Combinatorial Bases in Systems of Simplices and Chambers

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Abstract

We consider a finite set E of points in the n-dimensional affine space and two sets of objects that are generated by the set E : the system Σ of n-dimensional simplices and the system Γ of chambers. The relation $(A; \Sigma, \Gamma)$ introduced by the incidence matrix $M = \| a_{\sigma,\gamma} \|$, defines the notion of linear independence and rank in the system of simplices and system of chambers. We introduce the notion of a combinatorial basis. Combinatorial bases of chambers can be described in terms of a game. We describe the algorithm of decomposition of a convex polytope into shells. In case of the affine plane with the help of the game and the algorithm we construct combinatorial basis B of chambers. Using the algorithm we also construct a basis B' of simplices that together with the basis B of chambers form a "triangular pair".

Abstract in French

Nous considerons l'ensemble fini E des points situés dans l'espace affine de dimension n et deux systémes d'objets qui sont generées par l'ensemble E, c'est á dir, le systéme Σ de simplexes de dimension n et le systéme de chambres Γ. La relation $(A; \Sigma, \Gamma)$ introduite par la matrice d'incidence $M = \|a_{\sigma,\gamma}\|$, determine la notion de base combinatorielle. Des bases combinatorielles de chambres peuvent être formulées comme le resultat d'un jeux. Nous presentons un algorithme de decomposition d'un polytope convexe en " shells ". En cas de plan affine nous utilisons ce jeux et l'algorithme pour construire une base combinatorielle B de chambres. Avec l'aide de l'algorithme nous contruisons aussi une base B' de simplexes qui avec la base B de chambres form " une pair trianguliére".

1 Extended abstract

Consider a finite set of points $E = (e_1, e_2, \ldots, e_N)$ in the n-dimensional affine space V, $N > n$ and let $P = conv(E)$ be the convex hull of E. Let us assume that there are at least $n+1$ points in general position. If the points $e_{i_1}, \ldots, e_{i_{n+1}}$ are the points in general position we denote by $\sigma_{i_1,\ldots,i_{k+1}}$ (or sometimes simply by σ) the n-dimensional simplex spanned by these points. Denote by $\Sigma = \{\sigma\}$ the set of all these simplices. All simplices σ (as a rule overlapping) cover the polytope P. Simplices σ divide the polytope P into a finite number of chambers γ.

Let us give the definition of a chamber. Denote by $\tilde{\sigma}$ the boundary of the simplex σ and by $\tilde{\Sigma}$ the union of the boundaries of all the simplices σ, i.e. $\tilde{\Sigma} = \bigcup_{\sigma\in\Sigma} \tilde{\sigma}$ and let us denote $\check{P} = P \setminus \tilde{\Sigma}$. Let $\check{\gamma}$ be a connected component of $\check{P}$ and γ be the closure of $\check{\gamma}$.

Definition 1.1 *We will call γ a chamber and $\check{\gamma}$ an open chamber.*

Denote by Γ the set of all chambers in P. Open chambers do not overlap and all the chambers cover the polytope P. Note that every chamber is a polytope. We will say that a point is a *vertex of a chamber* if it is a vertex of the corresponding polytope.

Thus, for a finite set of points E we have constructed two sets of objects: a system of overlapping simplices Σ and a system of chambers Γ. We can consider a relation [1] $(A; \Sigma, \Gamma)$ defined by the incidence matrix M,

$M = \| a_{\sigma,\gamma} \|, \quad \sigma \in \Sigma, \quad \gamma \in \Gamma$, where

$$a_{\sigma,\gamma} = 1, \quad if \quad \gamma \subset \sigma, \quad a_{\sigma,\gamma} = 0, \quad if \quad \gamma \not\subset \sigma \tag{1}$$

For a relation a notion of rank is defined.

Let f_σ be the row of M that corresponds to a simplex σ, g_γ be the column of M that corresponds to a chamber γ. Denote by V_Σ the linear space spanned by simplices $\sigma \in \Sigma$ and by V_Γ - the linear space spanned by chambers $\gamma \in \Gamma$.

Note, that $dimV_\Sigma = dimV_\Gamma = rank(M)$.

Definition 1.2 *Rank of the relation $(A; \Sigma, \Gamma)$ is the dimension of the subspace V_Σ or subspace V_Γ.*

[1] A notion of relation was first introduced by MacLane.

Since the rows of matrix M are in one-to-one correspondence with the elements $\sigma \in \Sigma$, we can consider linear combinations of simplices σ instead of linear combinations of the corresponding rows of matrix M, and similarly, we can consider linear combinations of chambers $\gamma \in \Gamma$.

The systems of simplices and chambers appear in different problems (representation theory, Kostant partition functions , hypergeometric functions , etc) and two important questions arises: 1) how to construct a basis of simplices, 2) how to construct a basis of chambers. It is important for combinatorial problems to give an explicit construction of bases of chambers and bases of simplices and not of their linear combinations.

The combinatorial construction of a basis of simplices is in [AGZ]. In [AGZ] the theorem about the system of linear relations among simplices and the theorem about the system of linear relations among chambers are formulated. The approach in [AGZ] is based on the notion of marking (see also [B]) and is different from the geometrical approach used in the present paper. Note that differently from [AGZ], in this paper we use only two systems of objects: the system of simplices and the system of chambers, and we do not consider the system of hyperplanes.

In section 1 we prove that among chambers in the n-dimensional affine space there are linear relations that have simple geometrical meaning (Theorem 1.1 , that was formulated also in [AGZ]). All the linear relations among chambers are the linear combinations of these basic "geometrical " relations 2. In order to formulate these basic relations we will define a *new point*, and in order to determine the signs in the relation we will introduce an orientation around a new point.

Consider the set of all vertices of all chambers $\gamma \in \Gamma$. Some of these vertices are points from the set $E = \{e_i\}$ and some are not.

Definition 1.3 *A vertex w of a chamber $\gamma \in \Gamma$ is a new point if $w \notin E$.*

Let us denote by $W = \{w\}$ the set of all new points of all chambers $\gamma \in \Gamma$. The set of all vertices of all chambers $\gamma \in \Gamma$ is $E \cup W$.

Consider the case when through a new point $w \in W$ pass exactly n facets of simplices $\sigma \in \Sigma$, i.e. $(n-1)$-dimensional faces of simplices $\sigma \in \Sigma$. This means that all the facets of simplices $\sigma \in \Sigma$ are in general position. [2]

We introduce an orientation around a new point. Let $w \in W$ be a new point. The new point w lies in the intersection of exactly n facets of some simplices $\sigma \in \Sigma$. Therefore we can introduce a local coordinate system with the origin in the point w if we choose these n facets as the coordinate hyperplanes and choose an arbitraty orientation.

Let us denote by $\gamma(w)$ the set of all chambers that has the vertex w, i.e. $\gamma(w) = \{\gamma : w \in \gamma\}$. [3] Let $\gamma \in \gamma(w)$ and let $\xi \in \gamma$ be an arbitrary point with the local coordinates $\xi_1, \ldots, \xi_n$. Then the chamber $\gamma \in \gamma(w)$ can be characterized by the sequence of $+$ and $-$, i.e. by $(sign(\xi_1), \ldots, sign(\xi_n))$. It is clear that this sequence does not depend on the point $\xi \in \gamma$.

For a new point w we introduce the following function , $\varepsilon_w(\gamma)$

$$\varepsilon_w(\gamma) = sign(\xi_1) \cdot sign(\xi_2) \cdot \ldots \cdot sign(\xi_n)$$

Theorem 1.1 *For any new point $w \in W$ there is the following linear relation between chambers*

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma)\gamma = 0 \tag{2}$$

Note that the choice of another local system of coordinates around w can only change simultaneously the signs of functions $\varepsilon_w(\gamma)$ for all $\gamma \in \gamma(w)$ and therefore will not change the relation 2.

In section 2 we introduce combinatorial bases in V_Γ. We prove that such bases exist. Of course, any combinatorial basis is also a basis in terms of linear algebra. [4] Combinatorial basis consists of chambers (and not of their linear combinations).

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[2] We have this assumption in the Theorem 1.1 for simplicity of presenting the results that are used later. In sections 4,5,6 that contain the main results we do not have this assumption. Note also that this restriction about general position of facets of simplices is applied only to a new point. For an "old point" (i.e. a point from E) there are no assumptions.

[3] Chambers $\gamma \in \gamma(w)$ we will call chambers adjacent to the vertex w.

[4] In the space V_Σ we can also define a combinatorial basis using the relations among simplices introduced in [AGZ]. The linear space V_Σ will be considered in another paper.

Definition 1.4 *A basis of chambers is called combinatorial if any other chamber can be expressed via basis chambers by consequently applying the relation 2.*

Combinatorial bases of chambers naturally arise from some game. In section 2 we describe the game. This game will be also used in the proof of the Theorem 1.2 , which is an important step in the proof of the main Theorem 1.5.

Let us again assume [5] that through a new point pass exactly n facets of simplicies $\sigma \in \Sigma$, i.e. all the facets of simplices $\sigma \in \Sigma$ are in general position.

Game. Let $E = \{e_1, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space. Let Σ be the set of all n-dimensional simplices spanned by points $e_i \in E$ and Γ be the set of all chambers (see Definition 1.1). Let $W = \{w\}$ be the set of all new points (see Definition 1.3). One has to paint initially some chambers by , for example, blue color and pay for each blue chamber. After the initial painting has been completed, it is allowed to paint by green color some other chambers according to the following rule:

Rule. *If all except one chambers adjacent to a new point $w \in W$ are already painted (either by blue or green), then the last chamber adjacent to the new point w can be painted by green color.*

Green chambers are "for free", i.e. one does not pay for a green chamber [6].

Definition 1.5 *An initial painting B is called sufficient if all the chambers $\gamma \in \Gamma$, $\gamma \notin B$ can be painted by green color according to the rule above.*

The purpose of the game is to construct a sufficient initial painting that has the lowest price (number of blue chambers).

Note, that in the game the process of painting chambers by green color is actually the way of consequently applying the relations 2. Therefore, a combinatorial basis can be constructed from a sufficient initial painting that has the lowest price.

[5] In the section 4 we reformulate this game on the affine plane without this assumption.

[6] It is clear that in this game there are iterations of painting chambers by green color , because after we have painted some green chambers according to the rule, more chambers that can be painted by green color can appear.

In section 3 there is an algorithm of construction of some set of chambers B in the n-dimensional affine space. This set B will be studied in sections 4,5,6 in case of the affine plane. However, it is also important that this algorithm gives a decomposition of a convex polytope P into "shells" S_k (in case of the affine plane this decomposition into shells defines a triangulation).

Lemma 1.1 *Let $E = \{e_1, e_2, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space. A sequence of points $e_{i_1}, e_{i_2}, \ldots, e_{i_N}$, where $e_{i_k} \in E$ can be constructed such that*

$$F_k \cap P_k = \emptyset, \ for \ k = 1, \ldots, N, \tag{3}$$

where

$$P_k = conv(E \setminus (e_{i_1}, \ldots, e_{i_k}) \tag{4}$$

$$F_k = conv(e_{i_1}, \ldots, e_{i_k}) \tag{5}$$

We assume also $F_0 = \emptyset, P_N = \emptyset, P_0 = conv(E) = P, F_N = conv(E) = P.$

Algorithm. Let us construct the set of chambers B. Let $E = \{e_1, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space and let Σ be the set of simplices and Γ be the set of chambers. Construction of the set of chambers B will be made by steps. Let us denote by B_k the set of chambers that will be constructed on the k-th step. We define then

$$B = \{\gamma : \gamma \in \bigcup_k B_k\} \ .$$

First step. We choose a vertex of the polytope P and denote it by e_{i_1}. We define the set B_1 as a set of all chambers in P adjacent to the vertex e_{i_1} , i.e.

$$B_1 = \{\gamma : \ \gamma \subset P, \ e_{i_1} \in \gamma\} \ .$$

Step k. We choose a point $e_{i_k} \in E \setminus (e_{i_1}, \ldots, e_{i_{k-1}})$ such that $F_k \cap P_k = \emptyset$ and we define a set B_k as the set of all chambers in the polytope P_{k-1} that are adjacent to the point e_{i_k}, i.e.

$$B_k = \{\gamma : \ \gamma \in P_{k-1}, \ e_{i_k} \in \gamma\} \ .$$

Since in the set of polytopes $P_0 \supset P_1 \supset P_2 \supset \dots$ every polytope has less by one vertex than the preceding one, the algorithm ends after a finite number of steps. As the result of the algorithm we obtain the set B of chambers

$$B = \{\gamma : \gamma \in \bigcup_k B_k\}. \tag{6}$$

It follows from Lemma that the number of operations in the algorithm is $O(N)$.

In the algorithm we have constructed a sequence of points $\{e_{i_k}\}$ and two sets of polytopes $P_0 \supset P_1 \supset \dots \supset P_{N-n}$ and $F_0 \subset F_1 \subset \dots \subset F_{N-n}$. Let us consider the polytope

$$S_k = \overline{P_{k-1} \setminus P_k}, \tag{7}$$

where $\bar{A}$ means the closure of the set A. From the definition of polytopes S_k follows

Proposition 1.1 *There is a decomposition of the polytope P into the polytopes S_k,*

$$P = \bigcup_{i=1}^{N-n} S_i \quad . \tag{8}$$

The polytopes S_k we will call *shells.*

We describe the geometrical properties of shells S_k. Let us introduce a notion of visibility.

Definition 1.6 *Let V be an n-dimensional affine space , P be a convex polytope and a point e such that $e \notin P$. We say that the point $p \in V$ is P-visible from the point e if $(e,p) \cap P = \emptyset$, where (e,p) is an open segment.*

Proposition 1.2 *The point $e_{i_k} \in P_{k-1}$ is $P_{k-1}-$visible point from the point $e_{i_{k-1}}$.*

From this proposition and the algorithm follow that:

1. The polytope S_k has the following shape: S_k is a part of the convex cone with the vertex e_{i_k} bounded by the "cover" $\mathcal{L}_k$, where $\mathcal{L}_k$ is the set of all points of the polytope P_k that are P_k-visible from the point e_{i_k}.

2. The "cover" $\mathcal{L}_k$ satisfies the following condition: if $x_1,\ x_2 \in \mathcal{L}_k$, then any point $x \subset (x_1, x_2)$ either belongs to $\mathcal{L}_k$ or does not lie in S_k.

3. Inside S_k there are no points from E.

Particularly, if V is the affine plane, the shell S_k is a part of an angle with the vertex e_{i_k}, two sides q_1, q_2 and the "cover" $\mathcal{L}_k$ which is a convex polygon line.

Let $Q = \{q\}$ be the set of all faces of all simplices $\sigma \in \Sigma$ and $star(e) = \{q \in Q : e \text{ is the vertex of } q\}$.

Proposition 1.3 *Let S_k and S_{k+1} be two neighbor shells, e_{ik} be the vertex of S_k and $e_{i_{k+1}}$ be the vertex of S_{k+1}. Then*

$$S_k \cap S_{k+1} = star(e_{i_{k+1}}) \cap \mathcal{L}_k,$$

where $\mathcal{L}_k$ is the set of all P_k-visible points from the point e_{i_k}.

In sections 4, 5, 6 we study in details the case of the affine plane.

In section 4 we prove the following theorem.

Theorem 1.2 *Let B be the set of chambers constructed by the algorithm (see formula 6). Any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$.*

This theorem will follow from the following theorem that is formulated in terms of the game (defined in section 3) . Consider the set B as an initial painting, i.e. suppose that all the chambers $\gamma \in B$ are painted by blue color.

Theorem 1.3 *Let E be a finite set of points on the affine plane. The set of chambers $\{\gamma :\ \gamma \in B\}$ is a sufficient initial painting.*

In order to prove the Theorem 1.3 we introduce a partial ordering of new points $w \in W$ in the polygon S_k, (i.e. $w \in W \cap S_k$) and prove that this partial ordering is correctly defined. A polygon S_k is a part of an angle bounded by the convex polygon line $\mathcal{L}_k$. Let q_1, q_2 be the sides of the shell S_k, i.e. the sides of this angle.

Definition 1.7 *We will compare only new points that lie on some edge $q \in Q$. Let $w', w'' \in q$. There are two possibilities.*

1) The edge q passes through the point e_{i_k} , i.e. $e_{i_k} \in q$. Then we say that $w' < w''$, if $| e_{i_k}, w' | < | e_{i_k}, w'' |$, where $| e, w |$ is the length of the segment (e, w).

2) The edge q intersects one of the sides q_1, q_2 of the shell S_k, for example, q_1 and let w_0 be a new point in their intersection, i.e. $q \cap q_1 = w_0$, $w_0 \in W$. Then for any $w', w'' \in q$ we say that $w' < w''$, if $| w_0, w' | < | w_0, w'' |$.

We also define $e_{i_k} < w$, for any $w \in W \cap S_k$.

Definition 1.8 *Let $\gamma \in S_k$ be an arbitrary chamber $\gamma \in \Gamma$. A vertex $w \in \gamma$ is called the minimal new vertex of the chamber γ, if $w < w'$ for any $w' \in \gamma$ such that (w, w') is an edge of the chamber γ.*

Proposition 1.4 *In the polygon S_k any chamber γ has the minimal vertex.*

Let B be the set of chambers constructed in section 3. Consider the chambers $\gamma \in B$ as an initial painting for the game on the affine plane. In other words, let us paint all chambers $\gamma \in B$ by blue color. According to the game we can now paint some chambers by green color if they satisfy the rule[7].

Let e_{i_k} be the vertex and q_1, q_2 be the sides of the shell S_k. We will explain how to paint chambers in S_k adjacent to the sides q_1, q_2 of the shell S_k.

Proposition 1.5 *Suppose that any $\gamma \in S_1 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ is painted by green color. Let $\gamma' \in S_k$ be a chamber adjacent to a new point $w \in q_j$. The chamber γ' can be painted by green color according to the Rule 1 of the game.*

The Proposition 1.4 and Proposition 1.5 enables us to prove the following lemma.

Lemma 1.2 *Suppose that any chamber $\gamma \in S_1 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ is painted by green color. Then any chamber $\gamma \in S_k, \gamma \notin B$ can be painted by green color according to the Rule 1 of the game.*

[7]On the affine plane we use slightly different rule in the game (Rule 1) in order not to restrict ourselves by general position of edges of triangles that pass through a new point.

The Theorem 1.3 is then proved by induction on the polygons S_k.

In section 5 we prove that all the chambers $\gamma \in B$ are linearly independent and that the set B is a combinatorial basis of chambers. Let E be a finite set of points on the affine plane and B be the set of chambers constructed in section 3.

Theorem 1.4 *All the chambers $\gamma \in B$ are linearly independent.*

In the proof of Theorem 1.4 we correspond a simplex $\sigma \in \Sigma$ to each chamber $\gamma \in B$. The incidence between these simplices and chambers $\gamma \in B$ is given by the submatrix M' of the incidence matrix M and is defined by the following formulae:

$$a_{\sigma_j^k, \gamma_j^k} = 1, \quad a_{\sigma_j^k, \gamma_i^k} = 0, \; if \;\; i \neq j, \quad a_{\sigma_j^k, \gamma_i^m} = 0, \quad for \; k > m \; ,$$

where $\gamma^k \in B_k$, $\quad B = \{\gamma : \gamma \in \bigcup_k B_k$.

Since the matrix M' is a triangular matrix, the columns of M' are linearly independent. By construction of M' these columns correspond to the chambers $\gamma \in B$, therefore the columns g_γ, $\gamma \in B$ of matrix M are linearly independent.

From the Theorem 1.2 and Theorem 1.4 follows

Theorem 1.5 *Let E be a finite set of points on the affine plane and B be the set of chambers constructed in section 3. The set B is a combinatorial basis in the linear space V_Γ.*

Using the algorithm of construction of the set B we can calculate the rank r of the relation $(A; \Sigma, \Gamma)$.

Proposition 1.6

$$r = \binom{N-1}{2} - \sum_{q \in Q} \binom{m(q)-1}{2},$$

where $m(q)$ is the number of points $e \in E$ on the edge $q \in Q$.

In section 6 we construct a basis of simplices B' using the algorithm from section 3. We show that there is a "triangular relation" between the basis of chambers B and the basis of simplices B'.

Definition 1.9 *A pair of a basis $e_1, \ldots, e_n$ in the space V and a basis $f_1, \ldots, f_n$ in the dual space V' is called a triangular pair, if $(e_i, f_k) = 0$, for $i > k$, and $(e_i, f_i) = 1$.*

Theorem 1.6 *1. The set of simplices $B' = \{\sigma\}$ is the basis in V_Σ.*

2. The basis of chambers B and the basis of simplices B' form a triangular pair.

References

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