A Combinatorial Formula for Kazhdan-Lusztig polynomials

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Abstract

We give a simple, nonrecursive, combinatorial formula for any Kazhdan-Lusztig polynomial of any Coxeter group. As consequences of our main result we obtain combinatorial formulas for each coefficient of any Kazhdan-Lusztig polynomial, and we characterize those Bruhat intervals whose Kazhdan-Lusztig polynomial equals the g-polynomial (as defined by Stanley) of the dual interval. The characterization is purely combinatorial and depends only on the isomorphism type of the interval as a poset. In particular, we obtain explicit formulas for the Kazhdan-Lusztig polynomials of intervals which are lattices as well as for several other classes of intervals.

Dans cet article, on donne une formule simple, combinatoire, et nonrecursive pour n'importe quel polynôme de Kazhdan-Lusztig dans n'mporte quelle groupe de Coxeter. Par conséquent de notre résultat principal, on obtient des formules combinatoires pour chaque coefficient de n'importe quelle polynôme de Kazhdan-Lusztig et on caractērise les intervales de Bruhat dont le polynôme de Kazhdan-Lusztig est égal au g-polynôme (comme l'a defini Stanley) de l'intervale dual. Le caractērisation est totalement combinatoire et ne depend que de type d'isomorphisme d'un interval comme poset. Particuliērement, on obtient une formule explicite pour les polynômes de Kazhdan-Lusztig des intervales qui sont treillis et même pour plusieurs autres classes d'intervales.

Extended Abstract

In their fundamental paper [11] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [9], Chap. 7). These polynomials are intimately related to the Bruhat order of W and to the algebraic geometry of Schubert varieties, and are of fundamental importance in representation theory.

Our aim in this work is to give a simple, nonrecursive, combinatorial formula for any Kazhdan-Lusztig polynomial of any Coxeter group (Theorem 1.4) , and to study some consequences of it. The main idea involved in the proof and statement of this formula is that of extending the concept of the R-polynomial (see, e.g., [9], §7.5) to any (finite) multichain of W (so that, for multichains of length 1, one obtains, apart from sign, the usual R-polynomials). Once this has been done, then the Kazhdan-Lusztig polynomial of a pair

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u, v turns out to be just the sum, over all multichains from u to v, of the corresponding (generalized) R-polynomials. The R-polynomial of a multichain can be readily defined, and computed, from the ordinary R-polynomials (see (5), (6), and (7)). Since several combinatorial formulas and interpretations are known for these polynomials (see, e.g., [3], [7], and, for the case of symmetric groups, [2]) and simple recurrences exist for them, we feel that this formula is a significant step forward in the understanding of the Kazhdan-Lusztig polynomials. Though combinatorial formulas for Kazhdan-Lusztig polynomials have appeared before in the literature (see, e.g., [13], [17], [6], [4]), none of them hold in complete generality.

As consequences of our main result we obtain combinatorial formulas for each coefficient of any Kazhdan-Lusztig polynomial. Even in their simplest special cases these formulas are new (see Corollaries 1.9 and 1.10) and have interesting and non-trivial consequences. In fact, we use one of these special cases to characterize those Bruhat intervals whose Kazhdan-Lusztig polynomial equals the g-polynomial (as defined in [16]) of the dual interval (Theorem 1.13). Our characterization is purely combinatorial and depends only on the isomorphism type of the interval as a poset. As a consequence of it we obtain explicit formulas for the Kazhdan-Lusztig polynomials of intervals which are lattices (Theorem 1.14) as well as for several other classes of intervals. Finally, we briefly sketch how it is possible to obtain analogues of all our results for inverse Kazhdan-Lusztig polynomials.

We write $S = \{a_1, \ldots, a_r\}_<$ to mean that $S = \{a_1, \ldots, a_r\}$ and $a_1 < \ldots < a_r$. The cardinality of a set A will be denoted by $|A|$. For $S \subseteq \mathbf{P}$ and $j \in \mathbf{P}$ we let S_j be the j-th largest element of S, and $S_j \stackrel{\text{def}}{=} 0$ if $j > |S|$, (so $S = \{S_{|S|}, \ldots, S_1\}_<$). Given a polynomial $P(q)$, and $i \in \mathbf{Z}$, we will denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$. For $a \in \mathbf{R}$ we let $\lfloor a \rfloor$ (respectively, $\lceil a \rceil$) denote the largest integer $\leq a$ (respectively, smallest integer $\geq a$).

We follow [14], Chap. 3, for notation and terminology concerning partially ordered sets. In particular, given a finite graded poset P and $S \subseteq \mathbf{N}$ we let $P_S \stackrel{\text{def}}{=} \{x \in P : \rho(x) \in S\}$, where $\rho : P \to \mathbf{N}$ is the rank function of P, and $\alpha(P; S)$ be the number of maximal chains of P_S. We also let $P_i \stackrel{\text{def}}{=} P_{\{i\}}$ if $i \in \mathbf{N}$. We say that a finite graded poset P as above is *Eulerian* if P has a $\hat{0}$ and $\hat{1}$ and $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ for all $x, y \in P$, $x \leq y$. Recall (see, e.g., [14], §3.14, p. 138, or [15], §2, p. 190) that to any Eulerian poset P as above there are associated two polynomials, denoted $f(P; q)$ and $g(P; q)$, defined inductively as follows:

i) if $|P| = 1$ then $f(P; q) \stackrel{\text{def}}{=} g(P; q) \stackrel{\text{def}}{=} 1$;

ii) if P has rank $n + 1 \geq 1$ and $f(P; q) = \sum_{i \geq 0} k_i q^i$ then

$$g(P; q) \stackrel{\text{def}}{=} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (k_i - k_{i-1}) q^i \,, \tag{1}$$

(where $k_{-1} \stackrel{\text{def}}{=} 0$);

iii) if P has rank $n + 1 \geq 1$ then

$$f(P; q) \stackrel{\text{def}}{=} \sum_{a \in P \setminus \{\hat{1}\}} g([\hat{0}, a]; q)\, (q - 1)^{n - \rho(a)} \,. \tag{2}$$

The polynomials $f(P;q)$ and $g(P;q)$ were introduced in [15] and are two very subtle invariants of the Eulerian poset P (see [14], §3.14, and [15], §§2,3, for further information). We call $g(P;q)$ the *g-polynomial* of P, and $(h_0,\ldots,h_n)$, where $h_i \stackrel{\text{def}}{=} [q^{n-i}](f(P;q))$, for $i=0,\ldots,n$, the *h-vector* of P.

We follow [9] for general Coxeter groups notation and terminology. Given a Coxeter system (W,S) and $\sigma \in W$ we denote by $l(\sigma)$ the length of σ in W, with respect to S, and we let

$$\varepsilon_\sigma \stackrel{\text{def}}{=} (-1)^{l(\sigma)}.$$

We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{\sigma s \sigma^{-1} : \sigma \in W,\ s \in S\}$ be the set of reflections of W. We will always assume that W is partially ordered by (strong) *Bruhat order.* Recall (see, e.g., [9], §5.9) that this means that $x \leq y$ if and only if there exist $r \in \mathbf{N}$ and $t_1,\ldots,t_r \in T$ such that $t_r \ldots t_1 x = y$ and $l(t_i \ldots t_1 x) > l(t_{i-1}\ldots t_1 x)$ for $i=1,\ldots,r$. Given $u,v \in W$ we let $[u,v] \stackrel{\text{def}}{=} \{x \in W : u \leq x \leq v\}$. We consider $[u,v]$ as a poset with the partial ordering induced by W. In particular, we will often use notation such as $[u,v]_S$ or $[u,v]_i$ ($S \subseteq \mathbf{N}$, $i \in \mathbf{N}$) to denote the rank-selected subposets of $[u,v]$. For simplicity we let $c(u,v) \stackrel{\text{def}}{=} |[u,v]_{l(v)-l(u)-1}|$ and $a(u,v) \stackrel{\text{def}}{=} |[u,v]_1|$. It is well known (see, e.g., [1], Corollary 1) that intervals of W (and their duals) are Eulerian posets.

We denote by $\mathcal{H}(W)$ the *Hecke algebra* associated to W. Recall (see, e.g., [9], Chap. 7) that this is the free $\mathbf{Z}[q,q^{-1}]$-module having the set $\{T_w : w \in W\}$ as a basis and multiplication such that

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w, & \text{if } l(ws) < l(w), \end{cases} \tag{3}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see, [9], Proposition 7.4).

Proposition 1.1 *Let $v \in W$. Then*

$$(T_{v^{-1}})^{-1} = q^{-l(v)} \sum_{u \leq v} (-1)^{l(v)-l(u)} R_{u,v}(q)\, T_u\,,$$

where $R_{u,v}(q) \in \mathbf{Z}[q]$.

The polynomials $R_{u,v}$ defined by the previous proposition are called the *R-polynomials* of W. It is easy to see that $\deg(R_{u,v}) = l(v)-l(u)$, and that $R_{u,u}(q)=1$, for all $u,v \in W$, $u \leq v$. It is customary to let $R_{u,v}(q) \stackrel{\text{def}}{=} 0$ if $u \not\leq v$.

The R-polynomials can be used to define the Kazhdan-Lusztig polynomials. The following result is not hard to prove (and, in fact, holds in much greater generality, see [16], Corollary 6.7 and Example 6.9) and a proof can be found, e.g., in [9], §§7.9-11, or [11], §2.2.

Theorem 1.2 *There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v\in W} \subseteq \mathbf{Z}[q]$, such that, for all $u,v \in W$:*

i) $P_{u,v}(q) = 0$ *if $u \not\leq v$;*

ii) $P_{u,u}(q) = 1$;

iii) $deg(P_{u,v}(q)) \leq \lfloor \frac{1}{2}(l(v) - l(u) - 1) \rfloor$, *if* $u < v$;

iv)

$$q^{l(v)-l(u)}\, P_{u,v}\left(\frac{1}{q}\right) = \sum_{u \leq z \leq v} R_{u,z}(q)\, P_{z,v}(q)\,, \tag{4}$$

if $u \leq v$.

The polynomials $P_{u,v}(q)$ defined by the preceding theorem are called the *Kazhdan-Lusztig polynomials* of W. Note that parts iii) and iv) of Theorem 1.2 actually yield an inductive procedure to compute the polynomials $P_{u,v}(q)$ for all $u, v \in W$, taking parts i) and ii) as initial conditions.

Throughout this work (unless otherwise explicitly stated) (W, S) denotes a fixed (but arbitrary) Coxeter system.

We can now define the crucial concept of this work, namely the R-polynomial of a multichain. This is a polynomial which can be associated to any (finite) multichain of W and which reduces (essentially, see (7)) to the ordinary R-polynomial if the multichain has length one. The importance of this concept lies in the fact that it plays a fundamental role in the computation of the Kazhdan-Lusztig polynomials of a Coxeter group (see Theorem 1.4).

Given a polynomial $p(x) \stackrel{\text{def}}{=} \sum_{i=0}^{d} a_i x^i$ (with coefficients in some ring) and $j \in \mathbf{Z}$ we let

$$U_j(p) \stackrel{\text{def}}{=} \sum_{i=j}^{d} a_i x^i\,. \tag{5}$$

So, for example, $U_2(x - 3x^2 + 2x^3) = -3x^2 + 2x^3$, and $U_j(p) = p$, for any polynomial p, if $j < 0$.

Given a multichain $a_0 \leq a_1 \leq \ldots \leq a_{r+1}$ $(r \in \mathbf{N})$ in W we define a polynomial $\mathcal{R}_{a_0,a_1,\ldots,a_{r+1}}(q)$ inductively as follows,

$$\mathcal{R}_{a_0,a_1,\ldots,a_{r+1}}(q) \stackrel{\text{def}}{=} \mathcal{R}_{a_0,a_1}(q)\, U_{\lceil \frac{d+1}{2} \rceil}\left(q^d \mathcal{R}_{a_1,\ldots,a_{r+1}}\left(\frac{1}{q}\right)\right)\,, \tag{6}$$

(where $d \stackrel{\text{def}}{=} l(a_{r+1}) - l(a_1)$) if $r \in \mathbf{P}$, and

$$\mathcal{R}_{a_0,a_1,\ldots,a_{r+1}}(q) \stackrel{\text{def}}{=} (-1)^{l(a_1)-l(a_0)} R_{a_0,a_1}(q) \tag{7}$$

if $r = 0$. We call $\mathcal{R}_{a_0,a_1,\ldots,a_{r+1}}(q)$ the *R-polynomial* of the multichain $a_0 \leq a_1 \leq \ldots \leq a_{r+1}$. For example, in $W = S_4$ we have that

$$\begin{aligned}
\mathcal{R}_{2134,2314,2413}(q) &= \mathcal{R}_{2134,2314}(q)\, U_1\left(q\mathcal{R}_{2314,2413}\left(\frac{1}{q}\right)\right) \\
&= -R_{2134,2314}(q)\, U_1\left(-qR_{2314,2413}\left(\frac{1}{q}\right)\right) \\
&= (1-q)\, U_1(q-1) = q - q^2\,.
\end{aligned}$$

An important property of the R-polynomial of a multichain is the following.

Proposition 1.3 *Let $r \in \mathbf{P}$ and $a_0 \leq a_1 \leq \ldots \leq a_{r+1}$ be a multichain in W such that $\mathcal{R}_{a_0,\ldots,a_{r+1}} \neq 0$. Then $l(a_{r+1}) - l(a_r) \geq 1$, and $l(a_{i+1}) - l(a_i) \geq 2$ for $i = 1, \ldots, r-1$. In particular $l(a_{r+1}) - l(a_1) \geq 2r - 1$.*

We can now state the main result of this work.

Theorem 1.4 *Let $u, v \in W$. Then*

$$P_{u,v}(q) = \sum_{C \in \mathcal{C}(u,v)} \mathcal{R}_C(q) \tag{8}$$

where $C(u,v)$ denotes the set of all multichains from u to v.

Note that, by Proposition 1.3, the sum on the RHS of (8) is finite.
Given a finite Coxeter system (W, S) we denote by ω_0 the longest element of W.

Corollary 1.5 *Let (W, S) be a finite Coxeter system, and $u, v \in W$. Then*

$$P_{u,v}(q) = P_{\omega_0 u \omega_0, \omega_0 v \omega_0}(q).$$

The following result is known (see, e.g., [6], p. 356), but is made particularly transparent by Theorem 1.4.

Corollary 1.6 *Let $u, v \in W$. Then*

$$P_{u,v}(q) = P_{u^{-1},v^{-1}}(q). \square$$

We can use Theorem 1.4 to obtain explicit formulas for the coefficients of Kazhdan-Lusztig polynomials.

The next result shows that the summation set on the RHS of (8) becomes smaller when we extract the coefficient of q^k.

Theorem 1.7 *Let $u, v \in W$, and $k \in \mathbf{N}$. Then*

$$[q^k](P_{u,v}(q)) = [q^k]\Big(\sum_{C \in C_k(u,v)} \mathcal{R}_C(q)\Big),$$

where $C_k(u,v)$ denotes the set of all multichains from u to v of length $\leq k+1$.

Using the preceding result it is possible to express the coefficients of Kazhdan-Lusztig polynomials solely in terms of coefficients of (ordinary) R-polynomials.

Theorem 1.8 *Let $u, v \in W$, and $k \in \mathbf{N}$. Then*

$$[q^k](P_{u,v}(q)) = \sum_{S \subseteq [k]} \sum_{(a_0,\ldots,a_{|S|+1}) \in C_S(u,v)} \prod_{r=0}^{|S|} [q^{l(a_{r+1}) - l(v) + S_r + S_{r+1}}](R_{a_r,a_{r+1}}(q)) \tag{9}$$

where $C_S(u,v)$ is the set of all multichains $a_0 \leq a_1 \leq \ldots \leq a_{|S|+1}$ from u to v such that $S_{r+1} \leq l(v) - l(a_r) - S_r \leq S_r - 1$, for $r = 1, \ldots, |S|$, and where $S_0 \stackrel{\text{def}}{=} l(v) - l(u) - k$.

For small values of k the formula in Theorem 1.8 yields very explicit information on $[q^k](P_{u,v})$. Even for $k = 1$ and $k = 2$ the resulting formulas are new and have non-trivial consequences.

Corollary 1.9 *Let $u, v \in W$. Then*

$$[q](P_{u,v}(q)) = (-1)^{l(v)-l(u)} [q](R_{u,v}(q)) + c(u,v)$$

where $c(u,v)$ is the number of coatoms in $[u,v]$.

Corollary 1.10 *Let $u, v \in W$. Then*

$$\begin{aligned}[q^2](P_{u,v}(q)) &= (-1)^{l(v)-l(u)}[q^2](R_{u,v}(q)) + \sum_{a\in[u,v]_1^*} (-1)^{l(v)-l(u)-1}[q](R_{u,a}(q)) + |[u,v]_2^*| \\ &\quad - \sum_{a\in[u,v]_3^*} [q](R_{a,v}(q)) + \alpha([u,v]^*, \{1,3\}).\end{aligned}$$

A consequence of Corollary 1.9 is the following result which was conjectured by G. Kalai, [10].

Corollary 1.11 *Let $u, v \in W$ be such that $P_{u,v}(q) = 1$. Then*

$$l(v) - l(u) \geq c(u,v).$$

A further application of Corollary 1.9 is the following result which was first proved by Dyer (see [5]).

Corollary 1.12 *Let $v \in W$. Then*

$$[q](P_{e,v}(q)) = c(e,v) - a(e,v). \tag{10}$$

Note that (10) does not hold in general. For example, if $W = S_4$, $u = 2143$, and $v = 4231$ then $[q](P_{u,v}) = 1$ but $c(u,v) = 4$ and $a(u,v) = 4$.

We can also use Corollary 1.9 to characterize those pairs of elements $u, v \in W$ such that $P_{u,v}(q)$ equals the g-polynomial of $[u,v]^*$. Our characterization is purely combinatorial and depends only on the isomorphism type of the poset $[u,v]$.

The next result greatly generalizes Theorem 4.8 of [2]. We denote by S_3 the poset isomorphic to the Bruhat order on S_3.

Theorem 1.13 *Let $u, v \in W$, $u \leq v$. Then the following are equivalent:*

i) *$[u,v]$ does not contain any interval isomorphic to S_3;*

ii) *$R_{u,v}(q) = (q-1)^{l(v)-l(u)}$;*

iii) *$[q](R_{u,v}(q)) = (-1)^{l(v)-l(u)}(l(u) - l(v))$;*

iv) *$[q](P_{u,v}(q)) = c(u,v) - l(v) + l(u)$;*

v) *$P_{u,v}(q) = g([u,v]^*; q)$.*

The preceding theorem has many consequences. The main one is probably the following which is obtained by combining Theorems 1.13 and 1.8, and which gives, in particular, an explicit combinatorial formula expressing the Kazhdan-Lusztig polynomial of any interval which is a lattice solely in terms of its combinatorial structure.

Theorem 1.14 *Let $u, v \in W$, $u \leq v$, be such that $[u, v]$ does not contain any interval isomorphic to S_3. Then*

$$[q^k](P_{u,v}(q)) = \sum_{S \subseteq [k]} (-1)^{l(v)-|S|+k} \sum_{(a_0,\ldots,a_{|S|+1}) \in C_S(u,v)} (-1)^{\sum_{i=1}^{|S|} l(a_i)} \prod_{r=0}^{|S|} \binom{l(a_{r+1})-l(a_r)}{l(v)-l(a_r)-S_r-S_{r+1}}$$

where $C_S(u, v)$, and S_0, have the same meaning as in Theorem 1.8. □

Another consequence of Theorem 1.13 is the following result which was first proved, in the case of symmetric groups, by M. Haiman and G. Kalai ([8]).

Corollary 1.15 *Let $u, v \in W$, $u \leq v$, be such that $[u, v]$ is a lattice. Then $R_{u,v}(q) = (q-1)^{l(v)-l(u)}$.* □

Two further non-trivial consequences of Theorem 1.13 are the following.

Corollary 1.16 *Let $u, v \in W$, $u \leq v$, be such that $[q](R_{u,v}(q)) = (-1)^{l(v)-l(u)}(l(u) - l(v))$. Then $R_{x,y}(q) = (q-1)^{l(y)-l(x)}$ for all $u \leq x \leq y \leq v$.* □

Corollary 1.17 *Let $u, v \in W$, $u \leq v$, be such that $[q](P_{u,v}(q)) = c(u, v) - l(v) + l(u)$. Then $P_{x,y}(q) = g([x, y]^*; q)$ for all $u \leq x \leq y \leq v$.* □

Theorem 1.13 also enables us to use results from the theory of g-polynomials to compute explicitly the Kazhdan-Lusztig polynomials of some classes of intervals. For $n \in \mathbb{N}$ we denote by B_n and Q_n the Boolean algebra of rank n and the face lattice of an n-dimensional cube, respectively.

Corollary 1.18 *Let $u, v \in W$, $u \leq v$, be such that $[u, v] \cong B_{l(v)-l(u)}$ (as posets). Then*

$$P_{u,v}(q) = 1 .$$

Corollary 1.19 *Let $u, v \in W$, $u \leq v$, be such that $[u, v] \cong Q_d^*$ (as posets) where $d \stackrel{\text{def}}{=} l(v) - l(u) - 1$. Then*

$$P_{u,v}(q) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{1}{d-i+1} \binom{d}{i} \binom{2(d-i)}{d} (q-1)^i . \tag{11}$$

Equivalently, $[q^i](P_{u,v}(q))$ is the number of plane trees with $d+1$ vertices such that exactly i vertices have ≥ 2 sons, for all $i \in \mathbb{N}$. In particular, $deg(P_{u,v}(q)) = \lfloor \frac{d}{2} \rfloor$.

Let $k \in \mathbf{N}$. Recall (see, e.g., [15], p.191) that a finite graded poset P with $\hat{0}$ is said to be *k-simplicial* if $[\hat{0}, x]$ is isomorphic to a Boolean algebra for all $x \in P_k$. It is well known (see, e.g., [1]) that intervals of W, and their duals, are 2-simplicial.

For brevity, we say that an interval $[u,v]$ is *S_3-free* if it satisfies condition i) of Theorem 6.3.

Proposition 1.20 *Let $u, v \in W$, $u \leq v$, and $k \in \mathbf{N}$. Suppose that $[u,v]^*$ is k-simplicial and S_3-free. Then*

$$[q^j](P_{u,v}(q)) = \sum_{i=0}^{j} (-1)^{i-j} \binom{l(v)-l(u)-i}{l(v)-l(u)-j} \, |[u,v]_i^*|$$

for $j = 0, \ldots, \lfloor \frac{k+1}{2} \rfloor$.

For $k = 3$ we can also prove the converse of Proposition 1.20.

Proposition 1.21 *Let $u, v \in W$, $u \leq v$. Then the following are equivalent:*

i) $[q^j](P_{u,v}(q)) = \sum_{i=0}^{j} (-1)^{i-j} \binom{l(v)-l(u)-i}{l(v)-l(u)-j} \, |[u,v]_i^*|$, *for $j \leq 2$;*

ii) $[u,v]^*$ *is S_3-free and 3-simplicial.*

We now briefly outline how it is possible to obtain analogues of all the results in this work for the inverse Kazhdan-Lusztig polynomials. Recall (see, e.g., [9], §7.13, or [12], p.190) that these are the polynomials $\{P_{u,v}^*(q)\}_{u,v \in W}$ uniquely defined by the condition that

$$\sum_{u \leq a \leq v} (-1)^{l(a)-l(u)} \, P_{u,a}(q) \, P_{a,v}^*(q) = \delta_{u,v} \tag{12}$$

for all $u, v \in W$, $u \leq v$.

For any multichain $a_0 \leq a_1 \leq \ldots \leq a_{r+1}$ ($r \in \mathbf{N}$) of W we now define a polynomial $\mathcal{R}_{a_0,\ldots,a_{r+1}}^*(q)$ as follows. We let

$$\mathcal{R}_{a_0,\ldots,a_{r+1}}^*(q) \stackrel{\text{def}}{=} U_{\lceil \frac{d+1}{2} \rceil} \left(q^d \mathcal{R}_{a_0,\ldots,a_r}^* \left(\frac{1}{q} \right) \right) \mathcal{R}_{a_r,a_{r+1}}^*(q) \tag{13}$$

(where $d \stackrel{\text{def}}{=} l(a_r) - l(a_0)$), if $r \in \mathbf{P}$, and

$$\mathcal{R}_{a_0,\ldots,a_{r+1}}^*(q) \stackrel{\text{def}}{=} (-1)^{l(a_{r+1})-l(a_r)} \, R_{a_r,a_{r+1}}(q), \tag{14}$$

if $r = 0$. We call the polynomial $\mathcal{R}_{a_0,\ldots,a_{r+1}}^*(q)$ the *dual R-polynomial* of the multichain $a_0 \leq a_1 \leq \ldots \leq a_{r+1}$.

The analogue of our main result is the following.

Theorem 1.22 *Let $u, v \in W$. Then*

$$P_{u,v}^*(q) = \sum_{C \in \mathcal{C}(u,v)} \mathcal{R}_C^*(q) . \square$$

In the case that (W,S) is a finite Coxeter system there is a simple relation between R-polynomials and dual R-polynomials of multichains.

Proposition 1.23 *Let (W,S) be a finite Coxeter system, and $a_0 \leq a_1 \leq \ldots \leq a_{r+1}$ be a multichain in W. Then*

$$\mathcal{R}^*_{a_0,\ldots,a_{r+1}}(q) = \mathcal{R}_{w_0 a_{r+1},\ldots,w_0 a_0}(q).\square$$

The analogue of Corollary 1.9 is the following.

Corollary 1.24 *Let $u, v \in W$. Then*

$$[q](P^*_{u,v}(q)) = (-1)^{l(v)-l(u)}[q](R_{u,v}) + a(u,v),$$

where $a(u,v)$ is the number of atoms in $[u,v]$. $\square$

The analogue of Theorem 1.13 is the following.

Theorem 1.25 *Let $u, v \in W$, $u \leq v$. Then the following are equivalent:*

i) *$[u,v]$ does not contain any interval isomorphic to S_3;*

ii) *$R_{u,v}(q) = (q-1)^{l(v)-l(u)}$;*

iii) *$[q](R_{u,v}(q)) = (-1)^{l(v)-l(u)}(l(u)-l(v))$;*

iv) *$[q](P^*_{u,v}(q)) = a(u,v) - l(v) + l(u)$;*

v) *$P^*_{u,v}(q) = g([u,v];q)$.* $\square$

Similar analogues hold for all the other results.

References

[1] A. Björner, *Orderings of Coxeter groups*, Combinatorics and Algebra, Contemporary Math. vol. 34, Amer. Math. Soc. 1984, 175-195.

[2] F. Brenti, *Combinatorial properties of the Kazhdan-Lusztig R-polynomials for S_n*, Advances in Math., to appear.

[3] V. V. Deodhar, *On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells*, Invent. Math. **79** (1985), 499-511.

[4] V. V. Deodhar, *A combinatorial setting for questions in Kazhdan-Lusztig Theory*, Geom. Dedicata, **36** (1990), 95-120.

[5] M. Dyer, *Hecke algebras and reflections in Coxeter groups*, Ph. D. Thesis, University of Sydney, 1987.

[6] M. Dyer, *On some generalisations of the Kazhdan-Lusztig polynomials for "universal" Coxeter groups*, J. Algebra, **116** (1988), 353-371.

[7] M. Dyer, *Hecke algebras and shellings of Bruhat intervals*, preprint.

[8] M. Haiman, G. Kalai, private communication, April 1992.

[9] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, no.29, Cambridge Univ. Press, Cambridge, 1990.

[10] G. Kalai, private communication, September 1992.

[11] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.

[12] D. Kazhdan, G. Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator, Proc. Sympos. Pure Math. 34, Amer. Math. Soc., Providence, RI, 1980, pp. 185-203.

[13] A. Lascoux, M.-P. Schützenberger, *Polynômes de Kazhdan & Lusztig pour les grassmanniennes*, Young tableaux and Schur functions in algebra and geometry, Astérisque, **87-88** (1981), pp. 249-266.

[14] R. P. Stanley, *Enumerative Combinatorics* , vol.1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.

[15] R. P. Stanley, *Generalized h-vectors, intersection cohomology of toric varieties, and related results*, Adv. Studies Pure Math., **11** (1987), 187-213.

[16] R. P. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc., **5** (1992), 805-851.

[17] A. Zelevinski, *Small resolutions of singularities of Schubert varieties*, Funct. Anal. Appl., **17** (1983), 142-144.