-complex. Then:

(i) All $h_i^{(c)}$ $(0 \le i \le d)$ are integers.

(ii)

$$h_0^{(c)} = 2^{d-1}, \tag{7}$$

$$h_1^{(c)} = f_0 - 2^{d-1} \tag{8}$$

and

$$h_d^{(c)} = (-2)^{d-1}\tilde{\chi}(K), \tag{9}$$

where

$$\tilde{\chi}(K) = \sum_{j=0}^{d}(-1)^{j-1} f_{j-1}(K) \tag{10}$$

is the reduced Euler characteristic of K.

(iii) More generally,

$$h_i^{(c)} = (-1)^i 2^{d-1} f_{-1} + \sum_{j=1}^{i}(-1)^{i-j}2^{j-1}\left(\sum_{k=0}^{i-j}\binom{d-j}{k}\right) f_{j-1} \qquad (1 \le i \le d) \tag{11}$$

and

$$f_{j-1} = 2^{1-j}\sum_{i=1}^{j}\binom{d-i}{d-j}\left[h_i^{(c)} + h_{i-1}^{(c)}\right] \qquad (1 \le j \le d). \tag{12}$$

In particular, lower or upper bounds on the cubical h-numbers imply corresponding bounds on the f-numbers.

(iv) For the boundary complex of the d-cube:

$$h_0^{(c)} = \ldots = h_d^{(c)} = 2^{d-1}. \tag{13}$$

The corresponding results for the short cubical h-vector are:

Lemma 2
Let K be a cubical $(d-1)$-complex. Then:

(i) *All $h_i^{(sc)}$ $(0 \le i \le d-1)$ are integers.*

(ii)

$$h_K^{(sc)}(0) = h_0^{(sc)} = f_0 \tag{14}$$

and

$$h_K^{(sc)}(1) = \sum_{i=0}^{d-1} h_i^{(sc)} = 2^{d-1} f_{d-1}. \tag{15}$$

(iii) *Explicitly,*

$$h_i^{(sc)} = \sum_{j=0}^{i} \binom{d-1-j}{d-1-i} (-1)^{i-j} 2^j f_j \qquad (0 \le i \le d-1) \tag{16}$$

and

$$f_j = 2^{-j} \sum_{i=0}^{j} \binom{d-1-i}{d-1-j} h_i^{(sc)} \qquad (0 \le j \le d-1). \tag{17}$$

(iv) *For the boundary complex of the d-cube:*

$$h_0^{(sc)} = \ldots = h_{d-1}^{(sc)} = 2^d. \tag{18}$$

Following is a list of some more profound properties. For general terminology regarding partially ordered sets, refer to [S_2, Chapter 3].

Theorem 3
Let K be a cubical $(d-1)$-complex, and denote by $\hat{P}(K)$ its lattice of faces augmented by a maximal element. Then:

(i) *If $\hat{P}(K)$ is semi-Eulerian (e.g., if K is a cubical subdivision of a $(d-1)$-manifold without boundary) then its short cubical h-vector is symmetric:*

$$h_i^{(sc)} = h_{d-1-i}^{(sc)} \qquad (0 \le i \le d-1). \tag{19}$$

(ii) *If $\hat{P}(K)$ is Eulerian (e.g., if K is a cubical subdivision of a $(d-1)$-sphere) then its cubical h-vector is symmetric:*

$$h_i^{(c)} = h_{d-i}^{(c)} \qquad (0 \le i \le d). \tag{20}$$

Equivalently, its short cubical h-vector is symmetric and satisfies the equation

$$h_K^{(sc)}(-1) = \sum_{i=0}^{d-1} (-1)^i h_i^{(sc)} = 2^{d-1} + (-2)^{d-1}. \tag{21}$$

(iii) *Let $F_1, \ldots, F_m$ be a shelling of the facets of K. Define the t-th shelling step ($1 \leq t \leq m$) to be of* **type** (a_0, a_1, a_2) *if, out of the $d-1$ pairs of antipodal sub-facets (i.e., $(d-2)$-faces) in F_t, exactly a_i pairs ($i = 0, 1, 2$) have i sub-facets in common with the union $\bigcup_{s<t} F_s$ of preceding facets in the shelling. Necessarily $a_0 + a_1 + a_2 = d - 1$, and either $a_1 = a_2 = 0$, $a_0 = a_1 = 0$ or $a_1 \geq 1$. Then*

$$h_K^{(sc)}(q) = \sum_{t=1}^{m} \Delta_t h_K^{(sc)}(q), \tag{22}$$

where the contribution of a shelling step of type (a_0, a_1, a_2) is

$$\Delta_t h_K^{(sc)}(q) = 2^{a_0}(1+q)^{a_1}(2q)^{a_2}. \tag{23}$$

Similarly,

$$h_K^{(c)}(q) = \sum_{t=1}^{m} \Delta_t h_K^{(c)}(q), \tag{24}$$

where the contribution of a shelling step of type (a_0, a_1, a_2) is

$$\Delta_t h_K^{(c)}(q) = \begin{cases} q \cdot 2^{a_0}(1+q)^{a_1-1}(2q)^{a_2}, & \text{if } a_1 \geq 1; \\ 2^{d-1}, & \text{if } (a_0, a_1, a_2) = (d-1, 0, 0); \\ 2^{d-1}q^d, & \text{if } (a_0, a_1, a_2) = (0, 0, d-1). \end{cases} \tag{25}$$

In particular, the cubical h-vector of a shellable cubical $(d-1)$-complex is nonnegative.

Let us also restate, in cubical h-vector terminology, two recent results of G. Blind and R. Blind. The restatement of Theorem 5 actually reflects part of its original proof.

Theorem 4 [BB$_1$]
A cubical d-polytope has at least 2^d vertices, so that

$$h_0^{(c)} \leq h_1^{(c)}. \tag{26}$$

Theorem 5 [BB$_2$]
If P is a cubical polytope of an even dimension $d \geq 4$ then any shelling of the facets of P contains an even number of steps of type $(0, d-1, 0)$. Therefore, all the numbers $h_i^{(c)}$ and $h_i^{(sc)}$ (and in particular $f_0 = h_0^{(sc)}$) are even.

3 Generalized Dehn-Sommerville Equations

As an illustration, we shall now prove part *(ii)* of Theorem 3 when K is the boundary complex of a cubical d-polytope. These equations will be derived from the Generalized Dehn-Sommerville Equations for polytopes [G]. In fact, one of the (aesthetic) motivations for introducing definition (5), and subsequently also (1), has been the attempt to rephrase the Cubical Dehn-Sommerville Equations as a symmetry property like (20), in analogy with the simplicial case.

Let P be an arbitrary convex d-polytope. The well-known **Euler relation** states that the reduced Euler characteristic of the boundary of P, which is homeomorphic to a $(d-1)$-sphere, is

$$\sum_{j=0}^{d}(-1)^{j-1}f_{j-1}(P)=(-1)^{d-1}. \tag{27}$$

Again, the convention $f_{-1}=1$ is to be used here. For an arbitrary $(i-1)$-face F_{i-1} of P $(0\le i\le d)$, the **link** $P/F_{i-1}=\mathrm{lk}_P(F_{i-1})$ has the structure of a convex $(d-i)$-polytope, so that (upon multiplying by -1)

$$\sum_{j=i}^{d}(-1)^{j-i}f_{j-i-1}(P/F_{i-1})=(-1)^{d-i}. \tag{28}$$

Summing over all $(i-1)$-faces of P (for a fixed i), one obtains the **Generalized Dehn-Sommerville Equations** for an arbitrary d-polytope P:

$$\sum_{j=i}^{d}(-1)^{j-i}f_{i-1,j-1}(P)=(-1)^{d-i}f_{i-1}(P)\qquad(0\le i\le d). \tag{29}$$

Here $f_{i-1,j-1}$ is the number of flags $F_{i-1}\subseteq F_{j-1}$ where F_{i-1} (F_{j-1}) is an $(i-1)$-face (respectively, a $(j-1)$-face) of P.

If P is a *simplicial* d-polytope then

$$f_{i-1,j-1}(P)=\binom{j}{i}f_{j-1}(P)\qquad(0\le i\le j\le d) \tag{30}$$

and one obtains the **Simplicial Dehn-Sommerville Equations**:

$$\sum_{j=i}^{d}(-1)^{j-i}\binom{j}{i}f_{j-1}(P)=(-1)^{d-i}f_{i-1}(P)\qquad(0\le i\le d). \tag{31}$$

These are equivalent to the single polynomial equation

$$\sum_{j=0}^{d}f_{j-1}q^j(1-q)^{d-j}=\sum_{i=0}^{d}f_{i-1}(q-1)^{d-i} \tag{32}$$

which, using definition (6), may be stated as

$$h^{(s)}(q)=q^d h^{(s)}(q^{-1}). \tag{33}$$

This amounts to the symmetry of the simplicial h-vector:

$$h_i^{(s)}=h_{d-i}^{(s)}\qquad(0\le i\le d). \tag{34}$$

Similarly, let P be a *cubical* d-polytope. Then

$$f_{i-1,j-1}(P)=\begin{cases}\binom{j-1}{i-1}2^{j-i}f_{j-1}(P), & \text{if } 1\le i\le j\le d;\\ f_{j-1}(P), & \text{if } 0=i\le j\le d,\end{cases} \tag{35}$$

and this implies (with a shift in indices) the **Cubical Dehn-Sommerville Equations**

$$\begin{cases} \sum_{j=i}^{d-1}(-2)^{j-i}\binom{j}{i}f_j = (-1)^{d-1-i}f_i & (0 \le i \le d-1); \\ \sum_{j=-1}^{d-1}(-1)^{j+1}f_j = (-1)^d f_{-1} & (\text{Euler, } i=-1). \end{cases} \tag{36}$$

The last equation is Euler's relation for P. The other d equations may be rewritten as

$$\sum_{j=i}^{d-1}(-1)^{j-i}\binom{j}{i}2^j f_j = (-1)^{d-1-i}\,2^i f_i \qquad (0 \le i \le d-1), \tag{37}$$

so that the vector $(f_0, 2f_1, 2^2 f_2, \ldots, 2^{d-1}f_{d-1})$ for a cubical d-polytope satisfies the same linear equations (31) as the vector $(f_{-1}, f_0, f_1, \ldots, f_{d-2})$ for a simplicial $(d-1)$-polytope. This motivates definition (5), leading to the rewriting of (36) as

$$\begin{cases} h^{(sc)}(q) = q^{d-1}h^{(sc)}(q^{-1}); & \\ h^{(sc)}(-1) = 2^{d-1} + (-2)^{d-1} & (\text{Euler}) \end{cases} \tag{38}$$

or equivalently

$$\begin{cases} h_i^{(sc)} = h_{d-1-i}^{(sc)} & (0 \le i \le d-1); \\ \sum_{i=0}^{d-1}(-1)^i h_i^{(sc)} = 2^{d-1} + (-2)^{d-1} & (\text{Euler}). \end{cases} \tag{39}$$

Definition (4), together with

$$h_0^{(c)} = 2^{d-1}, \tag{40}$$

transforms these equations into the symmetry property

$$h_i^{(c)} = h_{d-i}^{(c)} \qquad (0 \le i \le d). \tag{41}$$

4 Remarks and Open Problems

If v is a vertex of a cubical $(d-1)$-complex K then the link (or vertex-figure) K/v is a simplicial $(d-2)$-complex. It was observed by Gabor Hetyei [H] that the short cubical h-polynomial of K is equal to the sum, over all vertices v of K, of the simplicial h-polynomials of K/v:

$$h_K^{(sc)}(q) = \sum_{v \in V} h_{K/v}^{(s)}(q). \tag{42}$$

From the known properties of simplicial h-vectors it thus follows that the short cubical h-vector is nonnegative for every locally Cohen-Macaulay (not necessarily shellable) cubical complex, and is unimodal for the boundary complex of a convex cubical polytope. Similar properties may be expected for the cubical h-vector.

Question 1: Is it true that

$$h_i^{(c)} \ge 0 \qquad (0 \le i \le d) \tag{43}$$

for every Cohen-Macaulay cubical $(d-1)$-complex?

Question 2: Is it true that

$$h_{i-1}^{(c)} \leq h_i^{(c)} \qquad (1 \leq i \leq d/2) \tag{44}$$

for the boundary complex of a convex cubical d-polytope?

Note that Theorem 4 above provides an affirmative answer to Question 2 for $i = 1$.

Finally, attention should be payed to a notion of h-vector for general convex polytopes (in fact, for arbitrary Eulerian posets) that was introduced by Stanley [S_3]. It is defined by recursion on the poset elements, is symmetric (for Eulerian posets), and is unimodal for rational convex polytopes. For (the boundary complex of) the d-cube, a rather complicated explicit formula was derived by I. Gessel [S_3, pp. 193–194]. It is different from our result in Lemma 1(iv) above. See also [C] for a discussion of the impact of shellability on this h-vector.

Further research into the various notions of h-vector for cubical polytopes will doubtlessly solve some of the current puzzles, replacing them by even more challenging problems.

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