ABOUT HOCHSCHILD HOMOLOGY

NANTEL BERGERON

ABSTRACT. We present here two recent results involving Hochschild homology. The first one, with H. L. Wolfgang (and independently by M. O. Ronco), is a characterization of the components of the decomposition of Hochschild (co)homology for commutative algebras over a field of characteristic zero. More precisely, for $H^n(A,M) = \oplus_k H^{k,n-k}(A,M)$, we show that

$$H^{k,n-k}(A,M) = H^n\left(\mathrm{Sh}^k_\bullet(A,M)/\mathrm{Sh}^{k+1}_\bullet(A,M)\right),$$

where $\mathrm{Sh}^k_\bullet(A,M) = \mathrm{Hom}_{\mathbb{F}}(\mathrm{Sh}^k_\bullet A^{\otimes n}, M)$ and $\mathrm{Sh}^k_n \subseteq \mathbb{Q}[\mathcal{S}_n]$ is the ideal of k-shuffles. This characterization permit us to show some conjectures of Gerstenhaber and Schack. The second result, with D. Bar-Nathan, is the computation of $H^{1,n-1}(\mathcal{A}_\bullet)$ of a certain simplicial object $\mathcal{A}_\bullet$. This is motivated by the ideas of Drinfel'd stating that the obstruction to the construction of the Vassiliev knots invariants is in $H^{1,3}(\mathcal{A}_\bullet)$. We show that $H^{1,3}(\mathcal{A}_\bullet) = 0$. Hence one can construct the Vassiliev invariants using a combinatorial (and algebraic) argument. This avoids the use of the Kontsevish integrals and the Knizhnik-Zamolodchnikov connection.

RÉSUMÉ. Nous présentons deux résultats liés a l'homologie de Hochschild. Le premier, en collaboration avec H. L. Wolfgang (et indépendamment par M. O. Ronco), est une caractérisation des composantes de la décomposition de l'homologie de Hochschild d'une algèbre commutative sur un corps de caractéristique zéro: pour $H^n(A,M) = \oplus_k H^{k,n-k}(A,M)$, nous montrons que

$$H^{k,n-k}(A,M) = H^n\left(\mathrm{Sh}^k_\bullet(A,M)/\mathrm{Sh}^{k+1}_\bullet(A,M)\right),$$

où $\mathrm{Sh}^k_\bullet(A,M) = \mathrm{Hom}_{\mathbb{F}}(\mathrm{Sh}^k_\bullet A^{\otimes n}, M)$ et $\mathrm{Sh}^k_n \subseteq \mathbb{Q}[\mathcal{S}_n]$ est l'ideal des k-mélanges. Cette caractérisation nous a permis de démontrer quelques conjectures de Gerstenhaber et Schack. Le second résultat, en collaboration avec D. Bar-Nathan, est le calcul de $H^{1,n-1}(\mathcal{A}_\bullet)$ pour un certain objet simplicial $\mathcal{A}_\bullet$. Ceci est motivé par les idées de Drinfel'd qui nous donnent que l'obstruction à la construction des invariants de Vassiliev sur les noeuds est contenue dans $H^{1,3}(\mathcal{A}_\bullet)$. Nous montrons que $H^{1,3}(\mathcal{A}_\bullet) = 0$. Les invariants de Vassiliev peuvent donc être construits par des méthodes combinatoires (et algébriques). Ceci permet de contourner l'utilisation des intégrales de Kontsevish et des connections de Knizhnik-Zamolodchnikov.

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1. ON THE DECOMPOSITION OF HOCHSCHILD HOMOLOGY

Let A be a commutative algebra over $\mathbb{F}$, and M a symmetric A-bimodule. Define $\mathfrak{B}_n A = A \otimes A^{\otimes n}$, where all tensors are taken over $\mathbb{F}$. This can be viewed as a symmetric A-bimodule by multiplication on the left A factor. Let $\mathcal{S}_n$ denote the symmetric group on n elements, and let $\mathbb{Q}[\mathcal{S}_n]$ denote the group algebra. We define a (left) action of $\mathbb{Q}[\mathcal{S}_n]$ on $\mathfrak{B}_n A$ by letting $\sigma \in \mathcal{S}_n$ act on $(a_0, a_1, a_2, \dots, a_n) \in \mathfrak{B}_n A$ by $(a_0, a_{\sigma_1^{-1}}, a_{\sigma_2^{-1}}, \dots, a_{\sigma_n^{-1}})$. We have that $\mathfrak{B}_\bullet A$ is a complex with boundary map $\partial = \partial_n : \mathfrak{B}_n A \longrightarrow \mathfrak{B}_{n-1} A$, given by

$$\begin{aligned}\partial_n(a, a_1, \dots, a_n) \;=\; & (aa_1, a_2, \dots, a_n) + \\ & \sum_{i=1}^{n-1} (-1)^i (a, a_1, \dots, a_i a_{i+1}, \dots, a_n) + \\ & (-1)^n (a_n a, a_1, \dots, a_{n-1}).\end{aligned}$$

Since A is commutative and M is a symmetric A-bimodule, it follows that $H_\bullet(A, M)$ is the homology of $\mathfrak{B}_* A \otimes_A M$, and $H^*(A, M)$ is the homology of $\mathrm{Hom}_A(\mathfrak{B}_\bullet A, M) \cong \mathrm{Hom}_{\mathbb{F}}(A^{\otimes n}, M)$ (See [1, 10, 11]). We write $C_*(A, M) = \mathfrak{B}_* A \otimes_A M$, and $C^\bullet(A, M) = \mathrm{Hom}_{\mathbb{F}}(A^{\otimes n}, M)$. Note that $C_0(A, M)$ and $C^0(A, M)$ can be identified with M in a natural way, and $\partial_1 = 0$ implies that $H_0(A, M) \cong H^0(A, M) \cong M$.

Let $e_n^{(k)}$ be the Eulerian idempotents defined by

$$\sum_{k=1}^{n} e_n^{(k)} x^k = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (x + d(\sigma))(x + d(\sigma) + 1) \cdots (x + d(\sigma) + n - 1)\, \mathrm{sgn}(\sigma) \sigma,$$

where $d(\sigma) = \mathrm{Card}\{i : \sigma_i > \sigma_{i+1}\}$ is the number of *descents* of σ. The Eulerian idempotents $e_n^{(k)}$ appear in two different forms in the literature. The first form, that we denote $\rho_n^{(k)}$, appears in [4, 7, 8, 17, 21]. They are defined by

$$\sum_{k=1}^{n} \rho_n^{(k)} x^k = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (x + d(\sigma))(x + d(\sigma) + 1) \cdots (x + d(\sigma) + n - 1) \sigma.$$

It is shown that the $\rho_n^{(k)}$ are projections into the direct summands of the symmetric powers of the free Lie algebra. In the second form, the $e_n^{(k)}$ defined above, are the image of the $\rho_n^{(k)}$ under the automorphism $\theta \colon \mathbb{Q}[\mathcal{S}_n] \to \mathbb{Q}[\mathcal{S}_n]$ defined by $\theta(\sigma) = \mathrm{sgn}(\sigma)\sigma$. They are used in [1, 10, 12, 13] and are the ones of primary interest to us.

In [8] we find that

$$\mathrm{id} = \rho_n^{(1)} + \rho_n^{(2)} + \cdots + \rho_n^{(n)} \tag{1.1}$$

$$\rho_n^{(i)} \rho_n^{(j)} = \delta_{ij} \rho_n^{(i)}, \tag{1.2}$$

where $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. That is, the $\rho_n^{(k)}$ are orthogonal idempotents. Let $\mathrm{Lie}\langle \mathcal{A} \rangle$ denote the free Lie algebra on the generators $\mathcal{A} = \{a_1, a_2, \dots, a_f\}$ and let $\mathbb{Q}\langle \mathcal{A} \rangle$ denote the free associative algebra generated by $\mathcal{A}$. The Poincaré-Birkhoff-Witt theorem states that

$$\mathbb{Q}\langle \mathcal{A} \rangle \cong S(\mathrm{Lie}\langle \mathcal{A} \rangle),$$

where $S(\mathrm{Lie}\langle\mathcal{A}\rangle)$ denotes the symmetric powers of $\mathrm{Lie}\langle\mathcal{A}\rangle$. If we set the rank of the elements of $\mathcal{A}$ to be 1, then the algebra $\mathbb{Q}\langle\mathcal{A}\rangle$ and $S(\mathrm{Lie}\langle\mathcal{A}\rangle)$ are both naturally graded

$$\mathbb{Q}\langle\mathcal{A}\rangle = \bigoplus_{n\geq 0} \mathbb{Q}_n\langle\mathcal{A}\rangle,$$
$$S(\mathrm{Lie}\langle\mathcal{A}\rangle) = \bigoplus_{k\geq 0} S^k(\mathrm{Lie}\langle\mathcal{A}\rangle) = \bigoplus_{k\geq 0}\bigoplus_{n\geq 0} S^k_n(\mathrm{Lie}\langle\mathcal{A}\rangle),$$

where $S^k_n(\mathrm{Lie}\langle\mathcal{A}\rangle)$ is the total degree n component of the kth symmetric power of $\mathrm{Lie}\langle\mathcal{A}\rangle$. One of the main results of [8] is that $\rho^{(k)}_n$ is an idempotent such that

$$\mathbb{Q}_n\langle\mathcal{A}\rangle\rho^{(k)}_m \cong S^k_n(\mathrm{Lie}\langle\mathcal{A}\rangle). \tag{1.3}$$

The properties (1.1) and (1.2) then follow from (1.3). We should also note that the characters of the $\mathcal{S}_n$-module $\mathbb{Q}[\mathcal{S}_n]\rho_n^(k)$ have been computed by Bergeron, Bergeron and Garsia in [4] and independently (for $\mathbb{Q}[\mathcal{S}_n]e^{(k)}_n$) in [12].

If we translate to the $e^{(k)}_n$ the above results, we get statements for super-Lie algebras [20][1]. Moreover we have [10, 13]

$$\partial_n e^{(k)}_n = e^{(k)}_{n-1}\partial_n.$$

Combined with the identities (1.1) and (1.2) for the $e^{(k)}_n$, we have that

$$\mathfrak{B}_*A = \oplus_k e^{(k)}_*\mathfrak{B}_*A$$

is a splitting. That is, the $e^{(k)}_*\mathfrak{B}_*A$ are subcomplexes. This shows:

Theorem 1. [10, 13]

$$H_n(A,M) = \oplus_k H_{k,n-k}(A,M),$$
$$H^n(A,M) = \oplus_k H^{k,n-k}(A,M),$$

where

$$\begin{aligned} H_{k,n-k}(A,M) &= e^{(k)}_n H_n(A,M) = H_n\big(e^{(k)}_*\mathfrak{B}_*A\otimes_A M\big),\\ H^{k,n-k}(A,M) &= e^{(k)}_n H^n(A,M) \cong H_n\big(\mathrm{Hom}_{\mathbb{F}}(e^{(k)}_* A^{\otimes *}, M)\big),\\ H_{0,0}(A,M) &= H_0(A,M), \quad \textit{and} \quad H^{0,0}(A,M) = H^0(A,M). \end{aligned}$$

This splitting is the finest possible for a general commutative algebra A.

Let us also recall an alternative expression for the $e^{(k)}_n$ that will be useful later on. For this we define a *composition* of n as a k-tuple of positive integers, $p = (p_1, p_2, \ldots, p_k)$, such that $p_1 + p_2 + \cdots + p_k = n$. We refer to k as the number of *parts* of p, and we denote this number by $\kappa(p)$. We will use the shorthand $p \models n$ for "p is a composition of n." For

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[1] In the literature, (e.g. [19]), these are sometime called *graded Lie algebras*. To avoid confusion with the fact that the usual Lie algebras might be graded as-well, we prefer not to use this notation.

$\sigma \in \mathcal{S}_n$, we define the *descent set* of σ as $D(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$, and for $p \models n$, we define $S(p) = \{p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{k-1}\}$. Let $p \models n$, let

$$X_p = \sum_{D(\sigma) \subseteq S(p)} \sigma \quad \in \quad \mathbb{Q}[\mathcal{S}_n]$$

and let $\widetilde{X}_p = \theta X_p$. Let us write $L^{(k)}_m$ for the coefficient of t^m in the expansion of $(\log(1+t))^k$. We note that $L^{(k)}_m = 0$ unless $m \geq k$. We have [8]

$$\rho^{(k)}_n = \sum_{p \models n} L^{(k)}_{\kappa(p)} X_p.$$

This implies

$$e^{(k)}_n = \sum_{p \models n} L^{(k)}_{\kappa(p)} \widetilde{X}_p. \tag{1.4}$$

In the following, we refer to the element X_p as the p-shuffles. The name *shuffle* is motivated by the fact that when X_p acts on $\mathfrak{B}_n A$ we actually shuffle the entries 1, 2, …, n.

Example 2.

$$\begin{aligned} X_{(2,2)}.(a_0, a_1, a_2, a_3, a_4) \;&= (a_0, a_1, a_2, a_3, a_4) + (a_0, a_1, a_3, a_2, a_4) + (a_0, a_1, a_3, a_4, a_2) + \\ &\quad (a_0, a_3, a_1, a_2, a_4) + (a_0, a_3, a_1, a_4, a_2) + (a_0, a_3, a_4, a_1, a_2). \end{aligned}$$

We will make use of the following notations:

$$X_p.(a_0, a_1, a_2, \ldots, a_n) = a_0 \otimes (a_1 \cdots a_{p_1} \sqcup\!\sqcup\, a_{p_1+1} \cdots a_{p_1+p_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, a_{p_1+\cdots+p_{k-1}+1} \cdots a_n)$$

$$\widetilde{X}_p.(a_0, a_1, a_2, \ldots, a_n) = a_0 \otimes (a_1 \cdots a_{p_1} \overline{\sqcup\!\sqcup}\, a_{p_1+1} \cdots a_{p_1+p_2} \overline{\sqcup\!\sqcup} \cdots \overline{\sqcup\!\sqcup}\, a_{p_1+\cdots+p_{k-1}+1} \cdots a_n).$$

As the notation suggests, we can define $\sqcup\!\sqcup$ and $\overline{\sqcup\!\sqcup}$ as binary associative operations on $A^{\otimes n}$, which we will call the *shuffle* and *signed shuffle* operations. (By convention, we take $w \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup w = w \overline{\sqcup\!\sqcup} \emptyset = \emptyset \overline{\sqcup\!\sqcup} w = w$.) Moreover, the shuffle operation is commutative, and the signed shuffle operation is *signed-graded commutative*. That is $u_1 \ldots u_m \overline{\sqcup\!\sqcup}\, w_1 \ldots w_n = (-1)^{mn} w_1 \ldots w_n \overline{\sqcup\!\sqcup}\, u_1 \ldots u_m$.

Let $\mathrm{Sh}^k_n = \mathbb{Q}[\widetilde{X}_p \sigma : \kappa(p) = k,\ \sigma \in \mathcal{S}_n]$. We note that

$$\mathrm{Sh}^{l+1}_n \subseteq \mathrm{Sh}^l_n$$

since a $(l+1)$-shuffle can be expanded as a linear combination of l-shuffles. Moreover, we have the interesting fact that the map ∂ is a derivation for the signed shuffles.

Proposition 3. (see [14])

$$\begin{aligned} \partial(a_0 \otimes (a_1 \cdots a_i \overline{\sqcup\!\sqcup}\, a_{i+1} \cdots a_n)) \;&=\; \partial(a_0 \otimes (a_1 \cdots a_i)) \overline{\sqcup\!\sqcup} (a_{i+1} \cdots a_n) + \\ &\quad (-1)^i (a_1 \cdots a_i) \overline{\sqcup\!\sqcup}\, \partial(a_0 \otimes (a_{i+1} \cdots a_n)). \end{aligned}$$

This implies that

$$\partial\, \mathrm{Sh}^k_n(A) \subseteq \mathrm{Sh}^k_{n-1}(A),$$

where $\mathrm{Sh}^k_n(A) = \mathrm{Sh}^k_n\, \mathfrak{B}_n A$. Hence $\mathrm{Sh}^k_*(A)$ are subcomplexes of $\mathfrak{B}_n A$ indexed by k such that

$$\mathfrak{B}_* A = \mathrm{Sh}^1_*(A) \supseteq \mathrm{Sh}^2_*(A) \supseteq \mathrm{Sh}^3_*(A) \supseteq \cdots .$$

In particular we have that $\mathrm{Sh}^k_*(A) \otimes_A M$ and $\mathrm{Sh}^k_*(A, M) = \mathrm{Hom}_A(\mathrm{Sh}^k_*(A), M) \cong \mathrm{Hom}_{\mathbb{F}}(\mathrm{Sh}^k_*\, A^{\otimes n}, M)$ form chains of included subcomplexes. The main result of this section is the following theorem. This was independently proved by Ronco [18].

Theorem 4.

$$H_{k,n-k}(A,M) = H_n\left(\mathrm{Sh}^k_*(A) \otimes_A M \big/ \mathrm{Sh}^{k+1}_*(A) \otimes_A M\right), \tag{1.5}$$

$$H^{k,n-k}(A,M) = H_n\left(\mathrm{Sh}^k_*(A,M) \big/ \mathrm{Sh}^{k+1}_*(A,M)\right), \tag{1.6}$$

This will be an immediate consequence of our next theorem.

Theorem 5.

$$\bigoplus_{r=1}^{k} H_{r,n-r}(A,M) = H_n\left(C_*(A,M) \big/ \mathrm{Sh}^{k+1}_*(A) \otimes_A M\right), \tag{1.7}$$

$$\bigoplus_{r=1}^{k} H^{k,n-k}(A,M) = H_n\left(C^*(A,M) \big/ \mathrm{Sh}^{k+1}_*(A,M)\right), \tag{1.8}$$

Proof: We first show that

$$\mathrm{Sh}^{k+1}_n = \ker\left(\sum_{r=1}^{k} e^{(r)}_n\right) \tag{1.9}$$

in $\mathbb{Q}[\mathcal{S}_n]$. From this it will follow that

$$\left(\sum_{r=1}^{k} e^{(r)}_n\right) C_*(A,M) = C_*(A,M) \big/ \mathrm{Sh}^{k+1}_*(A) \otimes_A M,$$

$$\left(\sum_{r=1}^{k} e^{(r)}_n\right) C^*(A,M) = C^*(A,M) \big/ \mathrm{Sh}^{k+1}_*(A,M).$$

and the theorem will be proved. To show (1.9), we will need a lemma from [8]:

Lemma 6. [8] *If $p \models n$ and $\kappa(p) > r$ then $\rho^{(r)}_n X_p = 0$.*

This gives us that $e^{(r)}_n \overline{X}_p = 0$ if $\kappa(p) = k + 1$ and $r \le k$. Hence

$$\mathrm{Sh}^{k+1}_n \subseteq \ker\left(\sum_{r=1}^{k} e^{(r)}_n\right).$$

Now we note that

$$\ker\left(\sum_{r=1}^{k} e_n^{(r)}\right) = \mathrm{Im}\left(\sum_{s=k+1}^{n} e_n^{(s)}\right)$$

since the $e_n^{(r)}$ are orthogonal idempotents and $e_n^{(1)} + \cdots + e_n^{(n)} = 1$. So to get equality in (1.9). it suffices to show that $e_n^{(s)} \in \mathrm{Sh}_n^{k+1}$ for $s \geq k+1$. But this follows easily from (1.4) since $L^{(s)}_{\kappa(p)} = 0$ unless $\kappa(p) \geq s \geq k+1$. □

Remark 7. Loday [13] shows that the decomposition of Theorem 1 is valid for any functor $\Delta^{op} \longrightarrow$ $\mathbb{F}-$**Module** which factors through the category **Fin**$'$ of the sets $[n] = \{0, 1, 2, \ldots, n\}$ with morphism $f \colon [n] \longrightarrow [n]$ such that $f(0) = 0$. Theorem 4 relies only on the identity (1.9). If we let $\mathbb{Q}[\mathbf{Fin}']$ be the algebra of morphisms of **Fin**$'$, the identity (1.9) was shown inside $\mathbb{Q}[\mathcal{S}_n] \subseteq \mathbb{Q}[\mathbf{Fin}']$. Hence Theorem 4 is also valid for any functor $\Delta^{op} \longrightarrow \mathbb{F} -$ **Module** which factors through the category **Fin**$'$.

We close this section, stating some of the results we can prove using Theorem 4. Some of these results were conjectured in Gerstenhaber and Schack [11]. For $f \in C^n(A, A)$ and $g \in C^m(A, A)$, define $f \cup g \in C^{n+m}(A, A)$ by

$$f \cup g(a_1 \ldots a_{n+m}) = f(a_1 \ldots a_n) g(a_{n+1} \ldots a_{n+m}).$$

This defines a signed-graded commutative product [9] on $H^*(A, A)$, i.e. if $f^n \in H^n(A, A)$ and $g^m \in H^m(A, A)$, then $f^n \cup g^m = (-1)^{nm} g^m \cup f^n$. Gerstenhaber also defines, for $f^n \in C^n(A, A)$ and $g^m \in C^m(A, A)$ a *composition product* $f^n \bar{\circ} g^m \in C^{n+m-1}(A, A)$, as follows: For $i = 1, \ldots, n$, let

$$(f^n \circ_i g^m)(a_1, \ldots, a_{n+m-1}) = f^n(a_1, \ldots, a_{i-1}, g^m(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1}).$$

If $m = 0$, the above definition holds, with $g^m()$ interpreted as a fixed element of A. and if $n = 0$, $f^n \circ_i g^m$ is defined to be 0. Then let $f^n \bar{\circ} g^m = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f^n \circ_i g^m$. As Gerstenhaber points out, if f and g are cocycles, then $f \bar{\circ} g$ needs not be a cocycle. However, defining $[f^n, g^m] = f^n \bar{\circ} g^m - (-1)^{(n-1)(m-1)} g^m \bar{\circ} f^n$ yields a well-defined Super Lie product on the cohomology. Note that the grading is by degree, which is the dimension -1, i.e.

$$[f^n, g^m] = -(-1)^{(n-1)(m-1)} [g^m, f^n].$$

Let $\mathcal{F}_q = \bigoplus_{r \geq q} H^{*,r}(A, A)$. Gerstenhaber and Schack [10] show that $\mathcal{F}_1$ is an ideal of $H^*(A, A)$ for the cup product by exhibiting it as the kernel of a natural map $H^*(A, A) \mapsto H^*_{\mathrm{CE}}(A, A)$. In [11], they conjecture that $\mathcal{F}_q$ gives a decreasing filtration of $H^*(A, A)$ by ideals for the cup product, possibly with $\mathcal{F}_p \cup \mathcal{F}_q \subseteq \mathcal{F}_{p+q}$. In fact, we can show this using Theorem 4 and a generalization of the method of [10, 11]. Furthermore, we can show that the $\mathcal{F}_q$ are ideals for the Lie bracket and $[\mathcal{F}_p, \mathcal{F}_q] \subseteq \mathcal{F}_{p+q}$. We can also use Theorem 4 to compute the homology $H^{k,n-k}(A,A)$ for $A = \mathbb{Q}[x_1, x_2, \ldots]/I(2)$ where $I(2)$ is the ideal generated by polynomials of degree 2. This example can be used to show that in general

$$H^{2,0}(A,A) \cup H^{2,0}(A,A) \not\subseteq H^{4,0}(A,A).$$

This means that the cup product is not bi-graded in general.

2. VASSILIEV KNOT INVARIANTS

Knots invariants of finite type (Vassiliev invariants) are known to be at least as powerful as the Jones polynomial and its generalizations from quantum groups. As Vassiliev invariants are much easier to define and manipulate than quantum group invariants, it is likely that they will play a more fundamental role than the various knot polynomials.

Any numerical knot invariant V can be inductively extended to be an invariant $V^{(m)}$ of immersed circles that have exactly m *transversal* self intersections using the formulas

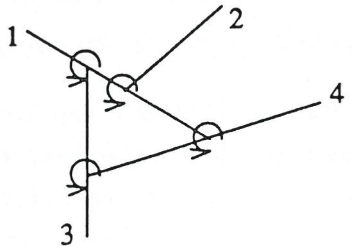
$$V^{(0)} = V,$$

We can think of the equation above as the definition of the mth partial derivative of a knot invariant in terms of its $(m-1)$st partial derivatives. In a knot projection there can be many crossings, and so one can *differentiate* with respect to many different *variables*. A Vassiliev invariant is one for which $V^{(m+1)} = 0$ for some $m \geq 0$. If $V^{(m)} \neq 0$ and $V^{(m+1)} = 0$, we say that the invariant is of type m. One fundamental question in knot theory is: *is there a Taylor theorem*? In other words, do Vassiliev invariants separate knots.

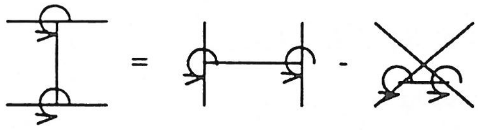
We do not address the question above. We concentrate our investigation on the *construction* of Vassiliev invariants. In [2], D. Bar-Natan showed how Kontsevish [personal communication] constructed (*integrated*) the Vassiliev invariants from its (constant) weight systems over $\mathbb{R}$ (mth derivative) using Knizhnik-Zamolodchnikov connection. This construction uses rather sophisticated integrals with values in an associative algebra of *graphs*.

Sparked by the work of Piunikhin [16], helped by the ideas of Drinfel'd [5, 6], D. Bar-Natan [3] developed a construction of the Vassiliev invariants which is combinatorial and algebraic. He reduced the problem to the computation of $H^{1,3}(\mathcal{A})$ for A a specific simplicial object. We were left showing that $H^{1,3}(\mathcal{A}) = 0$.

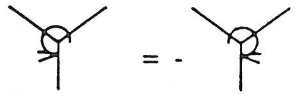
More precisely, let $G_u^{1,3}$ be the set of graphs with vertices of degree 1 and 3, with an orientation on each vertex of degree 3, and with u labeled vertices of degree 1. Below is an example of a graph in $G_4^{1,3}$.



Let $\mathcal{G}_u = \mathbb{F}G_u^{1,3}\big/\mathfrak{I}$ be the $\mathbb{F}$-module spanned by $G_u^{1,3}$ modulo the ideal $\mathfrak{I}$ generated by the *local* relations on graphs depicted as follows (see [2]):



and



Letting $\mathcal{S}_u$ act by permutation on the u labels of the graphs in $G_u^{1,3}$, we have a structure of $\mathcal{S}_u$-module on $\mathcal{G}_u$. Notice that $\bigoplus_{u\geq 0}\mathcal{G}_u$ is a graded algebra.

Consider now the $\mathbb{F}$-module $(\mathbb{F}^n)^{\bar{u}}$. We have a $\mathcal{S}_u$ on this space by letting $(v_1, v_2, \ldots, v_u)\sigma = (v_{\sigma_1}, v_{\sigma_2}, \ldots, v_{\sigma_u})$. Let $\mathcal{B}_{n,u} = (\mathbb{F}^n)^u\big/\mathcal{S}_u$ be the $\mathbb{F}$-module $(\mathbb{F}^n)^u$ modulo the action of $\mathcal{S}_u$. We put on $\mathcal{B}_*$ a structure of symmetric (co)-simplicial objects (i.e. a functor $\Delta \to \mathbb{F}$-module that factors through $\mathbf{Fin}'$) as follows. Let $\delta_i\colon \mathcal{B}_{n,u} \to \mathcal{B}_{n+1,u}$ be defined by $\delta_i(v_1, \ldots, v_u) = (\delta_i v_1, \ldots, \delta_i v_u)$ where for the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{F}^n$ we have

$$\delta_i(e_j) = \begin{cases} e_j & \text{if } j < i, \\ e_i + e_{i+1} & \text{if } j = i, \\ e_{j+1} & \text{if } j > i. \end{cases}$$

Similarly, one defines $s_i\colon \mathcal{B}_{n,u} \to \mathcal{B}_{n-1,u}$ with

$$s_i(e_j) = \begin{cases} e_j & \text{if } j < i, \\ 0 & \text{if } j = i, \\ e_{j-1} & \text{if } j > i. \end{cases}$$

The simplicial objects we are concerned with are $\mathcal{A}_n = \bigoplus_{u\geq 0}\mathcal{G}_u \otimes_{\mathbb{F}} \mathcal{B}_{n,u}$, and we want to show that $H^{1,n-1}(\mathcal{A}_*) = 0$ for $n \geq 2$.

To this end, notice first that $\mathcal{A}_n$ is a graded algebra (ranked by u) and the maps δ_i and s_i preserve the degrees. Hence $H^{i,j}(\mathcal{A}_*) = \bigoplus_{u\geq 0} H_u^{i,j}(\mathcal{A}_{*,u})$, where $\mathcal{A}_{n,u} = \mathcal{G}_u \otimes_{\mathbb{F}} \mathcal{B}_{n,u}$. Second, notice that $\mathcal{A}_{n,u}$ is of the form $(\mathcal{S}_u\text{-module}) \otimes_{\mathbb{F}} \mathcal{B}_{n,u}$. This will follow, if we can show that $H^{1,n-1}(R_\lambda \otimes_{\mathbb{F}} \mathcal{B}_{*,u}) = 0$ for $n \geq 2$ and for any irreducible $\mathcal{S}_u$-module R_λ. Since the right regular $\mathcal{S}_u$-module R contains every irreducible $\mathcal{S}_u$-module R_λ, it is enough to show that $H^{1,n-1}(R \otimes_{\mathbb{F}} \mathcal{B}_{*,u}) = 0$ for $n \geq 2$. Now, we have

$$R \otimes_{\mathbb{F}} \mathcal{B}_{n,u} \cong (\mathbb{F}^n)^u.$$

Using Künneth formula and Eilenberg-Zilber Theorem [14], we have

$$H^n(\mathbb{F}^*)^{\otimes u} \cong H^n\big((\mathbb{F}^*)^{\otimes u}\big) \cong H^n\big((\mathbb{F}^*)^u\big),$$

where the isomorphism from left to right is given by the map $[h_1] \otimes [h_2] \otimes \cdots \otimes [h_u] \mapsto [h_1 \varpi h_2 \varpi \cdots \varpi h_u]$. From theorem 4, it is clear that $H^{1,n-1}\big((\mathbb{F}^*)^u\big) = 0$ for $u \geq 2$. We are left with proving the result for $u = 1$. But in this case it is easy to check that $H^n(\mathbb{F}^*) = 0$ for $n \geq 2$. We have proved:

Proposition 8. $H^{1,n-1}(\mathcal{A}_*) = 0$ *for* $n \geq 2$.

Remark 9. Although we have concluded our program, we would like to point out that the construction of Vassiliev invariants depends on direct computations in $H^{1,2}(\mathcal{A}_*)$. The vanishing of $H^{1,3}(\mathcal{A}_*)$ garanties our computations will succeed but this may be very difficult. To avoid this [2], one can compute in $H^{1,2}(\mathcal{A}'_*)$, where $\mathcal{A}'_*$ are simpler simplicial objects described below. But for this, one would need to show that $H^{1,3}(\mathcal{A}'_*) = 0$. This is a beautifull combinatorial problem to look at.

Let $\mathcal{A}'_n = \mathbb{F}\langle t_{i,j} : 1 \leq i < j \leq n\rangle \big/ \mathfrak{I}'$ where $\mathfrak{I}'$ is the ideal generated by the elements

$$[t_{i,j}, t_{k,l}] \quad \text{and} \quad [t_{i,k} + t_{j,k}, t_{i,j}],$$

with i, j, k, l distinct and $[f, g] = fg - gf$. For simplicity, we have assumed that $t_{i,j} = t_{j,i}$ in the defining relations of $\mathfrak{I}'$ above.

Let

$$\delta_i t_{k,l} = \begin{cases} t_{k,l} & \text{if } l < i, \\ t_{k,i} + t_{k,i+1} & \text{if } l = i, \\ t_{k,l+1} & \text{if } k < i < l, \\ t_{i,l+1} + t_{i+1,l+1} & \text{if } k = i, \\ t_{k+1,l+1} & \text{if } i < k, \end{cases} \qquad s_i t_{k,l} = \begin{cases} t_{k,l} & \text{if } l < i, \\ 0 & \text{if } l = i, \\ t_{k,l-1} & \text{if } k < i < l, \\ 0 & \text{if } k = i, \\ t_{k-1,l-1} & \text{if } i < k, \end{cases}$$

and extend these maps algebraically to $\mathcal{A}'_n$. We leave it to the reader to check that δ_i and s_i are well defined.

Conjecture 10. $H^{1,3}(\mathcal{A}'_*) = 0$ *!*

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DEPT. OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138
E-mail address: nantel@math.harvard.edu