Harvey-Wiman hypermaps

Laurence Bessis

L.I.T.P., Institut Blaise Pascal, Université Paris 7
2, Place Jussieu, Paris Cedex 05
email : bessis@litp.ibp.fr

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Abstract

We show that on a hypermap (α, σ) of genus $g \geq 2$, an automorphism ψ is either of order $o(\psi) = p(1+2g/(p-1))$ if $(p, 1+2g/(p-1)) = 1$ or $o(\psi) \leq 2pg/(p-1)$, where p is the smallest divisor of the order of $Aut(\alpha, \sigma)$. We also give bounds on $| Aut(\alpha, \sigma) |$, namely, $| Aut(\alpha, \sigma) | \leq 2p(g-1)/(p-3)$ if $p \geq 5$, $| Aut(\alpha, \sigma) | \leq 15(g-1)$ if $p = 3$; thus, only when $p = 2$, the Hurwitz bound $| Aut(\alpha, \sigma) | \leq 84(g-1)$ is effective . We define p-Harvey hypermaps as hypermaps admitting an automorphism of order $p(1+2g/(p-1))$ (type I) or $2pg/(p-1)$ (type II) and characterise them as p-elliptic hypermaps.

1 Introduction

On a compact Riemann surface of genus $g > 1$ the maximal order for an automorphism is $4g + 2$ (Wiman [Wi]). Harvey [Ha] generalized the result by arithmetic methods to the following theorem:

Theorem 1.1 *Let S be a Riemann surface of genus $g \geq 2$, p be the smallest divisor of $Aut(\alpha, \sigma)$ and let ψ be an automorphism of S. Then :*
i) either $o(\psi) \leq p(1 + 2g/(p-1))$, if $o(\psi) = pm$ where p and m are coprime,
ii) or $o(\psi) \leq 2pg/(p-1)$, in all the other cases.

In this paper we first give an improvement of this result by showing that it can be seen as a generalization of results concerning a restricted type of surfaces: the so-called p-elliptic surfaces. These can be viewed as p-sheeted coverings of the sphere, where p is a prime (see below, or [Be2]). Our approach is combinatorial in nature; we represent a surface with a pair of permutations (α, σ) such that the group they generate is transitive; such a pair is called a *hypermap*. Then $Aut(S)$ becomes $Aut(\alpha, \sigma)$ the centralizer of the two permutations . As in the classical case, Machìproved that, for $g \geq 2$, where g is the genus of the hypermap (see section 2). Using his technique we prove a refinement of this result, namely that when $Aut(\alpha, \sigma)$ is of odd order, then $|Aut(\alpha, \sigma)| \leq 15(g-1)$ if $p = 3$ and $|Aut(\alpha, \sigma)| \leq 2p(g-1)/(p-3)$ if $p \geq 5$ where p is the smallest divisor of $|Aut(\alpha, \sigma)|$ (Theorem 4.2). This fact will be needed in proving the following result, that generalizes Harvey's theorem (see Theorem 4.3):

In [Be2] we have generalized the notion of a hyperelliptic hypermap to that of a p-elliptic hypermap: this is a hypermap admitting an automorphism of prime order p such that it is normal in $Aut(\alpha, \sigma)$ and fixes the maximum of points that an element of order p can fix, that is $2 + 2g/(p-1)$ (see below for a detailed explanation).

It is then possible to demonstrate the following theorem:

Theorem 1.2 *Let (α, σ) be a hypermap of genus $g \geq 2$, p be the smallest divisor of $Aut(\alpha, \sigma)$ and let ψ be an automorphism of (α, σ). Then*
i) either $o(\psi) = p(1 + 2g/(p-1))$, where p and $1 + 2g/(p-1)$ are coprime and the quotient hypermap with respect to the subgroup of order p is planar,
ii) or $o(\psi) \leq 2pg/(p-1)$, in all the other cases.

As in the case of a Riemann surface, a hypermap is *hyperelliptic* if it admits an involution ϕ fixing $2g+2$ points; This implies that ϕ is central in $Aut(\alpha, \sigma)$ (see [CoMa] p. 459). When p is fixed, both bounds are sharp since they are reached on p-elliptic hypermaps for infinitly many g, as we already showed in [Be4].

Kulkarni [Ku] defines a Wiman curve as Riemann surface which admits automorphisms of order $4g+2$ (type I) or $4g$ (type II). Accordingly, we define a Wiman hypermap as a hypermap admitting automorphisms of order $4g+2$ (type I) or $4g$ (type II). We then give the combinatorial equivalents to Kulkarni's results :

A Wiman hypermap of type I is hyperelliptic and its automorphism group is exactly the cyclic group C_{4g+2} .

A Wiman hypermap of type II is hyperelliptic and its automorphism group is exactly C_{4g} or D_{4g} (dihedral) except for $g = 2$ and $g = 3$.

Finally, we define a p-Harvey hypermap as hypermap admitting automorphisms of order $p(1+2g/(p-1))$ (type I) or $2pg/(p-1)$ (type II). Of course, Wiman hypermaps are just 2-Harvey hypermaps.

Now, Propositions 6.6 and 6.7 below show that these hypermaps admit an automorphism of order p fixing $2 + 2g/(p-1)$ points. Proposition 6.1 shows that such an automorphismis in the center of the p-Sylow subgroup containing it. Proposition 6.3 shows that such an automorphism generates a subgroup which is normal in the whole group $Aut(\alpha, \sigma)$, for $p \neq 3$. Thus, for $p \neq 3$:

it A p-Harvey hypermap of type I is p-elliptic and its automorphism group is exactly $C_{p(1+2g/(p-1))}$.

· *A p-Harvey hypermap of type II is p-elliptic and its automorphism group is exactly $C_{2pg/(p-1)}$ or $D_{2pg/(p-1)}$.*

To help proving these results, we characterize the automorphisms of the torus.

Finally, we recall a result which is well known in the theory of Riemann surfaces: *an automorphism of prime order cannot fix only one point.* For a proof of this in the case of hypermaps see [BiCo].

2 Hypermaps and automorphisms

For a general introduction to the theory of hypermaps see [CoMa]. In this section we recall a few definitions and results that will be needed in the sequel.

Definition 2.1 *A hypermap is a pair of permutations (α, σ) on B (the set of brins) such that the group they generate is transitive on B. When α is a fixed point free involution, (α, σ) is a map. The cycles of α,σ and $\alpha^{-1}\sigma$ are called edges, vertices and faces, respectively; but if there specification in termes of edges, vertices or faces is not needed, we will refer to them as points.*

Euler's formula gives the relationship between the numbers of cycles of these three permutations:

$$z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g$$

where $n = card(B)$, g is a non-negative integer, called the *genus* of (α, σ), and where for any permutation θ, $z(\theta)$ denotes the number of its cycles (cycles of length 1 are included) (see [CoMa], p.422). If $g = 0$, then (α, σ) is *planar*

Definition 2.2 *An automorphism ϕ of a hypermap (α, σ) is a permutation commuting with both α and σ:*

$$\alpha\phi = \phi\alpha \qquad \text{and} \qquad \sigma\phi = \phi\sigma \quad .$$

Thus, the full automorphism group of (α, σ), denoted by $Aut(\alpha, \sigma)$, is the centralizer in $Sym(n)$ of the group generated by α and σ. A subgroup G of $Aut(\alpha, \sigma)$ is an *automorphism group* of (α, σ); the transitivity of (α, σ) implies that $Aut(\alpha, \sigma)$ is semi-regular.

We denote by $\chi_\theta(\phi)$ the number of cycles of a permutation θ fixed by an automorphism ϕ and by $\chi(\phi)$ the total number of cycles of α, σ, and $\alpha^{-1}\sigma$ fixed by ϕ; $o(\phi)$ will be the order of ϕ. If (α, σ) is planar ($g = 0$) then $\chi(\phi) = 2$ for all non trivial automorphisms ϕ. Moreover, $Aut(\alpha, \sigma)$ is one of C_n (cyclic), D_n (dihedral), A_4, S_4 and A_5 (see [CoMa] p.464). We shall need this result later.

We now define an equivalence relation R on the set B.

Definition 2.3 *Let G be an automorphism group of the hypermap (α, σ). Two brins b_1 and b_2 are equivalent, $b_1 R b_2$, if they belong to the same orbit of G.*

This leads to the following definition.

Definition 2.4 *The quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ of (α, σ) with respect to an automorphism group G, is a pair of permutations $(\overline{\alpha}, \overline{\sigma})$ acting on the set $\overline{B}$, where $\overline{B} = B/R$ and $\overline{\alpha}$, $\overline{\sigma}$ are the permutations induced by α and σ on $\overline{B}$.*

The following Riemann-Hurwitz formula relates the genus γ of $(\overline{\alpha},\overline{\sigma})$ to the genus g of (α,σ) (see citeMa):

$$(RH1) \qquad 2g-2 = card(G)(2\gamma-2) + \sum_{\phi \epsilon G-\{id\}} \chi(\phi) \quad .$$

It follows that $\gamma \leq g$. In case G is a cyclic group, $G = <\phi>$, $(RH1)$ becomes

$$(RH2) \qquad 2g-2 = card(G)(2\gamma-2) + \sum_{i=1}^{o(\phi)-1} \chi(\phi^i) \quad .$$

As mentioned above one can prove that for $g \geq 2$ $|Aut(\alpha,\sigma)| \leq 84(g-1)$.

If ϕ is an automorphism of order m, then, for all integers i, $\chi(\phi) \leq \chi(\phi^i)$, and when m and i are coprime $\chi(\phi) = \chi(\phi^i)$.

Let (α,σ) be a hypermap, G an automorphism group of (α,σ) and let $(\overline{\alpha},\overline{\sigma})$ be the quotient hypermap of (α,σ) with respect to G. The proof of the following results can be found in [Be3]. For any element ψ in the normalizer of G in $Aut(\alpha,\sigma)$, the permutation $\bar{\psi}$, defined as $\bar{\psi} = \psi/G$, is an automorphism of $(\overline{\alpha},\overline{\sigma})$. The two following operations on (α,σ) are equivalent:

(i) quotienting (α,σ) first by G and then by $\bar{\psi}$

(ii) quotienting (α,σ) by $< G,\psi >$.

Definition 2.5 *The permutation $\bar{\psi}$ is called the projection of ψ on $(\overline{\alpha},\overline{\sigma})$. We also say ψ induces $\bar{\psi}$ on $(\overline{\alpha},\overline{\sigma})$.*

We consider now the case in which an automorphism ϕ of prime order p is normal $< \psi,\phi >$, where ψ is any element of $Aut(\alpha,\sigma)$.

Proposition 2.6 *Let ψ commute with ϕ.*

i) If ψ is of order m where p and m are coprime, then

$$\chi(\overline{\psi})p = \chi(\psi) + (p-1)\chi(\phi\psi).$$

ii) If ψ is of order pn, p and n coprime, and ϕ belong to $< \psi >$, then

$$\chi(\overline{\psi})p = \chi(\psi^p) + (p-1)\chi(\psi).$$

iii) If ψ is of order $p^m n$, $m > 1$, p and n coprime, and ϕ belong to $< \psi >$, then

$$\chi(\psi) = \chi(\overline{\psi}).$$

iv) If ψ is of order pm, m being any integer, and ϕ does not belong to $< \psi >$, then

$$\chi(\overline{\psi})p = \sum_{i=0}^{p-1} \chi(\psi\phi^i)$$

and

$$\chi(\psi\phi^i) \equiv 0 (mod \quad p).$$

v) If ψ does not commute with ϕ, then

$$\chi(\psi) = \chi(\overline{\psi}).$$

In the classical theory of Riemann surfaces, a hyperelliptic surface S is a surface admitting an involution which is central in $Aut(S)$ and fixes $2 + 2g$ points. This notion applies to hypermaps [CoMa]. In the next definition we consider automorphisms of prime order p to generalize the idea of hyperellipicity.

Definition 2.7 *A hypermap (α, σ) of genus $g > 1$ is said to be p-elliptic if it admits an automorphism ϕ of prime order p such that:*
(1) the quotient hypermap $(\overline{\alpha}, \overline{\sigma})$ with respect to ϕ is planar,
(2) $< \phi >$ is normal in $Aut(\alpha, \sigma)$.

Remark 2.8 *This definition is equivalent to that given in section 1*

Since an automorphism on the sphere fixes exactly 2 points, an automorphism ψ on a p-elliptic hypermap of genus g fixes $\chi(\psi) = 0, 1, 2, p, p + 1, 2p$ or $2 + 2g/(p - 1)$ points. It is a consequence of Proposition 2.6 together with the fact that a planar automorphism fixes exactly 2 points (see [Be4]).

Proposition 2.9 *Let (α, σ) be a hypermap and G an automorphism group; let $N(G)$ be the normalizer of G in $Aut(\alpha, \sigma)$ and $t > 0$ the number of points fixed by non trivial elements of G. Then there exists a homomorphism h from $N(G)$ to S_t and whose kernel is a cyclic group.*

We remark that when $G =< \phi >$, then the image of$h$ is contained in $S_{\chi(\phi)}$.

For complete proofs of these results see [Be3].

Theorem 2.10 *Let (α, σ) be a p-elliptic hypermap and let ψ be an automorphism of (α, σ). Then either $o(\psi) = p(1 + 2g/(p-1))$, where p and $1 + 2g/(p-1)$ are coprime, or $o(\psi) \leq 2pg/(p-1)$.*

Theorem 2.11 *Let (α, σ) be a p-elliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{pn} (cyclic) where n is a divisor of $1 + 2g/(p-1)$; C_{pn} or D_{pn} (dihedral) where n is a divisor of $2g/(p-1)$; a semi-direct product of either C_n or a lifting of D_n by C_p, where n is a divisor of $2+2g/(p-1)$; or is of order $12p$, $24p$ or $60p$ (extentions of A_4, S_4, A_5 respectively).*

Corollary 2.12 *Let (α, σ) be a hyperelliptic hypermap. Then $Aut(\alpha, \sigma)$ is either C_{2n} where n is a divisor of $2g + 1$, C_{2n} or D_{2n} where n is a divisor of $2g$; $C_n \times C_2$ or an extention of D_n by C_2, where n is a divisor of $2g + 2$; or $Aut(\alpha, \sigma)$ is of order 24, 48 or 120 (liftings of A_4, S_4, A_5 respectively).*

Proposition 2.13 *Let (α, σ) be a hypermap of genus $g > 1$ such that there exists an automorphism ϕ of prime order $2g + 1$. Then, except for the case $g = 3$ and $Aut(\alpha, \sigma) = PSL_2(\mathbf{Z}_7)$ (the simple group of order 168), (α, σ) is a $(2g + 1)-$elliptic hypermap.*

Proposition 2.14 *Let (α, σ) be a hypermap of genus $g > 30$ such that there exist an automorphism ϕ of prime order $g+1$. Then (α, σ) is a $(g+1)$-elliptic hypermap.*

Corollary 2.15 *Let (α, σ) be a hypermap of genus $g = 2$. Then $Aut(\alpha, \sigma)$ is either 1 or C_2, or else (α, σ) is $5-$elliptic, $3-$elliptic or hyperelliptic.*

3 Automorphisms of the torus

Proposition 3.1 *Let (α, σ) be a hypermap of genus 1 and ψ an automorphism. Then only two cases can happen:*

i) either ψ fixes nothing and neither does any non trival power of ψ.

ii) or ψ fixes at least one point and then ψ is of order $2, 3, 4$ or 6 with $4, 3, 2$ or 1 fixed points respectively.

4 Bounds on automorphism groups orders

By the Riemann-Hurwitz formula , we know that if a hypermap (α, σ) of genus $g > 1$ admits an automorphism group G such that the quotient hypermap with respect to it is $\gamma > 1$, then $\mid G \mid \leq g - 1$;

We now give a bound when $\gamma = 1$.

Theorem 4.1 *Let (α, σ) be a hypermap of genus $g > 1$ and G an automorphism group such that the quotient hypermap with respect to it is of genus $\gamma = 1$ then:*

$\mid G \mid \leq \frac{2p}{p-1}(g-1)$ *where p is the smallest prime that divides the order of $Aut(\alpha, \sigma)$.*

In the next theorem we show that if $Aut(\alpha, \sigma)$ is of odd order, then in the Hurwitz bound $84(g-1)$, 84 can be replaced by 15 if $\mid Aut(\alpha, \sigma) \mid$ is dividable by 3 and $\frac{2p}{p-3}$ if its smallest divisor $p \geq 5$.

Theorem 4.2 *Let (α, σ) be a hypermap of genus $g > 1$, G an automorphism group such that the quotient hypermap with respect to it is of genus $\gamma = 0$ and p the smallest prime that divides the order of $Aut(\alpha, \sigma)$. Then:*

If $p \geq 5$ $\mid G \mid \leq \frac{2p}{p-3}(g-1)$

If $p = 3$, $\mid G \mid \leq 15(g-1)$

If $p = 2$, *we have the Hurwitz bound* $\mid G \mid \leq 84(g-1)$

The following improvement of Harvey's theorem can now be obtained:

Theorem 4.3 *Let (α, σ) be a hypermap of genus $g \geq 2$, p be the smallest divisor of $Aut(\alpha, \sigma)$ and let ψ be an automorphism of (α, σ). Then either*
i) $o(\psi) = p(1 + 2g/(p-1))$, where p and $1 + 2g/(p-1)$ are coprime and the quotient hypermap with respect to the subgroup of order p is planar,
ii) or $o(\psi) \leq 2pg/(p-1)$.

5 Wiman hypermaps

Definition 5.1 *A* Wiman hypermap, *is a hypermap of genus $g \geq 2$ admitting an automorphism of order $4g + 2$ (type I) or an automorphism of order $4g$ (type II).*

Theorem 5.2 *Let (α,σ) be a Wiman hypermap of type I; then (α,σ) is hyperelliptic and $Aut(\alpha,\sigma) = C_{4g+2}$.*

Theorem 5.3 *Let (α,σ) be a Wiman hypermap of type II. Then, two cases may occur:*

i)(α,σ) is hyperelliptic and $Aut(\alpha,\sigma) = C_{4g}$, D_{4g}, or $\mid Aut(\alpha,\sigma) \mid = 48$, an extention of S_4 by C_2 and $g = 2$.

ii) (α,σ) is not hyperelliptic then $g = 3$ and $Aut(\alpha,\sigma) = C_{12}$ or $\mid Aut(\alpha,\sigma) \mid = 48$.

6 Harvey hypermaps

We recall that a normal subgroup of order p in a p-group it contained in the center of the p-group.

Proposition 6.1 *Let (α,σ) be a hypermap of genus $g \geq 2$, p a prime dividing the order of $Aut(\alpha,\sigma)$, and $\mathcal{P}$ a p-group. Let $\phi \in \mathcal{P}$ be an automorphism of prime order p such that $\chi(\phi) = 2 + 2g/(p-1)$. Then ϕ is in the center of $\mathcal{P}$.*

Corollary 6.2 *Let (α,σ) be a hypermap of genus $g \geq 2$ and G a nilpotent automorphism group. Let $\phi \in G$ be an automorphism of prime order p such that $\chi(\phi) = 2 + 2g/(p-1)$. Then ϕ is in the center of G.*

Theorem 6.3 *Let (α,σ) be a hypermap of genus $g \geq 2$, $p \neq 3$ the smallest prime dividing the order of $Aut(\alpha,\sigma)$. Let ϕ be an automorphism of order p such that $\chi(\phi) = 2+2g/(p-1)$. Then (α,σ) is p-elliptic for the automorphism ϕ.*

Definition 6.4 *A* p-Harvey hypermap, *is a hypermap of genus $g \geq 2$ admitting an automorphism of order $p(1+2g/(p-1))$ (type I) or an automorphism of order $2pg/(p-1)$ (type II) where p is the smallest prime dividing the order of $Aut(\alpha,\sigma)$.*

Remark 6.5 *A Wiman hypermap is a 2-Harvey hypermap.*

Theorem 6.6 *Let (α,σ) be a p-Harvey hypermap of type I where $p \neq 3$. Then (α,σ) is p-elliptic and $Aut(\alpha,\sigma) = C_{p(1+2g/(p-1))}$.*

Proposition 6.7 *A p-Harvey hypermap of type II admits an automorphism of order p fixing* $2 + 2g/(p-1)$ *points.*

Theorem 6.8 *A p-Harvey hypermap of type II where* $p \neq 3$ *is p-elliptic and* $Aut(\alpha, \sigma) = C_{2pg/(p-1)}$ *or* $D_{2pg/(p-1)}$,

References

[Be1] L.BESSIS, *A note on the fixed points of a hypermap automorphism*, Europeen Journal of combinatorics,(13),1992, p.65-69.

[Be2] L.BESSIS, *Induced automorphisms of hypermaps and p-elliptic hypermaps*, submitted.

[Be3] L.BESSIS, *Fixed points of induced automorphisms*, submitted.

[Be4] L.BESSIS, *p-elliptic hypermaps and a theorem of Wiman*, submitted.

[BiCo] G. BIANCHI and R. CORI, *Colorings of hypermaps and a conjecture of Brenner and Lyndon*, Pac. J. of Math., (110), 1, 1984, 41-48.

[Bu] W. BURNSIDE, *Theory of groups of finite order*, (2d edition), Dover Publications Inc., New York, 1955.

[CoMa] R.CORI AND A.MACHÌ, *Maps, hypermaps and their automorphisms: a survey*,I,II,III, Expositiones Matematicae, (10),1992, 403-427,429-447,449-467.

[Ha] W. J. HARVEY, *Cyclic groups of automorphisms of a compact Riemann surface*, Quart. J. Math Oxford, 2, (17), 1966, 86-97.

[Ku] R.KULKARNI, *Some investigations on symetries of Riemann surfaces*, Institut Mittag-Leffer, report No.8, 1989-90.

[Ma] A.MACHÌ, *The Riemann-Hurwitz formula for a pair of permutations*, Arch. d. Math. (42), 1984, 280-288.

[Wi] A.WIMAN, *Ueber die Hyperelliptischen Curven und diejenigen von Geschelchte p = 3 welche eindeutige Transformationen in sich zulassen*, Bihang Till K. Vet.-Akad. Handlingar (Stockholm 1895-6) bd.21, 1-23.